Toward Accounting for Stakes in Voting

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Abstract

When a society collectively makes a political decision, its members inevitably have different stakes in the decision—that is, some may stand to gain or lose a lot, while others are similarly affected by all potential outcomes. It is known—both colloquially and in the literature—that failing to account for people’s differing stakes can lead to highly socially-suboptimal decisions. Despite this, standard decision processes in social choice generally do not account for stakes, instead aiming to treat people symmetrically. Building on distinct, stakes-related ideas from across the literature, we introduce a general framework for formally measuring and accounting for voters’ stakes. Applying this framework, we use the established notion of distortion to give bounds on the possible welfare impact of accounting for stakes in ranking-based voting. We then apply these bounds to define and analyze a proof-of-concept voting mechanism designed to account for stakes. Finally, we explore how this mechanism design concept—and our stakes framework—can be extended to transform a multitude of existing decision processes into processes that account for stakes, thus illuminating a rich mechanism design space for future work.

1 Introduction

At the backbone of a democratic society are collective decisions: society-level choices over, e.g., policies or candidates, on which a society’s members weigh in and are in turn potentially affected by the outcome. We start this paper from the fundamental idea that a given collective decision can affect different members of society to differing degrees. It is not hard to come up with salient examples: consider, for instance, the decision of whether to instate a city-wide COVID masking requirement. Immunocompromised people and/or essential workers are likely much more affected by the outcome of this decision than people who work from home and can easily shelter in place. In mainstream language, we often talk about these disparities in terms of stakes, where someone with high stakes in a decision stands to gain or lose significantly depending on its outcome, while someone with low stakes is relatively unaffected by the decision.

It is intuitive not only that people’s stakes may differ, but that accounting for these differing stakes is important to the quality of decisions. This is often seen in critiques of how processes fail to do so: for instance, many critique the fact that the global south, despite being disproportionately affected by climate change, has thus far been granted far less power in deciding global climate policy (Strazzante, Rycken, and Winkler 2022). The importance of accounting for stakes is even codified—if coarsely—in the design of modern democracy: we restrict who can vote in elections based on residency because, say, people who reside in Kansas are not impacted by local elections in California.

Despite the omnipresent sense that stakes are an important factor in collective decisions, most decision processes studied in social choice do not account for stakes. Instead, most aim for some notion of equality across people, allotting them equal voting power regardless of their stakes: for example, approval voting, ranking-based voting, and even the more expressive approach of range voting (Pivato 2014) allot each person one vote (for details, see Appendix A.1).

1.1 A motivating example: why stakes matter

We now present Example 1.1, which, within the standard voting model, formalizes the intuition that ignoring voters’ differing stakes can lead to poor democratic outcomes. From this example, the model we will work with arises naturally: there are $n$ voters, $m$ alternatives, and each voter $i$ has some nonnegative utility $u_i(a)$ for each alternative $a$. Conceptually, $u_i(a)$ measures the extent to which voter $i$ stands to gain from $a$ being chosen. Voters express their preferences by ranking the alternatives in order of their utilities. These $n$ rankings are then aggregated via a voting rule: any mapping from $n$ rankings to a winning alternative (we focus mostly on deterministic voting rules, due to their widespread use). We evaluate the quality of the winner by its utilitarian social welfare—the sum of voters’ utilities for that alternative.

Example 1.1. Residents will vote to decide the placement of a new bus stop in their neighborhood. There are possible locations, $a$ and $b$. The residents fall into one of two types: Type 1 residents compose 10% of the population and live in the area directly surrounding $a$, but are far from $b$. They do not have cars and rely on the bus, and thus would benefit significantly from $a$. As such, they have utility vector $(10, 0)$, specifying utilities in the order $a, b$. Type 2 residents compose the remaining 90% of voters and live in the area surrounding bus stop $b$. Type 2 voters have cars and essentially never take the bus; in the rare case they do, they are inclined to take it from $b$ due to geographic and demographic prox-
iminity. Their weak preference for $b$ is reflected in their utility vector $(0, \epsilon)$, where $\epsilon > 0$ is some small constant.

**The election:** Based on their utilities, 90% of residents will vote for $b$ over $a$, and any majority-consistent voting rule must choose $b$ as the winner. This seems like a poor outcome: the election has chosen a bus stop that virtually no one will ever use, while the other alternative would have helped a lot of people. This intuitive suboptimality is captured in the fact that social welfare of $b$, the winner, is more than $1/\epsilon$ times smaller than that of $a$, the highest-welfare alternative.

Although this is a hypothetical example, it paints a picture of why stakes matter. The chosen numbers did not matter much: the key problem was that Type 1 voters had disproportionately high stakes, but lacked the voting power to sway the election. In contrast, Type 2 voters had very low stakes—i.e., they stood to gain very little—so when they got their way, little value was generated for the population. Given that people are likely to have disparate stakes in real issues—and sometimes minority groups may have much higher stakes than the majority—this example seems practically pressing: it suggests that such welfare loss could occur in real elections. This is especially concerning because, as illustrated in both Example 1.1 and preceding real-world examples, the highest-stakes voters—who stand to lose the most when welfare loss occurs—are likely to be already-marginalized, having high stakes precisely because they have less resources with which to adjust to policies that are suboptimal for them.

### 1.2 “Stakes” within the literature

Although standard social choice mechanisms do not account for stakes, the concept has been conceived of in multiple disciplines. From the social sciences, there is the philosophical notion of proportionality—the idea that “power should be distributed in proportion to people’s stakes in the decision under consideration” (Brighouse and Fleurbaey 2010).

In the context of binary decisions, Azrieli (2018) explores the welfare cost of treating agents symmetrically when they have different stakes. Finally, the results of Fleurbaey (2008) suggest that accounting for stakes can help: they show that in elections over two alternatives, reweighting each voter’s vote by the difference of their utilities—a measurement of their “stakes”—increases the welfare of majority voting.

Slightly less directly, the pitfalls of ignoring stakes are also recognized in the literature on distortion—the competitive ratio between the welfare of the winner and the max-welfare alternative, as in Example 1.1. In fact, two of the most pervasive assumptions used in this literature to circumvent otherwise unbounded welfare loss both implicitly relate to stakes. First, some work assumes utility queries (e.g., Abramowitz, Anshelevich, and Zhu 2019)—where the voting mechanism can access limited information about voters’ utilities via queries—which permits some knowledge of preference intensities, i.e., stakes. Another common assumption is unit sum utilities—where each voter’s utilities must add to 1 (Procaccia and Rosenschein 2006). As we will formalize later, this is akin to assuming voters have identical “stakes”, as measured by the sum of their utilities.

Although these approaches point out—and sometimes in theory circumvent—the pitfalls of ignoring stakes, there remains a gap between these approaches and practically accounting for stakes. Designing voting systems that account for stakes is challenging for two reasons: first, voters’ stakes are unobservable. Simply querying them, as in the related work, seems unlikely to work: voters would be strongly incentivized to inflate their stakes, and even if they were honest, they may not evaluate their utilities on the same scale. The unit-sum utilities assumption suggests treating all voters’ stakes as the same, but per Example 1.1, this assumption seems unlikely to carry over to practice. Secondly, even if we had stakes information, it is not clear how to use it: while in some contexts we could simply reweight votes, in many cases this could be politically unpalatable.

### 1.3 Our Approach and Contributions

Via the following contributions, we work toward closing this gap by laying the groundwork for a mechanism design approach to accounting for stakes. We lay this groundwork within perhaps the most ubiquitously-studied decision process—ranking-based voting—but as we discuss in Section 5, our formalisms extend naturally to other decision processes.

**C1: a formal stakes framework.** In Section 2.2 we lay the groundwork for studying stakes in full generality by embedding a formal model of stakes into the standard voting model. This entails answering Q1 and Q2:

**Q1:** How should we measure a voter’s stakes? Example 1.1 illustrates that intuitively, a voter’s stakes are captured in their utility vector. In a binary decision, it seems natural to measure voters’ stakes as the difference between their two utilities, as do Fleurbaey (2008); Azrieli (2018); however, how to measure stakes becomes less obvious when $m > 2$ (Consider the utility vectors $(1, 1, 0)$ and $(1, 0, 0)$. Which reflects higher stakes?) We thus define and study general stakes functions $s$: any mapping from a voter’s utility vector to a scalar measure of their stakes in the election.

**Q2:** How does a decision process account for stakes? Colloquially, we say that a decision process accounts for stakes if it grants voters “representation” in a stakes-dependent way. Here, we consider perhaps the simplest such dependence, s-proportionality, where voters are represented in proportion to their stakes as measured by $s$. While we will show that proportional dependence is optimal among any type of dependence, we show how to extend our framework to generic notions of stakes-dependent representation in Appendix A.2.

**Remark:** As we alluded to before, two common assumptions in the distortion literature—utility queries and unit sum utilities—are conceptually unified by the idea of stakes. In Section 2.3, we establish a formal connection between our framework and the unit-stakes assumption: that assuming unit-sum utilities is equivalent, from a distortion perspective, to achieving s-proportionality when $s$ is the sum of a voter’s utilities. In fact, we show this correspondence to hold.
for any 1-homogeneous stakes function, permitting useful transference of bounds within our model and existing work. More broadly, our bounds described in C2 will recover, relate to, and sometimes improve upon bounds in the unit-sum utilities literature (e.g., in [Caragiannis et al. 2017]) and the utility queries literature (e.g., in [Amanatidis et al. 2021]). We describe these connections in detail in Appendix A.3.

C2: the welfare impact of accounting for stakes. Within the stakes framework we have defined, we next study the welfare impact of accounting for stakes in full generality, at least within the ranking-based voting context. This means upper- and lower-bounding the distortion under s-proportionality of all voting rules, for all stakes functions s, in elections with arbitrarily many alternatives m. Focusing first on deterministic rules, in Section 3 we find that across stakes functions, many voting rules are not helped by accounting for stakes, but the voting rule PLURALITY is. Further, we show that when voters’ stakes are measured by their maximum utility, PLURALITY has the optimal distortion of m over all possible deterministic rules and stakes functions. While this shows that accounting for stakes can drop the distortion of deterministic voting from unboundedly large to m, this finding is surprising, as PLURALITY is generally considered to be a “bad rule” for its lack of expressiveness. We interpret this in Section 3. Finally, we give analogous (but abbreviated) results for randomized rules in Section 3.1.

C3: a mechanism design proof-of-concept. Finally, in Section 4, we begin an initial exploration of the mechanism design space. This analysis is just a proof-of-concept, whose goal is it to illustrate the core idea behind a much broader class of mechanisms: to require voters to make trade-offs in how much influence they exert in a given election, thereby maximizing, the distortion is order $m^2$. This analysis is just a proof-of-concept, whose winner is the alternative ranked in the j-th position in ranking π. When voter i “votes”, they submit a ranking πi. This ranking is determined by $u_i$: i ranks alternatives in decreasing order of their utilities, so that $u_i(a) > u_i(a') \implies a >_\pi a'$ for all $a, a' \in [m]$.2

A collection of n voters’ rankings is called a preference profile π. As in prior work (e.g., [Xia 2020]), instead of working with profiles π, we will work with histograms, which summarize collections of rankings by their frequencies. A generic preference histogram is a vector indexed by rankings, $h = (h_\pi | \pi \in S_m)$, where $h_\pi \in [0, 1]$ is the fraction of rankings in a given collection equal to π. As such, $||h||_1 = 1$. The space of all possible preference histograms is thus the simplex of all valid distributions over $S_m$, which we call $Δ(S_m) := \{ h \in [0, 1]^{S_m} : \sum_{\pi \in S_m} h_\pi = 1 \}$. Connecting profiles and histograms, we say π is consistent with h if each π in $S_m$ appears in $π$ exactly $n \cdot h_π$ times. Let $Π^h$ be the set of all profiles consistent with a histogram h. Note that $Π^h$ is non-empty iff all entries of h are rational.

Since voters’ rankings are fully implied by U, we let $U$ constitute an instance. We denote the histogram implied by $U$ as $hist(U)$, whose π-th entry is given by $hist_U(\pi) := \frac{1}{n} \sum_{i \in [n]} \mathbb{1}\{\pi_i = \pi\}$, for all $\pi \in S_m$.

