Envy-Free Division of Sellable Goods

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Abstract

We study the envy-free allocation of indivisible goods between two players. Our novel setting includes an option to sell each good for a fraction of the minimum value any player has for the good. To rigorously quantify the efficiency gain from selling, we reason about the *price of envy-freeness* of allocations of sellable goods — the ratio between the maximum social welfare and the social welfare of the best envy-free allocation. We show that envy-free allocations of sellable goods are significantly more efficient than their unsellable counterparts.

1 Introduction

After decades of unresolved communication problems (Kushilevitz and Nisan 1996), Alice and Bob have decided to get a divorce. Their worldly goods include a well-worn (shared) blackboard, a museum-quality collection of private keys, and a 19th century French vase. Can Alice and Bob divide these goods in a way that is fair to both sides?

To answer this question we must be more specific about what we mean by "fair". The notion of *envy-freeness* provides a natural interpretation: Alice (weakly) prefers her own bundle of goods to Bob's bundle, and Bob is likewise convinced that he got the better deal. In other words, when the allocation is envy free, neither Alice nor Bob is interested in swapping bundles.

While envy-freeness is a compelling ideal, envy may clearly be unavoidable when the goods are indivisible. But envy-freeness can nevertheless be achieved if we are willing to split one of the goods. This concession enables the famous *Adjusted Winner (AW)* protocol (Brams and Taylor 1996) — an envy-free protocol that has been patented by New York University and licensed to the law firm *Fair Outcomes, inc.*

In their book, Brams and Taylor (1996, pp. 102–108) apply AW to the real divorce case of *Jolis vs. Jolis*, which was decided in 1981 (for convenience let us call the wife Alice Jolis, and the husband Bob Jolis). The marital property included a Paris apartment, a Paris studio, a New York City coop, a farm, cash and receivables, securities, a profit-sharing plan, and a life insurance policy. Deducing Alice and Bob's values for these goods (from available data) and applying AW yields an allocation that gives the studio, coop, farm, securities, and life insurance policy to Bob, the Paris apartment to Alice, and splits the cash (giving a 1/11 fraction to Bob and the rest to Alice). In this case AW split a good that happens to be divisible, but this is by no means guaranteed: had Alice and Bob expressed different preferences, AW could have split one of the indivisible goods, say, the Paris apartment. In practice, this would typically mean selling the Paris apartment and splitting the cash. However, for example, 40% of the market price of the Paris apartment may not be equal to 40% of Bob's value for owning the entire apartment, invalidating the assumptions underlying AW and thus nullifying its guarantees. Moreover, if we are indeed allowed to sell goods, perhaps there is a better envy-free allocation?

This paper is motivated by the preceding observations and questions, which, we believe, call for an explicit model of the envy-free division of *sellable* goods.

1.1 Our Approach and Formal Model

We consider a setting with two players, Alice (denoted A) and Bob (denoted B). Our approach cannot give rise to non-trivial positive results when there are more than two players, as we discuss in Section 4. There is also a set of m indivisible goods to be allocated, denoted by $[m] = \{1, \ldots, m\}$. For each $j \in [m]$ and $P \in \{A, B\}$, P's value for j is denoted $v_P(j) \in [0, 1]$.

We make two assumptions regarding the valuation functions:

1. *Additivity*: For $P \in \{A, B\}$ and $J \subseteq [m]$, $v_P(J) = \sum_{j \in J} v_P(j)$. In particular, $v_P(\emptyset) = 0$.
2. *Normalization*: For $P \in \{A, B\}$, it holds that $v_P([m]) = \sum_{j \in [m]} v_P(j) = 1$.

An *allocation* (partition of the goods between Alice and Bob) $\mathcal{X} = \{X_A, X_B\}$ is *envy free* if $v_A(X_A) \geq v_A(X_B)$ and $v_B(X_B) \geq v_B(X_A)$. We are also interested in the economic efficiency of allocations, which we measure via their (utilitarian) *social welfare*: $\mathtt{SW}(\mathcal{X}, v_A, v_B) = v_A(X_A) + v_B(X_B)$.

Our main conceptual contribution is the idea that indivisible goods can be sold, and thereby converted into an infinitely divisible cash value (which Alice and

Bob value equally). We assume that there is a universal constant $c \in (0, 1]$ such that the selling price of a set of goods $J \subseteq [m]$ is $c \cdot \sum_{j \in J} \min\{v_A(j), v_B(j)\}$. The rationale is that Alice and Bob both value each good at least at its market value, because they can always sell a good they have obtained. But their value for a good can be strictly higher than its market value — this is captured by the constant c. While we assume, for ease of exposition, that the price is exactly a c-fraction of the minimum value, our results naturally hold if this expression is just a lower bound; this observation is important when different goods can be sold for different fractions of the minimum value.