Voting rules. Let $Δ([m])$ denote the set of all probability distributions over the alternatives $[m]$. Then, a voting rule f is a function $f : Δ(S_m) \rightarrow Δ([m])$ that maps a preference histogram to a distribution over winning alternatives. We refer to this class of functions as randomized rules to distinguish them from their sub-class, deterministic voting rules, which map a histogram to a distribution with singleton support. Among deterministic rules, the only specific rule we study is PLURALITY, whose winner is the alternative that is ranked first by the most voters. Among randomized rules, we consider the STABLE LOTTERY RULE [Ebadian et al. 2022], which draws a winner either at random or from a stable lottery—a randomization over a subset of $[m]$ that is preferred by voters to other such subsets. We will not apply this rule’s precise definition, so we defer it to Appendix C.8.

2For simplicity of our lower bounds, we will assume worst-case rankings when $u_i(a) = u_i(a')$: one could instead tie-break explicitly by perturbing the utilities by arbitrarily small amounts.

3Histograms are inherently anonymized, so we study only anonymous voting rules (encompassing essentially all rules).

2 Model and Preliminaries

We introduce the model in two parts. Section 2.1 establishes the standard voting model; Section 2.2 embeds our stakes framework within it. Throughout the paper, we use the shorthand $1, 0, o$ to mean a vector containing $ℓ$ ones followed by a string of $ℓ'$ zeros. We let $\mathbb{1}\{\cdot\}$ be the indicator function.
Distortion. Let an alternative $a$’s utilitarian social welfare be $\text{sw}(a, U) := \sum_{i \in [n]} u_i(a)$. We benchmark the social welfare of the winner against that of $a^* := \arg \max_{a \in [m]} \text{sw}(a, U)$, the highest-welfare alternative. For any rule $f$, the value of this competitive ratio in instance $U$ is called the instance-specific distortion, defined as

$$\text{dist}_U(f) := \frac{\text{sw}(a^*, U)}{\mathbb{E}[\text{sw}(f(\text{hist}(U)), U)]},$$

where the expectation is over the draw of the winner from the distribution $f(\text{hist}(U))$. As is standard, we evaluate $f$ via its overall distortion, $\text{dist}(f)$, which is the worst-case competitive ratio over all possible instances $U$:

$$\text{dist}(f) := \sup_{n \geq 1} \sup_U \text{dist}_U(f).$$

Th supremum over $n$ is just to more conveniently deal with the fact that in worst-case instances, $n$ must be large enough relative to $m$ to realize utility matrices with $m$-dependent fractional compositions. We consider the distortion to be a function of $m$, as is standard in the literature.

2.2 A stakes framework within the voting model

Measuring stakes via stakes functions. A stakes function is any function $s : \mathbb{R}^m_+ \rightarrow \mathbb{R}$ that maps a utility vector to a scalar measure of the stake it reflects. Intuitively, a voter’s stake should depend on the relative magnitudes of their utilities, but not which alternatives they prefer; we thus restrict to functions $s$ which are permutation invariant. For example, utility vectors $(0, 1)$ and $(1, 0)$ reflect the same stake. We often apply this invariance to evaluate voters’ stakes on a sorted version of their utility vector.

In some results, we restrict our consideration to stakes functions that are $1$-homogeneous, i.e., for all scalars $\alpha$, $s(\alpha u) = \alpha s(u)$. This applies to $\alpha = 0$, implying that for all $1$-homogeneous $s$, $s(0) = 0$. This restriction on $s$ is natural in that it makes our notion of accounting for stakes, formalized below, invariant to rescaling $U$. Although many of our results apply for generic stakes functions, three in particular will come up frequently, so we define shorthand for them:

$$\text{range}(u) := \max_a u(a) - \min_a u(a)$$

$$\max(u) := \max_a u(a), \quad \text{sum}(u) := \sum_a u(a)$$

Accounting for stakes via stakes-proportionality. Colloquially, we say that a voting process “accounts for stakes” if it grants voters representation to an extent that depends on their relative stakes. We can think of this as a form of stakes-dependent reweighting: instead of voter $i$’s ranking contributing to the $\pi_i$-th entry of the histogram with weight $1/n$, its contribution is additionally weighted by some function of $s(u_i)$. We can also think of this as recomposing the electorate, by duplicating voters in proportion to some function of $s(u_i)$. While these weights could be defined by any such “recomposition function”, we focus here on perhaps the simplest: the unit-stakes function, which results in voters being represented in proportion to their stakes. Formally, given a stakes function $s$ and a utility matrix $U$, we let $\text{hist}^s(U)$ be the $s$-proportional histogram arising from $U$, with $\pi$-th entry

$$\text{hist}^s_\pi(U) := \frac{\sum_{i \in [n]} s(u_i) \cdot \mathbb{I}(\pi_i = \pi)}{\sum_{i \in [n]} s(u_i)}, \quad \pi \in S_m,$$

In words, each voter $i$’s ranking is represented with weight $s(u_i)/\sum_{i \in [n]} s(u_i)$. When we are not discussing a specific histogram, we will sometimes instead refer to a stakes-proportional electorate. In Section 5, we explore going beyond stakes proportionality to study more general stakes-dependent recomposition functions.

Distortion under stakes-proportionality. We define the $s$-distortion of $f$ as its distortion in the $s$-proportional electorate arising from $U$, $\text{hist}^s(U)$:

$$\text{dist}^s_U(f) := \frac{\max_{a \in [m]} \text{sw}(a, U)}{\mathbb{E}[\text{sw}(f(\text{hist}^s(U)), U)]}, \quad \text{and}$$

$$\text{dist}^s(f) := \sup_{n \geq 1} \sup_U \text{dist}^s_U(f).$$

2.3 Connection to unit-sum utilities

Here, we connect our framework to the common sum utilities assumption: that voters’ utilities are normalized to sum to 1 (Procaccia and Rosenschein 2006; Caragiannis and Procaccia 2011; Caragiannis et al. 2017) (or, in recent work, that voters have maximum utility 1 (Eladien et al. 2022)). First, we observe that such assumptions amount to assuming that voters have identical stakes, as measured by sum (resp. max). Of course, one could assume this normalization with respect to any stakes function $s$; we call this general class of assumptions $s$-unit-stakes assumptions.

Not only is stakes a useful conceptual generalization; surprisingly, there is actually an equivalence between assuming $s$-unit stakes and accounting for stakes. Via the bidirectional reduction illustrated in Figure 1, we show that for any voting rule $f$ and any 1-homogeneous stakes function, the distortion of $f$ under the $s$-unit-stakes assumption is equivalent to its $s$-distortion. We formalize this reduction over two theorems: Theorem B.1 handles rational-valued histograms, and Theorem B.3 extends the argument to real-valued histograms (at the cost of a mild technical condition on $f$).

This reduction allows upper and lower distortion bounds to be passed between the two models, which we will illustrate to be useful in both directions. The bounds we prove in our model improve upon existing results under the unit stakes assumption (Remark A.1). More broadly, our results close the open question would assuming unit stakes with respect to a stakes function other than sum or max permit better distortion bounds? Our results in Section 3 will imply that it is optimal to assume max-unit stakes, permitting distortion at best $m$ for deterministic rules using PLURALITY (Theorem 3.6), and $\sqrt{m}$ for randomized rules using STABLE LOTTERY. In the opposite direction, the related work will directly imply a useful bound in our context Theorem 3.8.

3 Distortion Under Stakes-Proportionality

We focus our exposition here on deterministic voting rules, as they are most commonly used in practice; we address randomized rules in Section 3.1. To begin, it is folklore that deterministic voting rules have unbounded distortion:

Fact 3.1. $\text{dist}(f) = \infty$ for all deterministic rules $f$. 
The proof is essentially Example 1.1; we formalize it in Appendix C.1. In that example, accounting for Type 2 voters’ disproportionately high stakes would have helped; however, the question remains: does accounting for stakes give bounded distortion for general instances? By our reduction in Section 2.3, existing results under sum-stakes already imply that at least for some rules, s-proportionality will help: For example, carrying over bounds from (Caragiannis et al. 2017), we know that dist⁴(PLURALITY) ∈ O(m²). However, it remains unclear if this is optimal across voting rules and stakes functions. Moreover, in a mechanism that uses incentives to elicit stakes, f can be hand-picked, but s may arise from incentives and thus cannot be chosen. For these reasons, we now bound the s-distortion of any deterministic rule f for any stakes function s.

First, the following lower bound shows that for any deterministic rule f and any stakes function s, accounting for stakes can lead to s-distortion of at best m − 1:

**Theorem 3.2 (lower bound).** For all s and deterministic f, dist⁴(f) ≥ m − 1.

The proof is in Appendix C.2, and constructs two instances U, U’ in which all voters have identical utility vectors, and thus identical stakes for any s. We show that any deterministic rule has s-distortion at least m − 1 in either U or U’.

We now prove our universal upper bound Theorem 3.3. To reason about all voting rules f and stakes functions s at once, we must determine: Given any pair (s, f), what properties of s and f will lead to low distortion? We now introduce two such properties. First, β̂ is the minimum fraction of voters that must rank the winner by f in first position:

\[ β̂ := \min_{h \in \Delta(S_m)} \sum_{s \in S_m} h_s \cdot \mathbb{1}\{\pi(1) = f(h)\}. \]

Second, κ-upper(s) and κ-lower(s) measure the extent to which s can over- or under-estimate max(u), respectively:

\[ \text{κ-upper(s)} := \sup_u \frac{u(s)}{\max(u)}, \quad \text{κ-lower(s)} := \inf_u \frac{u(s)}{\max(u)}. \]

While bounds in terms of other properties of (s, f) are conceivable, these quantities will permit optimal upper bounds.

In terms of these quantities, Theorem 3.3 gives an upper bound on the s-distortion for any s and any deterministic f. The proof relies on the insight that β̂ and the κ values are linked: β̂ lower bounds how often the winner is ranked first, while the κ’s links the stakes and maximum utility of any voter who ranks the winner first. This connection implies a lower-bound on the social welfare of the winner.

**Theorem 3.3 (upper bound).** For all s and deterministic f, dist⁴(f) ≤ β̂⁻¹ · κ-upper(s) / κ-lower(s).

**Proof.** Fix an instance U, a stakes function s, and a deterministic rule f. Let a’ = f(hist⁴(U)) be the winner of the s-proportional election. First, we have that the social welfare of any alternative a is upper-bounded:

\[ \text{sw}(a, U) \leq \sum_{i \in [n]} \max(u_i) \leq \sum_{i \in [n]} s(u_i) / \text{κ-lower}(s). \quad (1) \]

Now, let Nₐ be the set of voters who rank a first. All i ∈ Nₐ must at least some utility for a’:

\[ u_i(a’) = \max_{a} u_i(a) \geq s(u_i) / \text{κ-lower}(s). \quad (2) \]

Also, since a’ is the winner, Nₐ composes at least a β̂ fraction of the stakes-proportional electorate:

\[ \sum_{i \in Nₐ} s(u_i) / \sum_{i \in [n]} s(u_i) \geq \betâ. \]

This fact, combined with equation (2), gives that

\[ \text{sw}(a’, U) \geq \sum_{i \in Nₐ} u_i(a’) \geq \sum_{i \in Nₐ} s(u_i) / \text{κ-upper}(s) \geq \betâ \sum_{i \in [n]} s(u_i) / \text{κ-upper}(s). \]

Combining this with inequality (1) and denoting the maximum welfare alternative by a*, we obtain that

\[ \text{dist}⁴_U(f) = \frac{\text{sw}(a*, U)}{\text{sw}(a’, U)} \leq \betâ⁻¹ \cdot \frac{\text{κ-upper}(s)}{\text{κ-lower}(s)}. \]

**Remark 3.4.** Theorem 3.3 also holds if in the definitions of κ-upper(s) and κ-lower(s), max is replaced with range. This is because the worst-case distortion can always be realized by instances where every voter has minimum utility 0, in which case max = range. We prove this in Appendix C.3.

One may notice that when β̂ = 0, there is an unbounded gap between the lower bound in Theorem 3.2 and the upper bound in Theorem 3.3. We now find that in this case, unfortunately the distortion is indeed unbounded:

**Proposition 3.5.** For all s, all f with β̂ = 0, dist⁴(f) = ∞.
This lower bound is practically significant because, as shown via a simple instance, most popular voting rules have \( \beta_f = 0 \) (see Appendix C.4 for related proofs).

This negative result motivates our study of PLURALITY, whose \( \beta_f \) is at least \( 1/m \), and is in fact maximal over all deterministic rules (Lemma C.4). In fact, we can show by Theorem 3.3 that when stakes are measured with \( \max \) (or range), PLURALITY has \( s \)-distortion \( \Omega \) (see Appendix C.9 for details).

We necessarily choose the stakes function voters use, we more tightly tend easily to richer ballot formats. Since we cannot necessarily that the ranking-based ballot format may be insufficiently critical information: this is good news, as PLURALITY is often considered a “bad rule” due to its lack of expressiveness. One interpretation is that PLURALITY, when stakes are accounted for, actually accounts for the lack of expressiveness. One interpretation is that PLURALITY, when stakes are accounted for, actually accounts for the most critical information: this is good news, as PLURALITY-like voting methods are widely-used. Another takeaway is that the ranking-based ballot format may be insufficiently expressive: as we discuss in Section 5, our framework extends easily to richer ballot formats. Since we cannot necessarily choose the stakes function voters use, we more tightly lower-bound the \( s \)-distortion across \( f \) and \( 1 \)-homogeneous \( s \); then, we identify an \( f,s \) pair whose \( s \)-distortion matches our lower bound up to log factors. For comparison, randomized rules have at best \( \Omega(m) \) distortion without accounting for stakes (see Appendix C.9 for details).

**Theorem 3.6 (Optimality of PLURALITY, max, range).** Among all \( f \) and \( s \), \( f = \text{PLURALITY} \) and \( s = \max \) or \( \text{range} \) have optimal \( s \)-distortion of \( \text{dist}^\max(\text{PLURALITY}) \in \Theta(m) \).

Our results point to PLURALITY as a promising voting rule when stakes are accounted for. This may seem strange, as PLURALITY is often considered a “bad rule” due to its lack of expressiveness. One interpretation is that PLURALITY, when stakes are accounted for, actually accounts for the most critical information: this is good news, as PLURALITY-like voting methods are widely-used. Another takeaway is that the ranking-based ballot format may be insufficiently expressive: as we discuss in Section 5, our framework extends easily to richer ballot formats. Since we cannot necessarily choose the stakes function voters use, we more tightly lower-bound the \( s \)-distortion across \( f \) and \( 1 \)-homogeneous \( s \); then, we identify an \( f,s \) pair whose \( s \)-distortion matches our lower bound up to log factors. For comparison, randomized rules have at best \( \Omega(m) \) distortion without accounting for stakes (see Appendix C.9 for details).

**Theorem 3.7.** For all \( 1 \)-homogeneous \( s \), randomized \( f \),  
\[
\text{dist}^s(f) \geq \sqrt{m} / (10 + 3 \log m).
\]

The rather intricate construction of this lower bound is deferred to Appendix C.7. Our next result proves that a known rule, STABLE LOTTERY, paired with one of a few stakes functions, has optimal \( s \)-distortion up to a log factor.

**Theorem 3.8.** For \( s \) equal to sum, max, or range,  
\[
\text{dist}^s(\text{STABLE LOTTERY}) \in O(\sqrt{m}).
\]

Via the reduction in Section 2.3, this follows from bounds proven under the sum-, max-unit stakes assumptions (Thm. 3.4, Ebadian et al. (2022)). The case of range requires a minor technical extension – see Appendix C.8 for details.

4 Toward Stakes-Accounting Mechanisms

Our result in Section 3 – along with results in related models – suggests that accounting for stakes could substantially improve the welfare of voting outcomes. This motivates the question: how do we actually account for stakes?

In this section, we do not intend to close this vast mechanism design question, but rather to open it by showing that there is hope for mechanistically accounting for stakes. In Section 4.1, we extend our results from Section 3 to the case where only approximate stakes information is recoverable – an inevitable fact of eliciting stakes in the real world. Then, in Section 4.2, we present and analyze a preliminary mechanism concept. Finally, in Section 5, we discuss the broader space of mechanisms for accounting for stakes in practice.

4.1 Robustness to imperfect stakes information

Suppose we achieve \( s \)-proportionality according to an incorrect estimate of each voter \( i \)'s stakes \( s(u_i) \). Formally, let \( \delta_i \geq 1 \) be the factor by which we overestimate \( i \)'s stakes, so \( s(u_i) = \delta_i s(u_i) \). Let \( \delta := (\delta_i | i \in [n]) \) be the vector of all such errors. Given \( U \) and \( \delta \), we denote the \( \delta \)-approximately \( \delta \)-approximately stakes-proportional histogram as \( \text{hist}^\delta(\delta, U) \), with \( \pi \)-th entry \( \text{hist}^\delta(\delta, U)_\pi := \frac{\sum_{i \in [n]} \delta_i s(u_i) (\pi, i = \pi)}{\sum_{i \in [n]} [\delta_i s(u_i)]} \). For \( \delta := \max \delta_i \), the \( \delta \)-s-distortion of \( f \) is then given as follows; note that for fixed \( \delta \), \( \delta \in [1, \delta] \) is chosen adversarially.

\[
\text{dist}^\delta_s(f) = \max_{n \geq 1, \text{U}, \delta \in [1, \delta]^{n}} E[\text{sw}(f, \text{hist}^\delta(\delta, U), U)]
\]

We now prove strong robustness to such errors:

**Theorem 4.1.** For all \( \delta \geq 1 \), \( f \), \( 1 \)-homogeneous \( s \), and \( U \),  
\[
\text{dist}^\delta_s(f) \leq \delta \cdot \text{dist}^s(f)
\]

The proof is in Appendix D.1. The intuition is simple: since \( s \) is \( 1 \)-homogeneous, mis-estimating \( i \)'s stakes by up to \( \delta \) is the same as overestimating voters’ utilities by up to \( \delta \). Such overestimates can change the distortion by at most a \( \delta \) factor.

4.2 Initial exploration of multi-issue mechanisms

Here, we give introduce a simple multi-issue mechanism: we place \( k \) entire elections on a ballot and permit voters to allocate a budget of voting power across them. Applying results from Section 3, we show that although any election alone would have had unbounded distortion, within this mechanism, all elections have distortion at most \( m^2 \).

**Mechanism.** Let there be \( n \) voters and \( k \) elections, where each election \( \ell \) is over a set of \( m \) alternatives \( A \). For election \( \ell \), \( U^\ell \) is the instance and each \( i \) has utility vector \( u_i^\ell \). To vote, \( i \) submits to each election \( \ell \) a weighted basis vector \( w^\ell_i e^\ell_i \). Here, \( e^\ell_i \) is the \( |\ell| \)-length basis vector with a 1 at the \( \pi^\ell_i \)-th index corresponding \( i \)'s ranking over alternatives in election \( \ell \), and \( \pi^\ell_i \in [0, 1] \) is the weight \( i \) places behind this ranking. Voters have a total weight budget of 1 and are charged 

\[\text{ quadratically } \] for each marginal unit of weight per election.\( ^5 \)

\[\text{ if } u_i := (w_i^\ell | \ell \in [k]), \text{ then for all } i, \text{ it must be }\]

\[\text{ We can realize any type of errors with } \delta \geq 1, \text{ because the composition of the resulting electorate is relative.} \]

\[\text{ One may wonder if there is a correspondence between this mechanism and Quadratic Voting (QV). Technically, the cost quadraticity serves the same purpose, but our mechanism is over multiple elections, while QV occurs within a single election.}\]
Lemma 4.3. via Lemma 4.2.

We aim to study how voters make trade-offs between elections, so for simplicity, we assume they are strategic in only this respect. Subject to honestly ranking alternatives, voters are utility-maximizing: \( i \) chooses the \( \tilde{w}_i \) that maximizes their expected utility across elections. We also assume that, in computing this expectation, voters take their probability of pivotality in \( \ell \) to be linear in the weight they place on \( \ell \); we formalize and justify this assumption in Appendix D.2.

Analysis. First, we show that by utility maximizing, voters will naturally trade-off different elections using a simple stakes function \( s^*(\mathbf{u}) := \max(\mathbf{u}) - \frac{\sum(u_i - \max(u_i))}{m-1} \), the gap between their maximum utility and their average utility for the other alternatives. We use shorthand \( s_i^* := s^*(\mathbf{u}_i^*) \) to summarize \( i \)'s stakes across all \( k \) elections.

Lemma 4.2. To each election \( \ell \), each \( i \) weights their vote by
\[
\tilde{w}_i^\ell = s^*(\mathbf{u}_i^\ell) / \| s_i^* \|_2.
\]

The proof is in Appendix D.2. It involves making deductions via \( i \)'s priors, then optimizing \( w_i \), using the first-order condition. Promisingly, we next find that \( s^* \) is nearly optimal:

Lemma 4.3. \( \text{dist}^*(\text{PLURALITY}) \leq m^2. \)

Proof. We will use the alternative definitions of \( \kappa \)-upper and \( \kappa \)-lower from Remark 3.4, where \( \kappa \) is replaced with range. Characterizing these \( \kappa \) values, \( \kappa \)-upper(\( s^* \)) = 1, realized by the vector \( \mathbf{u} = 1_m0_{m-1} \), and \( \kappa \)-lower(\( s^* \)) = 1/(\( m-1 \)), realized by the vector \( \mathbf{u} = 1_{m-1}0_1 \). Then, the bound follows from Theorem 3.3 and Remark 3.4.

We now conclude by bounding the distortion in each election within the mechanism. The remaining issue to deal with is that although each individual voter will spread their votes across elections in proportion to their stakes (Lemma 4.2), if some voters have substantially higher total stakes across the \( k \) elections, the uniform budgets across voters will undercount these voters’ stakes. It may seem that we are back where we started — where uniform voting power fundamentally cannot account for stakes. However, unlike before, we have another lever at our disposal: the design of our slate of \( k \) elections: we can choose these elections such that all voters are affected to a relatively high degree by at least one election.\(^6\) While we cannot hope for this approach perfectly equalize voters’ total stakes, it may bring them closer, e.g., within a factor of \( \delta \). Then, per Theorem 4.1, all elections in our mechanism will have distortion at most \( \delta m^2 \).

Theorem 4.4. Fix a \( \delta \geq 1 \) such that for all pairs of voters \( i, i' \in [n] \), \( \| s_i^* \|_2 / \| s_{i'}^* \|_2 \leq \delta. \) Then, for all \( \ell \in [k] \),
\[
\text{dist}^\ell_i(\text{PLURALITY}) \leq \delta m^2.
\]

Proof. Let \( \alpha := \max_{i \in [n]} \| s_i^* \|_2 \) be the maximum total stakes of any voter. Fix an \( \ell \in [k] \). Then, per Lemma 4.2, for all \( i \), and for some \( \delta_i \in [1, \delta] \), we have that \( w_i^\ell = s^*(u_i^\ell) / \| s_i^* \|_2 = \delta_i \cdot s^*(u_i^\ell) / \alpha \).

5 Discussion

While we focused on ranking-based voting, the same mechanistic approach could be used with essentially any election format: simply place multiple elections on the same ballot, across which voters must trade off a total allotment of voting power.\(^7\) In fact, there is a known mechanism that roughly does this: storable votes (Casella 2005), which allows voters to save up votes over a sequence of elections and spend them on the later elections they care about most. Like our mechanism above, this is just one within a massive design space, whose many levers we explore below.

Election format. Within purely voting-based elections, a key design parameter is the ballot format; our results suggest that going beyond ranking-based ballots might be beneficial, since many voting rules over rankings did not improve under stakes-proportionality. More broadly, one might consider ballot formats for making different types of decisions; e.g., deciding a budget allocation, as in participatory budgeting. Given a ballot format, one must subsequently also decide a voting rule. Zooming out beyond pure voting, many other practically-used processes – e.g., liquid democracy, deliberation – could be affected by accounting for stakes. As needed, our stakes framework extends easily to other election formats: stakes functions are defined whenever there are underlying utilities, and the conceptual notion of “stakes-based representation” applies in virtually any election.

Voter model. Each election format opens a vast space of possible assumptions about how voters will behave within it. One must specify, e.g., how voters reason about other voters’ behavior, with what goals they strategize, and how they estimate computationally complex decision parameters like probability of pivotality. One might base such assumptions on work showing how voters’ preference intensities impact their likelihood of voting (Downs 1957; Feddersen and P casendorfer 1996). Our mechanism charged voters quadratically per unit of weight placed on an election, but this pricing scheme must depend on the voter model.