To rigorously quantify the gain from selling, we use the notion of (utilitarian) *price of envy-freeness*, independently introduced by Caragiannis et al. (2012) and (in slightly different form) by Bertsimas et al. (2011). For valuation functions v_A, v_B, let $\texttt{OPT}(v_A, v_B)$ be the social welfare of the welfare-maximizing allocation; i.e., $\texttt{OPT}(v_A, v_B) \triangleq \max_{\mathcal{X}} \texttt{SW}(\mathcal{X}, v_A, v_B)$. Similarly, define $\texttt{OPT}_{\text{EF}}(v_A, v_B)$ to be the social welfare of the welfare-maximizing *envy-free* allocation *without selling*, and $\texttt{OPT}_{\text{EFS}}(v_A, v_B)$ to be the social welfare of the welfare-maximizing *envy-free* allocation *with selling*. The price of envy-freeness is the worst-case (over valuation functions v_A, v_B) ratio $\texttt{OPT}(v_A, v_B)/\texttt{OPT}_{\text{EF}}(v_A, v_B)$. With selling, it is the worst-case ratio $\texttt{OPT}(v_A, v_B)/\texttt{OPT}_{\text{EFS}}(v_A, v_B)$. Hereinafter the valuation functions will always be clear from the context, so we simply write $\texttt{OPT}$, $\texttt{OPT}_{\text{EF}}$ and $\texttt{OPT}_{\text{EFS}}$.

We can now formulate our primary research challenge:

Show that the option to sell goods provides a major boost to efficiency by establishing that the price of envy-freeness with selling is significantly lower than the price of envy-freeness without selling.

1.2 Our Results

Table 1 summarizes our results regarding the price of envy-freeness. The columns distinguish between two scenarios: (i) the general setting where an envy-free allocation without selling may not exist, and (ii) such an allocation does exist. The first row shows our bounds on the price of envy-freeness with selling (which are tight), the second row instantiates these bounds for $c = 1$, and the third row gives the price of envy-freeness without selling. In scenario (i), the price of envy-freeness without selling is ∞ (or, alternatively, it is not well defined). In contrast, our analysis (Theorem 1) gives a bound of $3/2$ for the case of $c = 1$. In scenario (ii), Caragiannis et al. (2009) show that the price of envy-freeness (without selling) is $3/2$; our bound (Theorem 2) instantiates to $6/5$ when $c = 1$.

We also investigate the problem of computing a social welfare maximizing allocation of sellable goods. While the problem is **NP**-complete (Theorem 3), we show that, when $c = 1$, it admits a fully polynomial time approximation scheme (Theorem 4).

	Is there an EF allocation?	
	No	**Yes**
Selling	$\max\{\frac{3-c}{c+c^2}, \frac{3}{1+c}\}$	$\max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\}$
Selling ($c = 1$)	$3/2$	$6/5$
No selling	∞	$3/2$

Table 1: Summary of our results.

1.3 Additional Context and Significance in AI

The rigorous study of fair division dates back to the work of Steinhaus (1948). Over the years a deep theory has been developed by economists, mathematicians, and political scientists; see, e.g., the books by Brams and Taylor (1996) and Moulin (2003). More recently, the study of fair division has attracted significant attention from the AI community. This relatively newfound interest is partly motivated by the idea that fair division theory can inform the design of multiagent systems (Chevaleyre et al. 2006).

The fair division literature makes a distinction between two typically disjoint cases, depending on whether the goods are divisible or indivisible. The divisible case usually involves a single, heterogeneous good, and the task of dividing this good is known as *cake cutting*. This setting has been extensively studied by AI researchers in recent years (Procaccia 2009; Chen et al. 2013; Caragiannis, Lai, and Procaccia 2011; Cohler et al. 2011; Brams et al. 2012; Bei et al. 2012; Kurokawa, Lai, and Procaccia 2013; Brânzei, Procaccia, and Zhang 2013); see the survey by Procaccia (2013) for an overview. In the context of indivisible goods, AI researchers have also studied issues like complexity, preference handling, and incentives (Bouveret and Lang 2008; 2011; Kalinowski et al. 2013).

Our work attempts to bridge these two worlds, by essentially allowing the division of indivisible goods — at a cost. In this sense, our paper is somewhat related to a line of work on reaching envy-free states through distributed negotiation over indivisible goods (Chevaleyre et al. 2007; Chevaleyre, Endriss, and Maudet 2007; 2010), because these papers allow players to pay each other in order to achieve envy-freeness (as long as the sum of payments is zero). That said, our motivation, questions, approach, and results are all fundamentally different.

2 Bounds on the Price of Envy-Freeness

To gain some intuition for price of fairness bounds, we start by discussing the price of envy-freeness *without* selling. As mentioned above, Caragiannis et al. (2009) restrict their attention to instances where an envy-free allocation does exist, and — for these instances — show that $\texttt{OPT}/\texttt{OPT}_{\text{EF}} < 3/2$. The proof is simple. First, notice that a player is not envious if and only if his or her bundle is worth at least $1/2$ (because the sum of values for the two bundles is 1). Now, if the optimal allocation is envy-free we are done. If not, on the one hand either Alice or Bob is envious under the optimal solution, so $\mathtt{OPT} < 1 + 1/2 = 3/2$; and on the other hand, $\mathtt{OPT}_{\text{EF}} \geq 1/2 + 1/2 = 1$. Crucially, Caragiannis et al. (2012) also show that this bound is tight; i.e., for every $\epsilon > 0$ there is an example where the ratio is $3/2 - \epsilon$.

In contrast, when the goods are sellable, there always exists an envy-free allocation, so the price of envy-freeness is always well defined. Interestingly, it turns out that the price of envy-freeness (with selling) when an envy-free allocation without selling is assumed to exist (Theorem 2) is significantly lower than the price of envy-freeness in the general case (Theorem 1), so we present these results as two separate theorems, starting with the (simpler, surprisingly) general case.

Theorem 1. *Even if an envy-free solution without selling does not exist for a particular instance, for any* $c \in (0, 1]$, $\frac{\mathtt{OPT}}{\mathtt{OPT}_{\text{EFS}}} \leq \max\{\frac{3-c}{c+c^2}, \frac{3}{1+c}\}$.