Slate of issues. Finally, as discussed in Section 4, multi-issue mechanisms rely on choosing a slate of issues that roughly balance voters’ total stakes. In many cases, this may be done via observable proxies for stakes (e.g., in Example 1.1, a proxy might be whether a person owns a car, or how often they take the bus). Given a set of proxies, how to design a slate of issues remains an open question.

\(^6\)For example, we might pair the election in Example 1.1 with one deciding where to fix potholes, which targets those who drive – the complementary group to those who rely on the bus.

\(^7\)One could also trade off voting power in an election with some other resource. For example, quadratic voting requires voters to purchase votes with money, thereby inducing similar trade-offs (Posner and Weyl 2015). However, such monetary mechanisms may disadvantage less wealthy voters with higher value for money.
References


A Supplemental Materials from Section 1.2

A.1 Social choice functions’ pursuit of “equality”

In ranking-based or approval-based voting, each voter is treated symmetrically, receiving one vote regardless of their stakes. Even voting rules that elicit cardinal preferences do not necessarily account for stakes: for instance, consider range voting (Pivato 2014), in which voters express their utilities through scores within a bounded range $[0, 1]$: in Example 1.1, even if voters’ reported scores reflected their true relative utilities for $a$ and $b$, type 1 and 2 voters would report scores of $(1, 0)$ and $(0, 1)$, and the election outcome would be the same. Even in nonstandard voting paradigms like Liquid Democracy (Gölz et al. 2021), each voter has one vote token to either delegate or use themselves (while it could be the case that people delegate their votes to those with high stakes, this is not at all guaranteed).

Even outside of strictly voting-based mechanisms, we still see the principle of equality: for example, a key desideratum of deliberative processes like citizens’ assemblies is to give everyone in the constituency equal chance of participation (Flanigan et al. 2021). While many such deliberative processes also aim for proportional demographic representation, this is distinct from—and can even be in tension with—ensuring that stakeholders have sufficient say. One study of deliberative democracy that does engage with stakes examines how voters become public-spirited due to deliberation, and as a result, vote in a way that can be seen as implicitly accounting for each others’ stakes. This work finds that the more voters account for each others’ stakes, the lower the distortion, in accordance with our results (Flanigan, Procaccia, and Wang 2023).

A.2 Beyond proportional recomposition

Because the mechanisms we discuss in this paper change the composition of the electorate, we find it more intuitive to think of stakes-based reweighting as stakes-based recomposition; however, the two terms are interchangeable.

There are a multitude of stakes-dependent ways, beyond proportionally, to recompose an electorate. Such recompositions with, e.g., submodular stakes-dependence might be practically desirable, or might result from certain mechanisms. To extend our model to encapsulate such more general recompositions, let $r: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a generic recomposition function, $s$ a stakes function, and $U$, and instance. Then, the $(r, s)$-recomposed histogram arising from $U$ has $\pi$-th entry

$$\text{hist}_{\pi}^{r,s}(U) = \frac{\sum_{i \in [n]} r(s(u_i)) \cdot \mathbb{I}(\pi_i = \pi)}{\sum_{i \in [n]} r(s(u_i))} \quad \forall \pi \in S_m.$$ 

The same approaches used in our paper can be applied to study these more generic histograms. However, because both our lower bounds (Theorems 3.2 and 3.7) use instances in which all voters have identical stakes, these lower bounds extend to any recomposition in the above class, implying that more sophisticated recompositions cannot be used to go beyond the lower bounds we have established.

A.3 Connections to existing results

We connect our results to those in three papers. The first two study distortion under the $s$-unit stakes assumption, and the third assumes utility queries.

Caragiannis et al. (2017) ($s$-unit stakes, deterministic rules)

This paper assumes $s$-unit-stakes. Although this paper proves distortion bounds for both deterministic and randomized rules, we do not discuss their analysis of randomized rules, as such bounds are more directly addressed in later work, described next. Another similarity between our bounds: $\beta_f$ that our bounds depend on is similar to the dependency of their analysis on alternatives’ plurality score.

Upper bounds (deterministic rules): Theorem 1 of their paper proves an $O(m^2)$ upper bound on the distortion of PLURALITY under $s$-unit-stakes (i.e., unit-sum utilities). We can recover this bound via our Theorem 3.3: First observe that $\kappa$-upper$(\text{sum}) = m$ and $\kappa$-lower$(\text{sum}) = 1$ (given by utility vectors $1$ and $1_1 0_{m-1}$, respectively). Recall also that $\beta_{\text{PLURALITY}} = 1/m$. By Theorem 3.3, it follows that $\text{dist}^{\text{sum}}(\text{PLURALITY}) \leq m^2$.

Lower bounds (deterministic rules): Theorem 1 of their paper proves an $\Omega(m^2)$ lower bound on the distortion of any voting rule under $s$-unit stakes (unit-sum utilities). We do show a lower-bound on the distortion of all deterministic voting rules, but due to its general across any stakes function (not just sum), our (tight) lower bound is $\Omega(m)$ (Theorem 3.2). However, we can recover the $\Omega(m^2)$ bound specifically for $s = \text{sum}$ for most voting rules by combining two of our results. First, by Appendix C.4, many voting rules have unbounded distortion under $s$-unit stakes with respect to any $s$, including $\text{sum}$. Among the remaining rules with $\beta_f > 0$, we can recover a lower bound of $\Omega(m^2)$ (with tighter constants) for PLURALITY via Proposition C.5. We have $\kappa$-upper$(\text{sum})$ from above, and $\kappa$-lower$(\text{sum}) = 2$, given by $1_2 0_{m-2}$. Then, by Proposition C.5, $\text{dist}^{\text{sum}}(\text{PLURALITY}) \geq (m - 1)m/2$. Our bounds here are tighter, improving upon the gap from a factor of 8 to a factor of 2.

Remark A.1 (placeholder). This remark is a placeholder for accuracy of cross-references — the original contents of this remark can now be found in the above two paragraphs.
Ebadian et al. (2022) (s-unit-stakes, randomized rules)
This paper studies only randomized rules, under both the sum-unit stakes assumption and the max-unit stakes assumption (which they call “range”).

Upper bounds (randomized rules): As discussed in Section 3.1, we use our reduction in Section 2.3 to directly apply their upper bounds on the distortion of STABLE LOTTERY under the sum- and max-unit stakes assumptions (their Theorem 3.4) to prove our upper bound in Theorem 3.7.

Lower bounds (randomized rules): In Theorem 3.7 of their paper, they show a lower bound of $\Omega(\sqrt{m})$ on the distortion of any randomized rule under max-unit-stakes. This complements a previously-known bound by Boutilier et al. (2012) of $\Omega(\sqrt{m})$ on the distortion of any randomized rule under sum-unit stakes. Our lower bound in Theorem 3.7 is weaker than these bounds by a $\log(m)$ factor, but it applies to s-unit stakes for any 1-homogeneous s (which includes both sum and max). We suspect our lower bound can be tightened to $\Omega(\sqrt{m})$, which would make it a strict generalization of the existing bound.

Amanatidis et al. (2021) (utility queries, deterministic rules) This paper considers deterministic voting rules with access to one of two kinds of queries: value queries, where the voting mechanism can directly ask agents about any one of their utilities; and comparison queries, where the voting mechanism can ask agents: “for alternatives a and b, is your utility for a at least d times your value for b?” Stakes information according to an arbitrary s can be recovered by some number of either type of these queries (trivially, m value queries, but in many cases, far less). Determining the optimal set of queries of these types to recover a given stakes function is outside the scope of this appendix. Thus, when thinking about upper bounds, we will restrict our consideration here to the stakes functions max, which can be recovered by 1 value query. Similar reasoning applies for range. Finally, we remark that their permission of noisy queries (i.e., queries within a constant factor of the truth) are related to our robustness results in Theorem 4.1, though due to the generality of our class of stakes functions (and the fact that errors are occurring on stakes functions’ output rather than utilities directly) requires us to handle additional technicalities.

Upper bounds (deterministic rules): Their upper bound in Theorem 1 shows that their mechanism 1-PRV – equivalent to PLURALITY under stakes-proportionality with respect to max – gives distortion $O(m)$. This result corresponds to our upper bound on the distortion of plurality under max-proportionality, proven via Theorem 3.3.

Lower bounds (deterministic rules): Their Theorem 7 shows that any single-value query can enable at best $\Omega(m)$ distortion. Our lower bound in Theorem 3.2 generalizes this lower bound, showing that any system of queries yielding the value of a scalar-valued stakes function, when paired with a deterministic voting rule, can achieve at best $\Omega(m)$ distortion. Of course, this is not to say that we generalize all their lower bounds – they prove several other lower bounds for their setting, which are incomparable to ours.

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8A necessary step in showing that our lower bound subsumes theirs is arguing that our lower bound actually applies to any stakes-dependent electoral recomposition, not just proportional recomposition. We explain why this is the case above in Appendix A.2.
B Supplemental Materials from Section 2.3

Theorem B.1: reduction for rational-valued histograms  Here, we state and prove the reduction assuming hist'(U) has only rational entries, which we ensure by restricting to rational utility matrices $U \in \mathbb{Q}_{\geq 0}^{n \times m}$ and rationality-preserving stakes functions $s$ (i.e., $s(u) \in \mathbb{Q}$ whenever $u \in \mathbb{Q}_{\geq 0}^m$).

**Theorem B.1.** Let $f$ be a voting rule, $s$ a rationality-preserving and 1-homogeneous stakes function, and let $U_n$ be the set of all rational utility matrices satisfying the $s$-unit-stakes assumption. Then,
\[
\sup_{n \geq 1} \sup_{U \in U_n} \text{dist}_u(f) = \sup_{n \geq 1} \sup_{U \in \mathbb{Q}_{\geq 0}^{n \times m}} \text{dist}_u(f).
\]

**Proof.** We show the claimed equality by separately proving the directions ‘$\leq$’ and ‘$\geq$’. In order to see the direction ‘$\leq$’, we note that for any unit-stakes utility matrix $U \in U_n$, hist(U) = hist'(U): the standard and stakes-proportional histograms are the same. Therefore, dist$_u(f) = \text{dist}'_u(f)$. Taking suprema over $n \geq 1$ and $U \in U_n$, we obtain the ‘$\leq$’ direction.

It remains to show ‘$\geq$’. In order to prove this direction, we fix any utility matrix $U \in \mathbb{Q}_{\geq 0}^{n \times m}$, and construct a unit-stakes utility matrix $\tilde{U}$ such that dist$_\tilde{U}(f) = \text{dist}'_u(f)$. We let
\[
\tilde{s}_i = \frac{s(u_i)}{\sum_{i \in [n]} s(u_i)}, \quad i \in [n]
\]
and now construct a utility matrix $\tilde{U} \in \mathbb{Q}_{\geq 0}^{n \times m}$ for which $f$ (without taking into account stakes) exhibits the same distortion as $U$ (while accounting for stakes).

- We divide the electorate of $\tilde{n}$ into $n$ groups, each of them of size $\tilde{s}_i \tilde{n}$. Call these groups $G_1, \ldots, G_n$.
- Within each group $G_i$, voters have the same ranking $\pi_i(U)$ as voter $i$ in $U$. However, they possess scaled utilities $u_i/s(u_i)$.

Then we notice that by definition, hist(\tilde{U}) = hist'(U), and therefore also $f(\text{hist}(\tilde{U})) = f(\text{hist}'(U))$. Moreover, since $s$ is 1-homogeneous, it holds that for all $i$,
\[
s\left(\frac{u_i}{s(u_i)}\right) = \frac{1}{s(u_i)} s(u_i) = 1,
\]
which yields that $U_n$ satisfies the unit-stakes property. Moreover, for all alternatives $a \in [m]$, it holds that
\[
\frac{\text{sw}(a, U)}{n} = \sum_{i \in [n]} u_i(a) = \frac{1}{n} \sum_{i \in [n]} \tilde{s}_i \frac{u_i(a)}{s(u_i)} = \sum_{i \in [n]} s(u_i) \cdot \frac{\text{sw}(a, \tilde{U})}{n} = \sum_{i \in [n]} s(u_i) \tilde{n} \cdot \frac{\text{sw}(a, \tilde{U})}{\tilde{n}}.
\]
Since $\sum_{i \in [n]} s(u_i) \tilde{n} = $ is a fixed constant independent of $i$ and $a$, it follows that the average utilities in $U$ and $\tilde{U}$ are equal up to multiplication with a fixed constant — thus distortion is preserved. $\square$

Theorem B.3: Extension of Theorem B.1 to real-valued histograms  Under an additional very mild restrictions on the voting rule $f$, it is possible to prove the correspondence between stakes-based procedures and unit-stakes assumptions from Theorem B.1 not just for rational utilities, but for all real-valued utility functions. We term this assumption for $f$ to be rationally approximable, which amount to the outcome of $f(h)$ for any preference histogram being well-approximated by some preference histogram $\tilde{h}$ with only rational entries.

**Definition B.2** (Rationally approximable rules). We say that a (deterministic or randomized) voting rule $f : \Delta(S_m) \rightarrow \Delta([m])$ is ‘rationally approximable’ if for every $h \in \Delta(S_m)$ and every $\varepsilon > 0$ there exists another histogram $\tilde{h} \in \mathbb{Q}_{\geq 0}^{n \times m}$ with only rational entries such that
\[
\sup_{\pi \in S_m} |h_{\pi} - \tilde{h}_{\pi}| \leq \varepsilon \quad \text{and} \quad \sup_{a \in [m]} |f_a(h) - f_a(\tilde{h})| \leq \varepsilon,
\]
where $f_a(h)$ denotes the win probability of $a$ in $f(h)$.

**Theorem B.3.** For any 1-homogeneous stakes function $s$ and any voting rule $f : \Delta([m]) \rightarrow \Delta([m])$, we have that
\[
\sup_{n \geq 1} \sup_{U \in U_n} \text{dist}_u(f) \leq \text{dist}'_u(f).
\]
If additionally $s$ is 1-homogeneous and $f$ is either (i) weakly locally constant or (ii) continuous, then the reverse inequality is also true,
\[
\sup_{n \geq 1} \sup_{U \in U_n} \text{dist}_u(f) \geq \text{dist}'_u(f).
\]
Proof of Theorem B.3. The first inequality is immediately implied by the fact that for any \( U \in \mathcal{U} \), the stakes-recomposed electorate is identical to the original electorate. Indeed, in this case stakes-based election yields the same outcome as the non-stakes-based election, \( f(\text{hist}(U)) = f(\text{hist}^*(U)) \), so that \( \text{dist}(f) = \text{dist}^*_U(f) \). It thus only remains to prove the reverse inequality.

Let us fix an arbitrary \( n \geq 1 \) and utility matrix \( U \in \mathbb{R}^{n \times m} \), and let \( \text{hist}^*(U) \in \Delta(S_m) \) denote the stakes-recomposed profile corresponding to \( U \). Without loss of generality, we may assume both \( \text{sw}_U(a^*, U) > 0 \) (since otherwise \( U = 0 \) and \( E[\text{sw}_U(f(\text{hist}^*(U)), U)] > 0 \),

since otherwise \( \text{dist}^*_U(f) = \infty \) and there remains nothing to prove. By Proposition B.4, given any \( \rho > 0 \) we may choose a unit-stakes utility matrix \( \tilde{U} \in \mathbb{R}^{\tilde{n} \times m} \) such that

\[
\sup_{a \in [m]} |f_a(\text{hist}^*(U)) - f_a(\text{hist}(\tilde{U}))| \leq \rho \quad \text{and} \quad \sup_{a \in [m]} \left| \frac{\text{sw}(a, U)}{n} - \frac{\text{sw}(a, \tilde{U})}{\tilde{n}} \right| \leq \rho.
\]

These two properties, taken together, imply the convergence

\[
\left| E\left[ \frac{\text{sw}(f(\text{hist}^*(U)), U)}{n} \right] - E\left[ \frac{\text{sw}(f(\text{hist}(\tilde{U})))}{\tilde{n}} \right] \right| \xrightarrow{\rho \to 0} 0,
\]

as well as the convergence

\[
\left| \max_{a \in [m]} \frac{\text{sw}(a, U)}{n} - \max_{a \in [m]} \frac{\text{sw}(a, \tilde{U})}{\tilde{n}} \right| \xrightarrow{\rho \to 0} 0.
\]

Taken together, this implies that

\[
|\text{dist}^*_U(f) - \text{dist}_{\tilde{U}}(f)| \xrightarrow{\rho \to 0} 0,
\]

which proves the claim.

\( \square \)

Proposition B.4 (Approximation of social welfares). Suppose \( f \) is a rationally approximable voting rule. Let \( U \in \mathbb{R}^{n \times m} \) be any non-zero utility matrix. Then, for any \( \rho > 0 \) there exists some large enough \( \tilde{n} \) and a unit-stakes utility matrix \( \tilde{U} \in \mathbb{R}^{\tilde{n} \times m} \) such that

- The election outcomes are close,

\[
\sup_{a \in [m]} |f_a(\text{hist}^*(U)) - f_a(\text{hist}(\tilde{U}))| \leq \rho.
\]

- For all \( a \in [m] \), the average utilities in \( U \) and \( \tilde{U} \) are close,

\[
\left| \frac{\text{sw}(a, U)}{n} - \frac{\text{sw}(a, \tilde{U})}{\tilde{n}} \right| \leq \rho.
\]

Proof. Let \( \varepsilon > 0 \) be arbitrary and fix any \( U \). By Definition B.2, we can choose some \( \tilde{h} \in Q^{S_m}_{\geq 0} \) with rational coefficients such that

\[
\sup_{\pi \in S_m} |\text{hist}_U^*(U) - \tilde{h}_\pi| \leq \varepsilon \quad \text{and} \quad \sup_{a \in [m]} |f(\text{hist}_U^*(U)) - f_a(\tilde{h})| \leq \varepsilon,
\]

Step 1: Construction of utility matrix which induces \( \tilde{h} \). Since \( \tilde{h} \in Q^{S_m}_{\geq 0} \) only has rational coefficients, there exists some electorate with \( \tilde{n} \) many voters and preferences (\( \tilde{\pi}_i : i \leq \tilde{n} \)) such that for each \( \pi \in S_m \), exactly a \( \tilde{h}_\pi \) fraction of the voters have ranking \( \pi \). Now, we construct a unit-stakes utility matrix \( \tilde{U} \in \mathcal{U}_n \cap \mathbb{R}^{\tilde{n} \times m} \) which induces those rankings to the \( \tilde{n} \) voters, and which in turn will induce the profile \( h, \text{hist}(\tilde{U}) = \tilde{h} \). To this end, let

\[
\tilde{s}_i := \frac{s(u_i)}{\sum_{i \in [n]} s(u_i)}, \quad \sum_{i \in [n]} \tilde{s}_i = 1,
\]

denote the weights corresponding to each voter \( i \)'s preferences in the stakes-recomposed electorate. Since \( s \) is 1-homogeneous, we may assume without loss of generality that \( \sum_{i \in [n]} s(u_i) = n \), by simply scaling the utilities (note that this leaves \( \text{hist}_U^*(U) \) and also \( \text{dist}^*_U(f) \) unchanged). We partition in the new 'unit-stakes electorate' (which consists of \( \tilde{n} \) voters) into \( n + 1 \) parts, which we denote by \( G_1, ..., G_{n+1} \). Within each of those groups, voters share the same ordered utility vector.

Groups \( G_1, ..., G_n \). The first \( n \) groups \( G_1, ..., G_n \) are specified as follows. Voters in group \( i \) have the utilities \( \frac{u_i}{s(u_i)} \), i.e., the same utilities as voter \( i \) in the original electorate, but scaled to unit-stakes. In particular, voters in group \( G_i \) will inherit the
same ranking \( \pi_i \) as the \( i \)-th voter from the original electorate. Let the (fraction) size of the \( i \)-th group be denoted by \( g_i \), i.e., 
\[ g_i = \frac{|G_i|}{\bar{n}}. \]
We now determine those sizes. Since 
\[ \sup_{\pi \in S_m} |\hat{h}_\pi - \text{hist}_\pi(U)| \leq \varepsilon, \]
we can now choose the \((g_i : i \in n)\) in such a way such that simultaneously, the following properties are satisfied. First, 
\[ g_i \in [\bar{s}_i - \varepsilon, \bar{s}_i], \]
and second, for every \( \pi \in S_m \), 
\[ \sum g_i I(\pi_i = \pi) \leq \hat{h}_\pi. \tag{3} \]

The first property states that the group size \( G_i \) does not exceed the amount of representation of voter \( i \) in the stakes-recomposed electorate \( \bar{s}_i \). The second property states that by assigning group sizes \( g_i \), compared to the histogram \( \hat{h} \), none of the rankings is overrepresented. Note that 
\[ \sum_i g_i \leq \sum_i \bar{s}_i \leq 1, \quad \text{and} \quad \sum_i g_i \geq \sum_i \bar{s}_i - \varepsilon \geq 1 - n\varepsilon. \]

**Group** \( G_{n+1} \). This group constitutes the remainder of the population. Within this group, everyone has the same ordered utility vector, but not the same rankings of alternatives. In this group, we assign the ordered utility vector \((x, 0, \ldots, 0)\), where \( x \) is given by \( x = s((1, 0, \ldots, 0))^{-1} > 0 \). Note that \( x \) is the (unique) constant such that \( s((x, 0, \ldots, 0)) = 1 \). In terms of the orderings of alternatives in group \( G_{n+1} \), we assign the exact rankings which are needed to complete the correct histogram \( \hat{h} \), which we aim to realize. Since from Groups \( G_1, \ldots, G_n \), none of the rankings \( \pi \in S_m \) was overrepresented compared to \( \hat{h} \) – see equation (3) – this is possible. The group \( G_{n+1} \) has size at most \( n\varepsilon \).

Let us denote the utility matrix which arises from this construction by \( \tilde{U} \in \mathbb{R}^\bar{n} \times m \).

**Step 2: Approximation of social welfares.** It remains to check that the distortion \( \text{dist}_f(U) \) induced by \( \tilde{U} \) approximates the distortion \( \text{dist}_f(U) \) for the stakes-based election. To this end, we upper and lower bound the difference in average utilities induced by \( U \) and \( \tilde{U} \), respectively. First, recalling that \( \sum_i s(u_i) = n \), we have the lower bound 
\[ \frac{\text{sw}(a, \tilde{U})}{\bar{n}} - \frac{\text{sw}(a, U)}{n} \geq \frac{1}{n} \sum_{i=1}^{n} g_i \frac{u_i(a)}{s(u_i)} - \frac{1}{n} \sum_{i=1}^{n} u_i(a) \]
\[ \geq \sum_{i=1}^{n} \left( \bar{s}_i - \varepsilon \right) \frac{u_i(a)}{s(u_i)} - \frac{1}{n} \sum_{i=1}^{n} u_i(a) \]
\[ \geq \sum_{i=1}^{n} \frac{s(u_i)}{\sum_{j \in [n]} s(u_j)} \frac{u_i(a)}{s(u_i)} - \frac{1}{n} \sum_{i=1}^{n} u_i(a) - \varepsilon \frac{1}{n} \sum_{i=1}^{n} \frac{u_i(a)}{s(u_i)} \]
\[ = -\varepsilon \sum_{i=1}^{n} \frac{u_i(a)}{s(u_i)}. \]

Similarly, we may derive an upper bound, recalling the constant \( x = s((1, 0, \ldots, 0))^{-1} \):
\[ \frac{\text{sw}(a, \tilde{U})}{\bar{n}} - \frac{\text{sw}(a, U)}{n} \leq \frac{1}{n} \sum_{i=1}^{n} g_i \frac{u_i(a)}{s(u_i)} + n\varepsilon x - \frac{1}{n} \sum_{i=1}^{n} u_i(a) \leq \sum_{i=1}^{n} \bar{s}_i \frac{u_i(a)}{s(u_i)} + n\varepsilon x - \frac{1}{n} \sum_{i=1}^{n} u_i(a) = n\varepsilon \cdot x. \]

Since \( \varepsilon > 0 \) was arbitrary, and since both of the latter two bounds tend to 0 as \( \varepsilon \to 0 \), we can now choose \( \varepsilon > 0 \) small enough to fulfill all of the inequalities in the Proposition B.4 for any prescribed threshold \( \rho > 0 \). This proves the claim.

Our result shows that, from the perspective of worst-case distortion, using a stakes-based recomposition is equivalent to assuming across the population that every voter has equal stakes.
C Supplemental Materials from Section 3

C.1 All deterministic rules have unbounded distortion

In this example, any voting rule must either choose \( a \) or \( b \). If it chooses \( a \), the distortion is unbounded as above. If it chooses \( b \) from this set of rankings, then let voters of types 1 and 2 have utility vectors \((\epsilon, 0)\) and \((0, 1)\) respectively – the distortion is again unbounded. We make this formal in Appendix C.1.