Before proving the theorem, we demonstrate that the bound is tight. For $c \in (0, 3/4]$ it holds that $\frac{3-c}{c+c^2} \geq \frac{3}{1+c}$. Let there be two goods, and let $v_A(1) = 1 - c/2 - \epsilon$, $v_A(2) = c/2 + \epsilon$, $v_B(1) = 1/2 + \epsilon$, $v_B(2) = 1/2 - \epsilon$, for an arbitrarily small $\epsilon > 0$. We have $\mathtt{OPT} = 3/2 - c/2 - 2\epsilon$. The only way to get an envy-free allocation is to sell both goods, and split the cash equally. The value of the resulting allocation is just the amount of cash: $\mathtt{OPT}_{\text{EFS}} = c \cdot (1/2 + \epsilon + c/2 + \epsilon) \approx (c + c^2)/2$. Then $\mathtt{OPT}/\mathtt{OPT}_{\text{EFS}} \approx (3 - c)/(c + c^2)$.

For $c \in [3/4, 1]$, $\frac{3}{1+c} \geq \frac{3-c}{c+c^2}$. Let $v_A(1) = 1$, $v_A(2) = 0$, $v_B(1) = 1/2 + \epsilon$, $v_B(2) = 1/2 - \epsilon$, for an arbitrarily small $\epsilon > 0$. For this instance, $\mathtt{OPT} = 3/2 - \epsilon$. To obtain an EF allocation, good 1 must be sold — we then allocate good 2 to Bob and divide that cash so that Alice and Bob are both satisfied. Therefore, $\mathtt{OPT}_{\text{EFS}} = c \cdot (1/2 + \epsilon) + (1/2 - \epsilon) \approx (c + 1)/2$, and $\mathtt{OPT}/\mathtt{OPT}_{\text{EFS}} \approx 3/(1 + c)$.

Proof of Theorem 1. Suppose we start with a social welfare maximizing allocation $\mathcal{S} = \{S_A, S_B\}$, and assume without loss of generality that $v_A(S_A) \geq v_B(S_B)$. It holds that $v_A(S_A) \geq v_B(S_A)$ and $v_B(S_B) \geq v_A(S_B)$, because to maximize social welfare each good is allocated to the player who values it more highly. Since no envy-free solution exists without selling, we have that $1 \geq v_A(S_A) \geq v_B(S_A) > 1/2 > v_B(S_B) \geq v_A(S_B)$. Thus, selling only S_B would leave the player that does not receive S_A envious. Hence, to obtain an envy-free allocation, it may be necessary to sell the entirety of S_A, in the worst case that it is a single good.

It may also be required to sell S_B to guarantee the existence of an envy-free allocation. If only S_A is sold, it is possible that the player that does not receive S_B will be envious, if $v_A(S_B) > c \cdot v_B(S_A)$.

Case 1: $v_A(S_B) \leq c \cdot v_B(S_A)$. In this case we only sell S_A. Alice's total remaining value is

$$\begin{aligned} v_A(S_B) + c \cdot v_B(S_A) &= 1 - v_A(S_A) + c \cdot v_B(S_A) \\ &\leq 1 - (1 - c) \cdot v_B(S_A), \end{aligned}$$

and Bob's total remaining value is $1 - (1 - c) \cdot v_B(S_A)$. We first show that there is enough cash from selling S_A to satisfy both Alice and Bob in an envy-free allocation.

Suppose that $v_B(S_B) \geq c \cdot v_B(S_A)$; i.e., Bob does not need to be given any cash to be envy-free. Alice's total (remaining) value is $v_A(S_B) + c \cdot v_B(S_A) \leq 2c \cdot v_B(S_A)$, so giving her the cash will make her envy-free.

Otherwise, suppose Bob is not immediately envy-free after selling S_A. We show a stronger statement that implies there is enough cash to satisfy both Alice and Bob:

$$\begin{aligned} c \cdot v_B(S_A) \geq \frac{1}{2}&(1 - (1 - c) \cdot v_B(S_A)) \\ &+ \left(\frac{1}{2}(1 - (1 - c) \cdot v_B(S_A)) - v_B(S_B)\right). \end{aligned}$$

Assume for the sake of contradiction that the above statement does not hold. Then

$$c \cdot v_B(S_A) < 1 - (1 - c) \cdot v_B(S_A) - v_B(S_B).$$

Equivalently, $v_B(S_A) + v_B(S_B) < 1$, which yields the contradiction.

We conclude that in Case 1 we can obtain an envy-free allocation by selling only S_A and letting Bob keep S_B. This yields

$$\mathtt{OPT}_{\text{EFS}} \geq c \cdot v_B(S_A) + v_B(S_B) = 1 - (1 - c) \cdot v_B(S_A).$$

Since $\mathtt{OPT} \leq 2 - v_B(S_A)$, and letting $x \triangleq v_B(S_A)$,

$$\frac{\mathtt{OPT}}{\mathtt{OPT}_{\text{EFS}}} \leq \frac{2 - x}{1 - (1 - c) \cdot x} \triangleq g(x, c).$$

We want the maximum of g over the entire range $v_B(S_A) \in [1/2, 1]$. The partial derivative of g with respect to x is

$$\frac{\partial g(x, c)}{\partial x} = \frac{1 - 2c}{(1 - (1 - c) \cdot x)^2}.$$