**Fact 3.1.** \( \text{dist}(f) = \infty \) for all deterministic rules \( f \).

While this fact can be technically proven with a simple instance, we prove it here in a more detailed, practically motivated example to illustrate such unbounded distortion can be a problem under simple, reasonable conditions.

C.2 Proof of Theorem 3.2

**Theorem 3.2 (lower bound).** For all \( s \) and deterministic \( f \),

\[
\text{dist}^s(f) \geq m - 1.
\]

**Proof.** We will define two instances, \( U \) and \( U' \), and show that all \( f \) must have at least \( m - 1 \) distortion in one of these two instances. We will construct \( U, U' \) in the following way: first, set aside one alternative \( a' \), and let the remaining alternatives be \( a_1, \ldots, a_{m-1} \). Divide voters in into \( m - 1 \) groups, and consider a voter \( i \) in group \( \ell \); we will assign utility vectors to these voters so that their ranking \( \pi_i = a_\ell \succ a' \succ a_1 \succ \ldots \succ a_{m-1} \). We display their utility vectors \( u_i \) and \( u'_i \), as given by \( U \) and \( U' \) respectively, in sorted order, to emphasize how their utilities correspond to their resulting ranking:

<table>
<thead>
<tr>
<th>alternative:</th>
<th>( a_\ell \succ a' \succ a_1 \succ \ldots \succ a_{m-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sorted ( u_i ) for ( i \in \text{group } \ell ):</td>
<td>1 \hspace{1cm} 1 \hspace{1cm} 0 \hspace{1cm} \ldots \hspace{1cm} 0</td>
</tr>
<tr>
<td>sorted ( u'_i ) for ( i \in \text{group } \ell ):</td>
<td>1 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} \ldots \hspace{1cm} 0</td>
</tr>
</tbody>
</table>

We now make three observations:

1. \( \text{hist}(U) \equiv \text{hist}(U') \) — that is, the utility matrices induce the same preference histogram. This is true because for every \( \ell \), voters in the \( \ell \)-th group of \( U \) and \( U' \) have the same ranking.
2. \( \text{hist}^s(U) \equiv \text{hist}(U) \) and \( \text{hist}^s(U') \equiv \text{hist}(U') \) — that is, the \( s \)-proportional profiles are identical to the standard profiles for both utility matrices. This is because within each utility matrix, all voters have the same ordered utility vector and thus have the same stakes.
3. \( \text{sw}(a', U) = n \) while \( \text{sw}(a', U') = 0 \). Moreover, \( \text{sw}(a_\ell, U) = \text{sw}(a_\ell, U) = n/(m - 1) \) for all \( \ell \in \{m - 1\} \).

We distinguish between two cases, depending on whether \( f(\text{hist}(U)) = a' \) or \( f(\text{hist}(U)) \neq a' \).

If \( f(\text{hist}(U)) = a' \), by (1), we also have that \( f(\text{hist}(U')) = a' \). Then, since \( \text{sw}(a', U') = 0 \),

\[
\text{dist}^s_U(f) \overset{(2)}{=} \text{dist}^s_U'(f) = \frac{\text{sw}(a_\ell, U')}{\text{sw}(a', U')} \overset{(3)}{=} \frac{n/(m - 1)}{0} = \infty.
\]

If \( f(\text{hist}(U)) \neq a' \), then there must exist some \( \ell \in \{m - 1\} \) such that \( f(\text{hist}(u)) = a_\ell \). Then, fixing this \( \ell \),

\[
\text{dist}^s_U(f) \overset{(2)}{=} \text{dist}^s_U(f) = \frac{\text{sw}(a', U)}{\text{sw}(a_\ell, U)} \overset{(3)}{=} \frac{1}{1/(m - 1)} = m - 1.
\]

C.3 Theorem 3.3 holds when \( \kappa \)'s are defined with range instead of max

**Observation C.1.** The bound in Theorem 3.3 remains true also for a slightly different definition of the coefficients \( \kappa \)-lower(\( s \)), \( \kappa \)-upper(\( s \)) where \( \text{max}(\cdot) \) is replaced by \( \text{range}(\cdot) \).

**Proof.** Let \( U \in \mathbb{R}^{m \times n}_{\geq 0} \) be any utility matrix. Then, let \( \hat{U} \) denote the utility matrix in which each agent \( i \)'s utility vector \( u_i \) is altered by

\[
\hat{u}_i(a) = u_i(a) - \min_{a \in [m]} u_i(a),
\]

i.e., the utilities are shifted down such that each voter’s minimum utility is 0. Then, letting \( c := \sum_{i \in [N]} \min_a u_i(a) \), we obtain that

\[
\frac{\text{sw}(a^*, U)}{\text{sw}(a', U)} \leq \frac{\text{sw}(a^*, U) - c}{\text{sw}(a', U) - c} = \frac{\text{sw}(a^*, \hat{U})}{\text{sw}(a^*, \hat{U})}.
\]
Then, we may restrict the arguments in the proof of Theorem 3.3 to utility vectors with zero minimum entry. This leads to a bound where we may use, instead of \(\kappa\)-upper\((s)\) and \(\kappa\)-lower\((s)\)

\[
\sup_{u \in \mathbb{R}_{\geq 0}^m: \min u(a) = 0} \frac{s(u)}{\max(u)} \quad \text{and} \quad \inf_{u \in \mathbb{R}_{\geq 0}^m: \min u(a) = 0} \frac{s(u)}{\max(u)}
\]

in place of \(\kappa\)-upper\((s)\) and \(\kappa\)-lower\((s)\). We may further upper and lower bound these last two quantities, respectively, by

\[
\begin{align*}
\sup_{u \in \mathbb{R}_{\geq 0}^m: \min u(a) = 0} \frac{s(u)}{\max(u)} &= \sup_{u \in \mathbb{R}_{\geq 0}^m: \min u(a) = 0} \frac{s(u)}{\text{range}(u)} \leq \sup_{u \in \mathbb{R}_{\geq 0}^m} \frac{s(u)}{\text{range}(u)}, \\
\inf_{u \in \mathbb{R}_{\geq 0}^m: \min u(a) = 0} \frac{s(u)}{\max(u)} &= \inf_{u \in \mathbb{R}_{\geq 0}^m: \min u(a) = 0} \frac{s(u)}{\text{range}(u)} \geq \inf_{u \in \mathbb{R}_{\geq 0}^m} \frac{s(u)}{\text{range}(u)},
\end{align*}
\]

and we then in particular obtain a distortion upper bound with the two expressions on the right hand side in place of \(\kappa\)-upper\((s)\) and \(\kappa\)-lower\((s)\).

\section*{C.4 Statement and proof of Proposition C.2}

\textbf{Proposition C.2.} Let \(f\) be any deterministic voting rule with \(\beta_f = 0\) and any stakes function \(s\), \(\text{dist}^\prime(f) = \infty\).

\textbf{Proof.} Let \(f\) satisfy \(\beta_f = 0\), and fix a histogram \(h\) in which the winner \(f(h)\) is never ranked first. Then, set the underlying \(U\) to realize this histogram while setting each voter’s ordered utility vector to \(1_0 m^{-1}\). Since the winner is never ranked first, it must get 0 average utility. Since each voter gives their respective first-ranked alternative utility 1, at least one alternative must have at least \(1/m\) average utility; thus, \(\text{dist}^\prime(f)\) is unbounded. Because all voters have identical utility vectors, \(\text{hist}(U) = \text{hist}(U)\). implying \(f(\text{hist}(U)) = f(\text{hist}(U))\); hence \(\text{dist}^\prime(f)\) is also unbounded.

\textbf{Observation C.3.} \(\beta_f = 0\) for many voting established rules.

This observation is shown via a simple instance. Before presenting this instance, we define the voting rules we will address.

\textbf{Voting rules.} \textsc{Borda Count} and \textsc{Veto} are positional scoring rules, which are rules defined by a scoring vector \(w \in [0, 1]^m\) with \(j\)-th entry \(w_j\). In these scoring rules, an alternative receives \(w_j\) points for each voter who ranks it \(j\)-th, and the winner in a given profile is the alternative with the most points. \textsc{Borda Count} is defined by the linearly-decreasing scoring vector \(w = (1, (m-2)/(m-1), \ldots, 1/(m-1), 0)\), and \textsc{Veto} is defined by \(w = 1_{m-1}0_1\). We also consider the entire class of Condorcet-consistent rules. To define this class, we say that \(a\) \textit{pairwise-dominates} \(a'\) in \(h\) if \(a\) is ranked ahead of \(a'\) in at least half of the electorate. We say that \(h\) has a \textit{Condorcet winner} \(a\) if \(a\) pairwise-dominates all other alternatives. A Condorcet-consistent rule is one which \(f(h)\) will be the Condorcet winner on all profiles \(h\) in which a Condorcet winner exists. We will consider this large class of voting rules as a whole, but will not consider any specific rule in this class.

\textbf{Instance.} Indeed, consider the following instance with 4 alternatives, \(a, b, c, d\):

- 1 voter has \(c \succ a \succ d \succ b\)
- \(n/3 - 1\) voters have \(c \succ a \succ b \succ d\)
- \(n/3\) voters have \(b \succ a \succ c \succ d\)
- \(n/3\) voters have \(d \succ a \succ b \succ c\)

Then, \(a\) is ranked ahead of any other alternative by \(2/3\) of voters, and is the Condorcet winner; it will also be the \textsc{Borda} winner, and the \textsc{Veto} winner. Yet, it is never ranked first.

\section*{C.5 Proof of Lemma C.4}

\textbf{Lemma C.4.} For any deterministic voting rule \(f\), it holds that \(\beta_f \leq 1/m\). Moreover, \(\beta_{\text{Plurality}}\) achieves this maximal value.

\textbf{Proof.} Fix any deterministic voting rule \(f\), and define the quantity

\[
\kappa_f = \min_{h \in \Delta(S_m)} \min_{a \neq f(h)} \sum_{\pi \in S_m} h(\pi) (f(\pi) \succ \pi a),
\]

which captures the minimum fraction of people by whom the winner \(f(\pi)\) ranked ahead of any other given alternative \(a\). In (Flanigan, Procaccia, and Wang 2023), it is shown that for any voting rule \(f\), we have that

\[
\kappa_f \leq \kappa_{\text{Minmax}} = 1/m,
\]
where MINIMAX is the voting rule which chooses the alternative $a$ that suffers the least severe worst pairwise defeat; see (Flanigan, Procaccia, and Wang 2023) for details. Moreover, we have that for any histogram profile $h$ and any alternative $a \neq f(h)$,

$$\sum_{\pi \in S_m} h_{\pi} \|\pi^{-1}(f(h)) = 1\| \leq \sum_{\pi \in S_m} h_{\pi} \|f(h) \succ \pi \ a\|
$$

It follows that $\beta_f \leq \kappa_f \leq 1/m$, which proves the first part of the claim.

Now, for the second part of the claim: the fact that $\beta_{\text{PLURALITY}} \geq 1/m$ follows immediately its definition: there always exists an alternative which is first-ranked in at least a $1/m$ fraction of the population – therefore, the PLURALITY winner also has to rank first at least in a $1/m$ fraction of the population.

\[\square\]

C.6 Proof of Proposition C.5

Proposition C.5. For all $s$,

$$\text{dist}^s(\text{PLURALITY}) \geq (m-1) \cdot \kappa\text{-upper}(s)/\kappa\text{-lower}(s).$$

Proof. Formally, we define $\kappa\text{-lower}$ as

$$\kappa\text{-lower} := \inf_{u \in U} \frac{s(u)}{\max u}, \quad U := \{ u \in \mathbb{R}^m_{\geq 0} : u_1 = u_2 = \cdots = u_m = 0 \}.$$ (4)

We will construct an instance which exhibits distortion of the desired order.

**Step 1: Designing the ordered utilities.** There are two population groups: one high-stake population group which we call $G_1$ and on low-stake population group which we call $G_2$. We denote the proportional group size of $G_1$ by $p = \frac{|G_1|}{n} \in (0, 1)$, $1 - p = \frac{|G_2|}{n}$. The exact value of $p$ will be determined later in Step 3 of this proof.