Thus, the maximum occurs at $g(1/2, c)$ for all $c \in [1/2, 1]$, since for such c, g is non-increasing as x increases, and at $g(1, c)$ for all $c \in (0, 1/2]$, since for these values, g is non-decreasing as x increases. Hence,

$$\frac{\mathtt{OPT}}{\mathtt{OPT}_{\text{EFS}}} \leq \max\left\{\frac{1}{c}, \frac{\frac{3}{2}}{1 - (1 - c) \cdot \frac{1}{2}}\right\} = \max\left\{\frac{1}{c}, \frac{3}{1 + c}\right\}.$$

Case 2: $v_A(S_B) > c \cdot v_B(S_A)$. It may be required to sell both S_A and S_B to guarantee the existence of an envy-free solution, if both are single goods. Take the allocation in which everything is sold and the resulting cash is split evenly. We have already established that $v_B(S_A) > 1/2$. As a result,

$$\mathtt{OPT} = 2 - (v_A(S_B) + v_B(S_A)) < 2 - c/2 - 1/2,$$

and

$$\texttt{OPT}_{\text{EFS}} \geq c \cdot (v_A(S_B) + v_B(S_A)) > c \cdot (1+c) \cdot v_B(S_A) > c \cdot (1+c) \cdot \frac{1}{2}.$$

Thus, $\frac{\texttt{OPT}}{\texttt{OPT}_{\text{EFS}}} < \frac{3-c}{c+c^2}$.

Wrapping up. Since $\frac{3-c}{c+c^2} \geq \frac{1}{c}$ for all $c \leq 1$, we proved $\frac{\texttt{OPT}}{\texttt{OPT}_{\text{EFS}}} \leq \max\{\frac{1}{c}, \frac{3-c}{c+c^2}, \frac{3}{1+c}\} = \max\{\frac{3-c}{c+c^2}, \frac{3}{1+c}\}$. □

Next, we assume that an envy-free allocation without selling exists. Recall that in this case, even without selling the price of envy-freeness is 3/2 (so Theorem 1 does not give a better bound). However, we are able to show that, with selling, the price of envy-freeness is significantly lower. In particular, for $c = 1$, the price of envy-freeness is only 6/5.

Theorem 2. *Suppose an envy-free solution without selling exists for a particular instance. Then for* $c \in (0, 1]$, $\frac{\texttt{OPT}}{\texttt{OPT}_{\text{EFS}}} \leq \max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\}$.

Once again, before proving the theorem we give an example showing that the bound is tight. For $c \in [1/2, 1]$ it holds that $\frac{6}{4+c} \geq \frac{3-2c}{2-c}$. Let there be four goods, with values given by the following table:

	1	2	3	4
v_A	$\frac{1-\epsilon}{2}$	$\frac{1-\epsilon}{2}$	ϵ	0
v_B	$1/4+\epsilon$	$1/4+\epsilon$	$1/4-\epsilon$	$1/4-\epsilon$

$\epsilon > 0$ is arbitrarily small. Note that an envy-free allocation without selling exists (Alice gets 1 and 3 and Bob gets 2 and 4). Moreover, we have that $\texttt{OPT} \approx 3/2$. The optimal envy-free allocation with selling would sell either 1 or 2, resulting in $\texttt{OPT}_{\text{EFS}} \approx 1 + c/4$. Thus, $\texttt{OPT}/\texttt{OPT}_{\text{EFS}} \approx 6/(4+c)$.

For $c \in (0, 1/2]$, the values are given by the following table:

	1	2	3	4
v_A	$\frac{1-\epsilon}{2}$	$\frac{1-\epsilon}{2}$	ϵ	0
v_B	$\frac{1/2}{2-c} - \epsilon$	$\frac{1/2}{2-c} - \epsilon$	$\frac{(1-c)/2}{2-c} + \epsilon$	$\frac{(1-c)/2}{2-c} + \epsilon$

The social welfare maximizing solution, where Alice receives 1 and 2 and Bob receives 3 and 4, yields a total value of $\texttt{OPT} \approx (3-2c)/(2-c)$. The envy-free with selling solution sells 1 or 2, and yields $\texttt{OPT}_{\text{EFS}} = 1$. The ratio is therefore roughly $(3-2c)/(2-c)$.

We are now ready to prove the theorem. The proof is quite long and intricate, so the proofs of several lemmas have been relegated to the appendix, which was submitted as supplementary material.

Proof of Theorem 2. Let $\mathcal{X} = \{X_A \cup C_A, X_B \cup C_B\}$ be an allocation where X_A and X_B are disjoint subsets of the goods and $C_A + C_B$ is the cash obtained from selling $[m] \setminus (X_A \cup X_B)$. We let

$$\underline{X_A} \triangleq \{j \in X_A : v_A(j) < v_B(j)\},$$

$$\underline{X_B} \triangleq \{j \in X_B : v_B(j) < v_A(j)\},$$

and define their complements, $\overline{X_A} \triangleq X_A \setminus \underline{X_A}$, and $\overline{X_B} \triangleq X_B \setminus \underline{X_B}$.

We have assumed that an envy-free allocation without selling exists, so let $\mathcal{Y} = \{Y_A, Y_B\}$ be a social welfare maximizing envy-free allocation without selling; that is, $\texttt{OPT}_{\text{EF}} = \texttt{SW}(\mathcal{Y}, v_A, v_B)$. Note that $\texttt{OPT}_{\text{EF}} \geq 1$, because $v_A(Y_A) \geq 1/2$ and $v_B(Y_B) \geq 1/2$ due to envy-freeness. Without loss of generality, assume that $v_A(\underline{Y_A}) \leq v_B(\underline{Y_B})$.

Note that when $v_A(\overline{Y_A}) \geq 1/2$, $\mathcal{Y}$ satisfies $\underline{Y_A} = \emptyset$ (so that $v_A(\underline{Y_A}) = v_B(\underline{Y_A}) = 0$), since giving those goods to Alice is not necessary for envy-freeness, and Alice values them strictly less than Bob.

Next, $\mathcal{S} = \{S_A, S_B\}$ will refer to a particular social welfare maximizing allocation: $S_A = \overline{Y_A} \cup \underline{Y_B}$, and $S_B = \overline{Y_B} \cup \underline{Y_A}$. Note that when $v_B(\underline{Y_B}) = 0$, $v_A(\underline{Y_A}) + v_B(\underline{Y_B}) = 0$, so $\texttt{SW}(\mathcal{S}) = \texttt{SW}(\mathcal{Y})$, in which case the price of fairness is 1. This is not an interesting case so assume henceforth that the price of envy-freeness is strictly greater than one, which is equivalent to assuming $v_B(\underline{Y_B}) > 0$.

Our first lemma, whose proof appears in Appendix A, is used throughout the theorem's proof.

Lemma 1. $v_A(S_A) \geq \frac{1}{2}$ *and* $v_B(S_B) < \frac{1}{2}$.

Consider the following two allocations. They both begin with the allocation $\mathcal{S}$, and involve Alice selling one of $\overline{Y_A}$ or $\underline{Y_B}$.

Allocation 1. Alice sells $\underline{Y_B}$ and gives Bob cash worth

$$C_B^1 \triangleq \max\left\{0, \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\underline{Y_B}) - v_B(\overline{Y_B}) - v_B(\underline{Y_A})\right\}.$$

Let $C_A^1 \triangleq c \cdot v_B(\underline{Y_B}) - C_B^1$ be Alice's remaining cash. Then define $\mathcal{Z}^1 \triangleq \{\overline{Y_A} \cup C_A^1, \overline{Y_B} \cup \underline{Y_A} \cup C_B^1\}$.

Note that Alice's value for all the remaining goods and the cash from the sale is $1 - v_A(\underline{Y_B}) + c \cdot v_B(\underline{Y_B}) \leq 1 - (1-c) \cdot v_B(\underline{Y_B})$, while Bob's value for everything is $1 - (1-c) \cdot v_B(\underline{Y_B})$.

Allocation 2. Alice sells $\overline{Y_A}$ and gives Bob cash worth

$$C_B^2 \triangleq \max\left\{0, \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\overline{Y_A}) - v_B(\overline{Y_B}) - v_B(\underline{Y_A})\right\}.$$

Let $C_A^2 \triangleq c \cdot v_B(\overline{Y_A}) - C_B^2$. Then define $\mathcal{Z}^2 \triangleq \{\underline{Y_B} \cup C_A^2, \overline{Y_B} \cup \underline{Y_A} \cup C_B^2\}$.

Observe that Alice's value for all the remaining goods and the cash from the sale is $1 - v_A(\overline{Y_A}) + c \cdot v_B(\overline{Y_A}) \leq 1 - (1-c) \cdot v_B(\overline{Y_A})$, while Bob's value for everything is $1 - (1-c) \cdot v_B(\overline{Y_A})$.

The next two lemmas show that, if the two allocations $\mathcal{Z}^1$ and $\mathcal{Z}^2$ have enough cash, then they are envy free and provide certain guarantees with respect to social welfare.

Lemma 2. *Assume that $C_B^1 \le c \cdot v_B(\underline{Y_B})$ (that is, there is enough cash to give to Bob under Allocation 1), and* $\texttt{OPT} > \frac{6}{4+c}$. *Then the allocation $\mathcal{Z}^1$ is envy free, and* $\texttt{SW}(\mathcal{Z}^1) = v_A(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) + c \cdot v_B(\underline{Y_B})$.

Proof. The statement about the value of $\texttt{SW}(\mathcal{Z}^1)$ is trivial. Turning to envy-freeness, clearly Bob has no envy in this transaction, since his value is

$$v_B(\overline{Y_B}) + v_B(\underline{Y_A}) + v_B(C_B^1) \ge \frac{1-(1-c)\cdot v_B(\underline{Y_B})}{2};$$

i.e., half his total (remaining) value. We need to show that Alice is not envious.

Case 1: $C_B^1 = 0$. Note that this case cannot happen if $c = 1$, since then $1/2 - v_B(\overline{Y_B}) - v_B(\underline{Y_A}) \le 0$, contradicting Lemma 1. Thus, suppose $c \in (0,1)$.

We show that if Alice receives $\overline{Y_A}$ and all the cash, then she would not envy Bob. Assume for the sake of contradiction that Alice would actually envy Bob: $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) > v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B})$.