Since we are considering proportional recomposition, we may assume without loss of generality that across agents, their maximal utility is equal to 1. Suppose that $u\text{-upper}$ is an ordered utility vector which maximizes the supremum in $\kappa\text{-upper}$, such that $\max_{a \in [m]} u\text{-upper}(a) = 1$. Similarly, let $u\text{-lower}$ denote the utility vector in $U$ that minimizes the infimum in (4). Now, we assign to $G_1$ the ordered utility vector $u\text{-upper}$, and to $G_2$ the ordered utility vector $u\text{-lower}$. Then, agents in these two population groups have respective stakes of

$$s(u\text{-upper}) = \kappa\text{-upper}, \quad s(u\text{-lower}) = \kappa\text{-lower}.$$ (5)

**Step 2: Designing the rankings.**

- In group $G_1$, we first-rank an alternative $a'$ – this alternative, by appropriate choice of $p$, will later turn out to be the winner of the plurality election. The second to last ranked alternatives in group $G_1$ can be chosen arbitrarily.
- In group $G_2$, the first-rank positions are divided up equally between the remaining $m - 1$ alternatives in $[m] \setminus \{a'\}$. Out of those $m - 1$ alternatives, we choose an arbitrary alternative which we will make the highest-welfare alternative, called $a^\ast$. This alternative $a^\ast$ is ranked second throughout the group $G_2$, whenever it does not rank first.
- Finally, we also specify that the alternative $a'$ is ranked last throughout group $G_2$. The remaining places in $G_2$'s preference profile may be filled arbitrarily.

**Step 3: Specifying the group size $p$.** It remains to calculate $p$. Since $G_1$ has stakes $\kappa\text{-upper}$ and $G_2$ has stakes $\kappa\text{-lower}$, the stakes-weighted plurality score obtained by $a'$ is $p\kappa\text{-upper}$. Any other alternative $a \neq a'$ obtains a stakes-weighted plurality score of $(1 - p)\kappa\text{-lower}/(m - 1)$. Thus, $a'$ winning the election amounts to the inequality

$$p\kappa\text{-upper} \geq \frac{1-p}{m-1} \kappa\text{-lower} \iff p\kappa\text{-upper} + \frac{\kappa\text{-lower}}{m-1} \geq \frac{\kappa\text{-lower}}{m-1} \iff p \geq \frac{\kappa\text{-lower}}{p\kappa\text{-upper} + (m-1)\kappa\text{-upper}}.$$ (6)

Thus, let us set $p$ to be equal to the last expression, i.e.

$$p = \frac{|G_1|}{n} = \frac{\kappa\text{-lower}}{p\kappa\text{-upper} + (m-1)\kappa\text{-upper}}.$$ (7)

With this choice of $p$, we notice that

$$\frac{\text{sw}(a', U)}{n} = p, \quad \text{and} \quad \frac{\text{sw}(a^\ast, U)}{n} \geq \frac{1}{n} \sum_{i \in G_2} u_i(a^\ast) = 1 - p,$$

since agents in $G_2$ have utility 1 for $a^\ast$, and agents in $G_1$ may have positive utility for $a^\ast$. In conclusion, the distortion in this instance is lower bounded by

$$\frac{\text{sw}(a^\ast, U)}{\text{sw}(a', U)} \geq \frac{1-p}{p} = \frac{(m-1)\kappa\text{-upper}}{\kappa\text{-lower} + (m-1)\kappa\text{-upper}} = \frac{(m-1)\kappa\text{-upper}}{\kappa\text{-lower}}.$$ (8)

\[\square\]
C.7 Proof of Theorem 3.7

Theorem 3.7. For all 1-homogeneous $s$, randomized $f$,
\[
\text{dist}^*(f) \geq \sqrt{m} / (10 + 3 \log m).
\]

Proof. Define the vector $1_z \cdot 0_z$ to be the vector consisting of $z$ ones followed by $z'$ zeroes.

**Case 1:** Suppose that there exists some $z \leq (\log m) - 1$ such that \( s(1_{z+1}0_{m-z-1}) \leq 1 \). Fix this $z$. We now design a utility instance and associated preference histogram which exhibits a distortion of the order $\sqrt{m}/\log m$.

**Step 1: Designing the rankings.** We begin by designing the preference histogram. We divide the population into $m/\log m$ groups
\[
G_1, \ldots, G_{m/\log m}.
\]
Let alternatives $1, \ldots, m/\log m$ occupy the first positions in each of the groups $G_1, \ldots, G_{m/\log m}$, respectively. Similarly, we occupy the second to $z$-th rank of those groups by following alternatives:

- **Rank:** 1 2 \ldots \ z
- **Group $G_1$:** $m/\log m + 1 \; \ldots \; (z-1)m/\log m + 1$
- **Group $G_{m/\log m}$:** $m/\log m \; 2m/\log m \; \ldots \; zm/\log m$.

Next, we also divide the population into $\sqrt{m}$ parts $H_1, \ldots, H_{\sqrt{m}}$ of equal size, based on which alternatives occupy the $(z+1)$-th position. We may design this partition in a way such that
\[
\forall k \in [\sqrt{m}], \; \{l \in [m/\log m] : H_k \cap G_l \neq \emptyset\} \leq \frac{\sqrt{m}}{\log m} + 2.
\]

Intuitively, this is because the groups $H_k$ are larger by a factor of $\sqrt{m}/\log m$ than the groups $G_i$. We may thus pick the partition into $H_k$ such that each $H_k$ overlaps with at most $\sqrt{m}/\log m$ groups $G_i$. For each $k \in [\sqrt{m}]$, we assign the $(z+1)$-th position in group $H_k$ to be occupied by the alternative $zm/\log m + k$. Finally, we fill the rest of the positions in the preference histogram — i.e. the $(z+2)$-th to last ranks — arbitrarily.

**Step 2: Designing the utilities.** Amongst the $\sqrt{m}$ alternatives which are ranked in the $(z+1)$-th position, there must exist one alternative which we call $\bar{a}$ which is chosen by the voting rule $f$ with probability at most $1/\sqrt{m}$. That is, if $h$ denotes the preference histogram constructed in Step 1, then
\[
f_h(\bar{a}) \leq 1/\sqrt{m}.
\]

Let $H_{\bar{a}}$ be the unique group which ranks $\bar{a}$ in the $(z+1)$-th position. Now, we assign utilities as follows. Define the following ratio of stakes:
\[
c_z := \frac{s(1_{z+1}0_{m-z-1})}{s(1_{1}0_{m-z})} \leq e.
\]

- **Group $H_{\bar{a}}$.** We assign to agents in $H_{\bar{a}}$ the ranked utilities $s(1_{z+1}0_{m-z-1})$.
- **Remainder.** In the remaining population $H_{\bar{a}}^c$, we assign the ranked utilities $c_z \cdot s(1_{z}0_{m-z})$.

These ordered utilities, together with the rankings designed in Step 1, determine a utility matrix which we call $U$.

1. The alternative $\bar{a}$ has average utility $\sw(\bar{a}, U) = 1/\sqrt{m}$.
2. All other alternatives $a \neq \bar{a}$ have average utility at most $\sw(a, U) = c_z \log m/m \leq e \log m/m$.
3. By the homogeneity of the stakes function $s(\cdot)$, all voters have equal stakes. Therefore, we have that $\text{hist}^*(U) = \text{hist}(U) = h$, and thus also
\[
f(\text{hist}^*(U)) = f(h).
\]

In particular, $\bar{a}$ is chosen by the voting rule with probability at most $1/\sqrt{m}$ in $f(\text{hist}^*(U))$.

Together, these observations yield that
\[
\mathbb{E} [\sw(f(\text{hist}^*(U))))] \leq \frac{e \log m}{m} + \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{m}} = \frac{e \log m + 1}{m},
\]
and thus the $f$ in Case 1 is at least
\[
\frac{\max_a \sw(a, U)}{\mathbb{E} [\sw(f(\text{hist}^*(U))))]} \geq \frac{1/\sqrt{m}}{1 + e \log m}/m = \sqrt{m} / (1 + e \log m).
\]
Case 2: It remains to treat the case when the premise of Case 1 is not fulfilled, that is, for every \( z \leq \log m - 1 \), it holds that \( s(1_{z+1}0_{m-z-1})/s(1_{z}0_{m-z}) \geq e \). By multiplying this equality for all \( z = 2, \ldots, \log m - 1 \), it follows that
\[
\frac{s(1_{\log(m)-1}0_{m-\log(m)+1})}{s(1_{1}0_{m-1})} \geq 2^{\log m - 2} \geq \frac{m}{e^2}.
\]
(5)

Now let us consider a histogram profile where the population is divided in \( \sqrt{m} \) many equal sizes groups, which first-rank alternatives \( 1, \ldots, \sqrt{m} \), respectively. We fill up the remaining positions in the histogram arbitrarily. Denote this histogram by \( h \).

We now assign utilities to induce \( \mathcal{A} \). There must exist one alternative among the \( \sqrt{m} \) first-ranked alternatives that receives \( \leq 1/\sqrt{m} \) probability of selection by \( f(h) \). Let us call this alternative \( a^* \), and let us call the group which ranks \( a^* \) first \( G \).

- **Group \( G \).** In this group, we assign the ordered utility vector \( 1_10_{m-1} \).
- **Group \( G^c \).** In the remainder of the population, we assign the ordered utility vector

\[
\frac{s(1_{1}0_{m-1})}{s(1_{\log(m)-1}0_{m-\log(m)+1})} \cdot 1_{\log(m)-1}0_{m-\log(m)+1}
\]

Let us denote the resulting utility matrix by \( U \). We observe the following.

1. The average utility of \( a^* \) is at least \( \text{sw}(a^*, U)/n \geq 1/\sqrt{m} \).
2. By equation (5), the average utility of any other alternative \( a \neq a^* \) is at most

\[
\frac{\text{sw}(a, U)}{n} \leq \frac{e^2}{m}.
\]

3. All voters have equal stakes. Therefore \( f(h) = f(\text{hist}(U)) = f(\text{hist}^*(U)) \) and we may estimate

\[
\mathbb{E}[\text{sw}(f(\text{hist}^*(U)), U)] \leq \frac{1}{\sqrt{m}} + \frac{e^2}{m^2} \leq \frac{10}{m}.
\]

We obtain an overall distortion of at least

\[
\text{dist}^*_U(f) \geq \frac{\sqrt{m}}{10},
\]

and the proof is complete. \( \square \)

C.8 Formalisms about the Stable Lottery Rule

We now define the Stable Lottery rule, following (Ebadian et al. 2022). Since only the case of a stable lottery of size \( \sqrt{m} \) is relevant to us, we shall restrict our definition to this special case. Let \( \mathcal{P}_{\sqrt{m}}([m]) \) be the set of all subsets (or ‘committees’) of \([m]\), of size \( \sqrt{m} \), and let \( \Delta(\mathcal{P}_{\sqrt{m}}([m])) \) be the set all of all distributions on \( \mathcal{P}_{\sqrt{m}}([m]) \). Given a subset \( A \subseteq [m] \) of alternatives, an alternative \( a \in [m] \) and a histogram profile \( h \in \Delta(S_m) \), let us denote the fraction of voters who rank \( a \) ahead of all of \( A \) by

\[
\text{Freq}_{a > A}(h) = \sum_{\pi \in S_m} h_{\pi}(a >_\pi A).
\]

If \( a \in A \), then we set \( \text{Freq}_{a > A}(h) = 0 \) for all \( h \).

**Definition C.6** (Stable lottery). Given a preference histogram \( h \), a stable lottery (of size \( \sqrt{m} \)) is a probability distribution \( P(h) \in \Delta(\mathcal{P}_{\sqrt{m}}([m])) \) (i.e., a random selection of a committee of size \( \sqrt{m} \)) such that for all \( h \),

\[
\max_{a \in [m]} \mathbb{E}_{A \sim P(h)}[\text{Freq}_{a > A}(h)] < \frac{1}{\sqrt{m}}.
\]

It is well-known that a stable lottery always exists, see, e.g. (Ebadian et al. 2022). Building on this definition, we define the Stable Lottery Rule in terms of histograms.

**Definition C.7** (Stable Lottery Rule). Given a histogram \( h \), let \( P(h) \) be a stable lottery. With probability \( 1/2 \), sample a committee \( A \) of size \( \sqrt{m} \) from \( P(h) \), and then choose an alternative uniformly at random from \( A \). Else, with the remaining probability \( 1/2 \), simply choose an alternative uniformly at random from \([m]\).