Note that, by Lemma 1, $v_B(\overline{Y_A}) + v_B(\underline{Y_B}) = v_B(S_A) > 1/2$. Since we are assuming $\texttt{OPT} > \frac{6}{4+c}$, and

$$\begin{aligned}\frac{6}{4+c} < \texttt{OPT} &= 2 - (v_B(\overline{Y_A}) + v_B(\underline{Y_B}) + v_A(\underline{Y_A}) + v_A(\overline{Y_B})) \\ &< \frac{3}{2} - (v_A(\underline{Y_A}) + v_A(\overline{Y_B})),\end{aligned}$$

it follows that $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) < \frac{3c}{2(4+c)}$. By our earlier assumption that Alice would envy Bob,

$$\begin{aligned}\frac{3c}{2(4+c)} &> v_A(\underline{Y_A}) + v_A(\overline{Y_B}) \\ &> v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) \\ &\ge v_B(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) \\ &= v_B(\overline{Y_A}) + v_B(\underline{Y_B}) - (1-c)\cdot v_B(\underline{Y_B}) \\ &> \frac{1}{2} - (1-c)\cdot v_B(\underline{Y_B}).\end{aligned}$$

Thus, $v_B(\underline{Y_B}) > \frac{2-c}{(1-c)(4+c)}$. This value is strictly greater than $1/2$ for all $c \in (0,1)$, which is a contradiction to the envy-freeness of $\mathcal{Y}$, since then $v_A(\underline{Y_B}) \ge v_B(\underline{Y_B}) > 1/2$.

Hence, when $C_B^1 = 0$, $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) \le v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B})$, so Alice does not envy Bob.

Case 2: $C_B^1 > 0$. Alice's value is

$$\begin{aligned}&v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) - \left(\frac{1}{2} - \frac{(1-c)}{2}\cdot v_B(\underline{Y_B})\right) \\ &\quad + v_B(\overline{Y_B}) + v_B(\underline{Y_A}) \\ &= v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) - \left(\frac{1}{2} - \frac{(1-c)}{2}\cdot v_B(\underline{Y_B})\right) \\ &\quad + (1 - v_B(\underline{Y_B}) - v_B(\overline{Y_A})) \\ &= \frac{1}{2} - \frac{(1-c)}{2}\cdot v_B(\underline{Y_B}) + v_A(\overline{Y_A}) - v_B(\overline{Y_A}) \\ &\ge \frac{1}{2}\left(1 - (1-c)\cdot v_B(\underline{Y_B})\right),\end{aligned}$$

where the last line follows from the assumption that $v_A(\overline{Y_A}) \ge v_B(\overline{Y_A})$. The right hand side is at least half of Alice's total (remaining) value. □

The proofs of the following two lemmas, 3 and 4, are relegated to appendices B and C, respectively.

Lemma 3. *Assume that $C_B^2 \le c \cdot v_B(\overline{Y_A})$ (that is, there is enough cash to give to Bob under Allocation 2), and* $\texttt{OPT} \ge \frac{6}{4+c}$. *Then the allocation $\mathcal{Z}^2$ is envy free, and* $\texttt{SW}(\mathcal{Z}^2) = v_A(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) + c \cdot v_B(\overline{Y_A})$.

At this point, it will be useful to define the maximum of Allocations 1 and 2, with respect to social welfare, as $\mathcal{Z}$. That is,

$$\mathcal{Z} \triangleq \begin{cases} \mathcal{Z}^1 & \text{if } v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) > v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}), \\ \mathcal{Z}^2 & \text{otherwise.}\end{cases}$$

We will use C_A^* to refer to the cash received by Alice in $\mathcal{Z}$, and C_B^* for the cash Bob receives in $\mathcal{Z}$.

Lemma 4. *When* $\texttt{OPT} \ge \frac{6}{4+c} \cdot \texttt{OPT}_{\text{EF}}$, *$\mathcal{Z}$ has sufficient cash: C_A^* and C_B^* are both non-negative.*

With all the lemmas in place, we can now complete the proof of Theorem 2. Observe that the lemmas imply $\mathcal{Z}$ is envy-free. Also note that $\texttt{OPT}_{\text{EF}} \ge 1$, so assuming $\texttt{OPT} > \max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\} \cdot \texttt{OPT}_{\text{EF}}$ allows Lemmas 2, 3, and 4 to apply.

Since the maximum of two numbers is at least their average, and

$$v_B(\underline{Y_A}) + v_B(\overline{Y_B}) = \texttt{OPT} - v_A(\overline{Y_A}) - v_A(\underline{Y_B}) \ge \texttt{OPT} - 1,$$

we have that

$$\begin{aligned}&\texttt{OPT}_{\text{EFS}} \ge \texttt{SW}(\mathcal{Z}) \\ &= \max\{v_A(\overline{Y_A}) + c\cdot v_B(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}), \\ &\qquad v_A(\underline{Y_B}) + c\cdot v_B(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B})\} \\ &\ge \frac{1}{2}\Big(v_A(\overline{Y_A}) + v_A(\underline{Y_B}) + c\cdot(v_B(\overline{Y_A}) + v_B(\underline{Y_B})) \\ &\qquad + 2(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))\Big) \\ &= \frac{1}{2}\Big(\texttt{OPT} - v_B(\underline{Y_A}) - v_B(\overline{Y_B}) \\ &\qquad + c\cdot(1 - v_B(\underline{Y_A}) - v_B(\overline{Y_B})) \\ &\qquad + 2(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))\Big) \\ &= \frac{1}{2}\Big(\texttt{OPT} + c + (1-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))\Big) \\ &\ge \frac{1}{2}\Big((2-c)\cdot\texttt{OPT} + 2c - 1\Big).\end{aligned}$$

This implies

$$\frac{\texttt{OPT}}{\texttt{OPT}_{\text{EFS}}} \le \frac{2\cdot\texttt{OPT}}{(2-c)\cdot\texttt{OPT} + 2c - 1} \triangleq f(\texttt{OPT}, c).$$

For every c, we want to re-write this bound to be only in terms of c and not $\texttt{OPT}$. Note that it suffices to consider $\texttt{OPT} \in (\max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\}, 3/2)$. This follows from the reasoning from the beginning of Section 2: if at least one player is envious, then $\texttt{OPT} < 1 + 1/2 = 3/2$. Since we want the bound to hold across the entire possible range of $\texttt{OPT}$, we need to take the maximum of $f(\texttt{OPT}, c)$ over the range. Thus, we take the derivative, with respect to $\texttt{OPT}$, of $f(\texttt{OPT}, c)$.

$$\frac{\partial f(\texttt{OPT}, c)}{\partial\, \texttt{OPT}} = \frac{4c-2}{((2-c)\cdot \texttt{OPT} + 2c - 1)^2}.$$

For $c \in [1/2, 1]$, $\frac{\partial f}{\partial \texttt{OPT}} \geq 0$, so

$$\frac{\texttt{OPT}}{\texttt{OPT}_{\text{EFS}}} \leq f\left(\frac{3}{2}, c\right) = \frac{6}{4+c}.$$

For $c \in (0, 1/2]$, $\frac{\partial f}{\partial \texttt{OPT}} \leq 0$ and $\max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\} = \frac{3-2c}{2-c}$, so

$$\frac{\texttt{OPT}}{\texttt{OPT}_{\text{EFS}}} \leq f\left(\frac{3-2c}{2-c}, c\right) = \frac{3-2c}{2-c}. \qquad \square$$

3 An Algorithmic Retrospective

Our theorems in Section 2 are *existence* results: they state that there always exists an envy-free allocation of sellable goods that yields a certain fraction of the optimal social welfare. The proof of Theorem 1 constructs an allocation that achieves the stated bound by selling portions of the social welfare maximizing allocation $\mathcal{S}$. The allocation $\mathcal{S}$ is easy to compute (just give each good to the player that values it more). In contrast, from a computational viewpoint, the guarantees of Theorem 2 may be hard to achieve, as the proof requires Alice to sell bundles from a welfare-maximizing *envy-free* (without selling) allocation $\mathcal{Y}$, which is far trickier to compute (Bouveret and Lang 2008).

The optimal envy-free allocation with selling is at least as good as the allocations constructed in the two proofs. And, in theory, the option to sell goods may actually make its computation easy. Our next result shows that this is not the case.[1] To be more formal, let us define the Max-EFS(c) problem as follows: the input is the set of goods $[m]$ that can be sold for a c-fraction of the minimum value, the valuation functions v_A and v_B, and $k \in \mathbb{R}^+$; the question is whether there is an envy-free allocation with social welfare at least k.

Theorem 3. *For any $c \in (0, 1]$, the* Max-EFS(c) *problem is* **NP***-complete.*

The theorem's proof is relegated to Appendix D, which was submitted as supplementary material. Intuitively, though, what makes the problem hard is that, starting from a social welfare maximizing allocation (which is not envy free), an optimal solution would have to sell a set of goods that is sufficient to satisfy the envious player, while losing as little value as possible. For $c = 1$, this can be formulated as a Min-Knapsack problem, which admits a fully polynomial time approximation scheme (FPTAS) (Kellerer, Pferschy, and Pisinger 2004). Leveraging this insight, we establish following result, whose proof appears in Appendix E.

Theorem 4. Max-EFS(1) *admits an FPTAS; i.e., there is an algorithm that, for any $\epsilon > 0$, returns an envy-free allocation $\mathcal{X}$ (which possibly includes cash) such that $\texttt{SW}(\mathcal{X}, v_A, v_B) \geq (1-\epsilon)\,\texttt{OPT}_{\text{EFS}}(v_A, v_B)$, and runs in polynomial time in the parameters of the problem and $1/\epsilon$.*

Moreover, it is easy to see that for any value of c, Max-EFS(c) can be formulated as an integer linear program (ILP), which can be solved using a variety of practical algorithms. To conclude, we do not view Theorem 3 as a serious obstacle to solving fair division problems in our framework, and, in particular, to achieving the guarantees given by Theorems 1 and 2.

4 Discussion

All of our results focus on the case of two players. This is because, when there are three or more players, the price of envy-freeness with selling is unbounded. To see why, let there be two goods and three players A, B, C; set $v_A(1) = v_B(1) = 1 - \epsilon$, $v_C(1) = 0$, $v_A(2) = v_B(2) = \epsilon$, and $v_C(2) = 1$. Both goods needs to be sold to prevent envy, but this only generates ϵ cash. In contrast, the case of two players (which is of special significance) gives rise to a rich collection of insights.

Another assumption — this one implicit — worth discussing is the conversion between value and cash. We have assumed that Alice and Bob's valuations for the complete bundle of goods are normalized to 1. This assumption is also made in many other fair division papers that reason about utilitarian social welfare (see, e.g., (Caragiannis et al. 2009; Cohler et al. 2011; Brams et al. 2012)). In practice, this could mean assigning Alice and Bob the same number of points to distribute between goods. But the conversion to cash means that the normalized valuation of a good can be compared to its market value. This is clearly possible, for example, if Alice and Bob have the same *actual* value for the complete bundle of goods. In any case, the fact that c can be any number in $(0, 1]$ gives us the flexibility to handle a range of conversion schemes, possibly at the cost of slightly weaker guarantees.