**Proof of Theorem 3.8.**

**Theorem 3.8.** For \( s \) equal to sum, max, or range,

\[
\text{dist}^*(\text{Stable Lottery}) \in O(\sqrt{m}).
\]
First, assume that \( s \in \{\text{max}, \text{sum}\} \), and let \( f = \text{STABLE LOTTERY RULE} \). Then, by a well-established result from Ebadian et al. (Ebadian et al. 2022), we know that both for \( s = \text{sum} \) and \( s = \text{max} \), the worst-case distortion over unit-stakes instances is of the order \( O(\sqrt{m}) \),

\[
\sup_{s \geq 1} \sup_{U \in \mathcal{U}_s} \text{dist}_U(\text{STABLE LOTTERY RULE}) \in O(\sqrt{m}),
\]

where we recall the notation \( \mathcal{U}_s \) for the set of utility matrices \( U \) where each voter has unit stakes, \( s(u_i) = 1 \). Our goal is to use Theorem B.1 to conclude that the stakes-proportional procedure also has distortion of the order at most \( O(\sqrt{m}) \). For this, we need to confirm that the \( \text{STABLE LOTTERY RULE} \) is rationally approximable in the sense of Definition B.2. Indeed, this is seen as follows. Let \( h \) be an arbitrary preference histogram. In (Ebadian et al. 2022), it is proven not just that a stable lottery always exists for \( h \); indeed, a slightly stronger requirement is validated, namely, that the lottery satisfies

\[
\max_{a \in [m]} \mathbb{E}_{A \sim P(h)}[\text{Freq}_{a \succ A}(h)] \leq \frac{1}{\sqrt{m} + 1}.
\]

Now, let \( \varepsilon > 0 \). Suppose that \( \tilde{h} \) is another histogram profile with rational entries such that

\[
\sup_{\pi \in S_m} |h_{\pi} - \tilde{h}_{\pi}| \leq \varepsilon.
\]

We may also choose \( \tilde{h} \) such that the difference \( |\text{Freq}_{a \succ A}(h) - \text{Freq}_{a \succ A}(\tilde{h})| \leq \varepsilon \) for any \( a \). Choosing \( \varepsilon \) small enough, \( P(h) \) is a permissible stable lottery also for \( \tilde{h} \). Using this stable lottery, we have that \( f(h) = f(\tilde{h}) \); thus \( f \) is rationally approximable; the statement follows for \( s \in \{\text{max}, \text{sum}\} \).

It remains to show the claim for \( s = \text{range} \). Here, we argue along the same lines as Observation C.1: The worst-case distortion both for \( s = \text{range} \) and for \( s = \text{max} \) can be realized while only considering utility matrices in which each voter has minimum utility \( 0 \). Let this set of utilities be denoted by \( V \). Then,

\[
\sup_{U \in \mathbb{R}^{n \times m}_{\geq 0}} \text{dist}_U^\text{range}(f) = \sup_{U \in V} \text{dist}_U^\text{range}(f) = \sup_{U \in V} \text{dist}_U^\text{max}(f) = \sup_{U \in \mathbb{R}^{n \times m}_{\geq 0}} \text{dist}_U^\text{max}(f).
\]

\[\square\]

### C.9 Folklore: all randomized rules have at least \( m \) distortion.

**Fact C.8.** For all voting rules \( f \), \( \text{dist}(f) \geq m \).

**Proof.** Consider a histogram in which each of the \( m \) alternatives occupies a \( 1/m \) fraction of the first positions and the second to last positions are occupied arbitrarily. There exists some alternative \( a \) which will be chosen by the randomized rule with probability at most \( 1/m \). Let \( G \) denote the group in which \( a \) is ranked first. In this group, let us assign the ordered utility vector \((1, 0, ..., 0)\). In the remainder of the population \( G^c \), we assign the zero utility vector. Let us denote this utility matrix by \( U \). Then, since \( f \) selects \( a \) with probability at most \( 1/m \), denoting the winner of the election by \( a' \), we obtain \( \mathbb{E}[\text{sw}(a', U)/n] \leq 1/m^2 \), while the maximum welfare alternative has average utility \( \text{sw}(a, U)/n = 1/m \); thus the distortion of \( f \) is at least \( m \). \[\square\]
D Supplemental Materials from Section 4

D.1 Proof of Theorem 4.1
Theorem 4.1. For all Δ ≥ 1, f, 1-homogeneous s, and U,
\[ \text{dist}_{\hat{s}}(f) \leq \Delta \cdot \text{dist}_{s}(f). \]

Proof. Fix a utility matrix U, a stakes function s and an error vector Δ. Then, let \( \hat{U} \) be the utility matrix where voter \( i \)'s utility vector is scaled by a factor \( \Delta \), i.e., \( \hat{u}_i = \Delta u_i \). Then, since \( s \) is 1-homogeneous, we have that \( s(u_i) = \Delta s(u_i) \), and therefore \( \text{hist}^s(\hat{U}) = \text{hist}^{\Delta s}(U) \).

This directly implies that \( f(\text{hist}^s(\hat{U})) = f(\text{hist}^{\Delta s}(U)) \). Moreover, for every alternative \( a \), it holds that \( \text{sw}(a, \hat{U}) \in [\text{sw}(a, U), \Delta \text{sw}(a, U)] \). It follows that \[ \mathbb{E}[\text{sw}(f(\text{hist}^s(\hat{U})) - \mathbb{E}[\text{sw}(f(\text{hist}^{\Delta s}(U)), U)]\]

from which in turn we deduce that \[ \frac{\max_a \text{sw}(a, U)}{\mathbb{E}[\text{sw}(f(\text{hist}^s(U))], U)] \leq \Delta \cdot \frac{\max_a \text{sw}(a, U)}{\mathbb{E}[\text{sw}(f(\text{hist}^{\Delta s}(U)), U)]} \leq \Delta \cdot \text{dist}^s(f). \]

Taking suprema on the left hand side then completes the proof. \( \square \)

D.2 Assumptions and Proof of Lemma 4.2
Voter model. We assume voters are honest: for all \( \ell \) and \( i \), \( \pi^i_\ell \) is consistent with \( u^i \). Within this constraint, voters are utility-maximizing: \( i \) chooses \( u_i \) to maximize their expected utility across elections. This expectation is taken over \( i \)'s prior on \( h^i_\ell \), the votes submitted by all other voters. Here, let \( i \) be oblivious: \( i \) assumes that \( h^i_\ell \) is drawn from the \( \mathcal{I} \), the uniform distribution over all possible such histograms (we call it \( \mathcal{I} \) for its resemblance to the Impartial Culture model). Then, \( i \) chooses \( \hat{u}_i \) so that \[ \hat{u}_i = \arg \max_{u_i} \mathbb{E}_{h^i_\ell \sim \mathcal{I}} \left[ \sum_{\ell \in [k]} u_i(b^\ell)|w_i^\ell \right] \text{ s.t. } \sum_{\ell \in [k]} (w_i^\ell)^2 \leq 1. \]

where the random variable \( b^\ell \), the winner, implicitly depends on both \( h^i_\ell \) and \( w_i^\ell \). Finally, we assume that voters believe their probability of pivotality in election \( \ell \) increases linearly in \( w_i^\ell \). While this is technically mathematically inconsistent with their uniform priors over others’ votes, we make this added assumption to simplify the analysis for both ourselves and for voters: even under uniform priors, computing the exact marginal increase in probability of pivotality per unit of weight is a complex calculation that we cannot expect voters to do. In light of this, linearity is a natural assumption.

Proof. (Lemma 4.2). Fix an \( i \). Let the alternatives in each election \( \ell \) be \( A^\ell \), and for all \( \ell \in [k] \), let \( \hat{a}^\ell \) be \( i \)'s favorite alternative in the \( \ell \)th election. For any given \( w_i \), define the following events:

- \( X^\ell \): \( \hat{a}^\ell \) wins among only the votes in \( h^i_\ell \) (and thus also wins with \( i \)'s ranking weighted by \( w_i^\ell \) added to the profile)
- \( Z^\ell \): some \( a \neq \hat{a}^\ell \) wins among only the votes in \( h^i_\ell \) by a margin of between 0 and \( w_i^\ell \) (and thus \( i \)'s vote is pivotal).

Note that these events are mutually exclusive.

Now, we compute voter \( i \)'s expected reward in terms of these quantities. In the first equality below, we use the observation that if neither \( X^\ell \) or \( Z^\ell \) occur, then some alternative \( a \neq \hat{a}^\ell \) wins among the votes in \( h^i_\ell \) by a margin of more than \( w_i^\ell \). Because \( i \) has uniform priors over other voters’ rankings, conditioned on \( \neg X^\ell \land \neg Z^\ell \), they assume uniform probability over all other alternatives winning and thus assess their expected utility in this event to be the average of all alternatives in \( A^\ell \setminus \{\hat{a}^\ell \} \). Here, we will use the shorthand \( h_{\ell-i} := (h^i_\ell : \ell \in [k]) \); also recall the shorthand \( s^i_\ell := s^i(u_i)_{\ell \in [k]} \).

\[ \mathbb{E}_{h_{\ell-i} \sim \mathcal{I}} \left[ \sum_{\ell \in [k]} u_i(b^\ell)|w_i^\ell \right] = \sum_{\ell \in [k]} \left( \text{Pr}[X^\ell] + \text{Pr}[Z^\ell|w_i^\ell] \right) \cdot u_i(\hat{a}^\ell) + \text{Pr}[\neg X^\ell \land \neg Z^\ell|w_i^\ell] \cdot \sum_{a \in A^\ell \setminus \{\hat{a}^\ell \}} \frac{u_i(a)}{m-1} \]

Again using \( i \)'s uniform priors over other voters’ behavior, \( \text{Pr}[X^\ell] = 1/m \). Simplifying,

\[ = \sum_{\ell \in [k]} \left( 1/m + \text{Pr}[Z^\ell|w_i^\ell] \right) u_i(\hat{a}^\ell) + \left( 1 - 1/m - \text{Pr}[Z^\ell|w_i^\ell] \right) \cdot \sum_{a \in A^\ell \setminus \{\hat{a}^\ell \}} \frac{u_i(a)}{m-1} \]

\[ = \sum_{\ell \in [k]} \left( \frac{1}{m} \sum_{a \in A^\ell} u_i(a) + \text{Pr}[Z^\ell|w_i^\ell] \left( u_i(\hat{a}^\ell) - \sum_{a \in A^\ell \setminus \{\hat{a}^\ell \}} \frac{u_i(a)}{m-1} \right) \right) \]
The first term of the summand is fixed in the instance, so it is not decision relevant. Then,

\[
\max_{w_i} \mathbb{E}_{h_i \sim I} \left[ \sum_{\ell \in [k]} u_i(b^\ell_i) | w_i \right] = \max_{w_i} \sum_{\ell \in [k]} Pr[Z^\ell_i | w_i] \left( u_i(\hat{a}^\ell_i) - \sum_{a \in A^\ell_i \setminus \{\hat{a}^\ell_i\}} \frac{u_i(a)}{m - 1} \right) \\
= \max_{w_i} \sum_{\ell \in [k]} Pr[Z^\ell_i | w_i] \left( \max(u_i^\ell_i) - \frac{\text{sum}(u_i^\ell_i) - \max(u_i^\ell_i)}{m - 1} \right) \\
= \max_{w_i} \sum_{\ell \in [k]} Pr[Z^\ell_i | w_i] \cdot s^*(u_i^\ell_i)
\]

Finally, we assumed that voters assume that their probability of pivotality increases linearly with each additional unit of weight placed behind their vote. Applying this assumption,

\[
= \max_{w_i} \sum_{\ell \in [k]} w_i^\ell_i \cdot s^*(u_i^\ell_i)
\]

Then, the final problem voters are solving is then

\[
\hat{w}_i := \arg \max_{w_i} \sum_{\ell \in [k]} w_i^\ell_i \cdot s^*(u_i^\ell_i) \quad \text{s.t.} \quad \sum_{\ell \in [k]} (w_i^\ell_i)^2 \leq 1. \quad (6)
\]

We can compute the optimizer of this program by simply projecting the vector \( s_i^* \) onto the unit sphere, concluding the claim.

\[
\hat{w}_i = \frac{s_i^*}{\|s_i^*\|_2}
\]