Finally, note that the Adjusted Winner protocol (Brams and Taylor 1996), discussed in Section 1, actually guarantees another fairness property called equitability: the players have equal values for their own bundles of goods. In principle, one can ask the same questions we have answered above about the price of equitability instead of envy-freeness — but that would be a bit repetitive!

[1] A related computational problem is whether $\texttt{OPT}_{\text{EFS}} = \texttt{OPT}$. This question does turn out to be tractable for $c = 1$; the proof is implicit in the arguments of Appendix E. In contrast, the question of whether $\texttt{OPT}_{\text{EF}} = \texttt{OPT}$ is **NP**-complete.

References

Bei, X.; Chen, N.; Hua, X.; Tao, B.; and Yang, E. 2012. Optimal proportional cake cutting with connected pieces. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI)*, 1263–1269.

Bertsimas, D.; Farias, V. F.; and Trichakis, N. 2011. The price of fairness. *Operations Research* 59(1):17–31.

Bouveret, S., and Lang, J. 2008. Efficiency and envy-freeness in fair division of indivisible goods: logical representation and complexity. *Journal of Artificial Intelligence Research* 32:525–564.

Bouveret, S., and Lang, J. 2011. A general elicitation-free protocol for allocating indivisible goods. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, 73–78.

Brams, S. J., and Taylor, A. D. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press.

Brams, S. J.; Feldman, M.; Morgenstern, J.; Lai, J. K.; and Procaccia, A. D. 2012. On maxsum fair cake divisions. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI)*, 1285–1291.

Brânzei, S.; Procaccia, A. D.; and Zhang, J. 2013. Externalities in cake cutting. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI)*, 55–61.

Caragiannis, I.; Kaklamanis, C.; Kanellopoulos, P.; and Kyropoulou, M. 2009. The efficiency of fair division. In *Proceedings of the 5th International Workshop on Internet and Network Economics (WINE)*, 475–482.

Caragiannis, I.; Kaklamanis, C.; Kanellopoulos, P.; and Kyropoulou, M. 2012. The efficiency of fair division. *Theory of Computing Systems* 50(4):589–610.

Caragiannis, I.; Lai, J. K.; and Procaccia, A. D. 2011. Towards more expressive cake cutting. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, 127–132.

Chen, Y.; Lai, J. K.; Parkes, D. C.; and Procaccia, A. D. 2013. Truth, justice, and cake cutting. *Games and Economic Behavior* 77:284–297. Preliminary version in AAAI'10.

Chevaleyre, Y.; Dunne, P. E.; Endriss, U.; Lang, J.; Lemaître, M.; Maudet, N.; Padget, J.; Phelps, S.; Rodríguez-Aguilar, J. A.; and Sousa, P. 2006. Issues in multiagent resource allocation. *Informatica* 30:3–31.

Chevaleyre, Y.; Endriss, U.; Estivie, S.; and Maudet, N. 2007. Reaching envy-free states in distributed negotiation settings. In *Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI)*, 1239–1244.

Chevaleyre, Y.; Endriss, U.; and Maudet, N. 2007. Allocating goods on a graph to eliminate envy. In *Proceedings of the 22nd AAAI Conference on Artificial Intelligence (AAAI)*, 700–705.

Chevaleyre, Y.; Endriss, U.; and Maudet, N. 2010. Simple negotiation schemes for agents with simple preferences: Sufficiency, necessity and maximality. *Autonomous Agents and Multi-Agent Systems* 20(2):234–259.

Cohler, Y. J.; Lai, J. K.; Parkes, D. C.; and Procaccia, A. D. 2011. Optimal envy-free cake cutting. In *Proceedings of the 25th AAAI Conference on Artificial Intelligence (AAAI)*, 626–631.

Kalinowski, T.; Narodytska, N.; Walsh, T.; and Xia, L. 2013. Strategic behavior when allocating indivisible goods sequentially. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence (AAAI)*, 452–458.

Kellerer, H.; Pferschy, U.; and Pisinger, D. 2004. *Knapsack Problems*. Springer.

Kurokawa, D.; Lai, J. K.; and Procaccia, A. D. 2013. How to cut a cake before the party ends. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence (AAAI)*, 555–561.

Kushilevitz, E., and Nisan, N. 1996. *Communication Complexity*. Cambridge University Press.

Moulin, H. 2003. *Fair Division and Collective Welfare*. MIT Press.

Procaccia, A. D. 2009. Thou shalt covet thy neighbor's cake. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI)*, 239–244.

Procaccia, A. D. 2013. Cake cutting: Not just child's play. *Communications of the ACM* 56(7):78–87.

Steinhaus, H. 1948. The problem of fair division. *Econometrica* 16:101–104.

A Proof of Lemma 1

We first argue that

$$v_A(\underline{Y_B}) > v_B(\underline{Y_B}) > v_B(\underline{Y_A}) \geq v_A(\underline{Y_A}). \tag{1}$$

The only inequality that does not following directly from the definitions of $\underline{Y_A}$ and $\underline{Y_B}$ is the middle inequality. Note that the last inequality is non-strict in case $\underline{Y_A} = \emptyset$. Assume for the sake of contradiction that $v_B(\underline{Y_A}) \geq v_B(\underline{Y_B})$. Recall that we assumed $v_B(\underline{Y_B}) \geq v_A(\underline{Y_A})$, and using the fact that $v_A(\underline{Y_B}) \geq v_B(\underline{Y_B})$, we get that $v_A(\underline{Y_B}) \geq v_A(\underline{Y_A})$. Therefore, the social welfare maximizing allocation satisfies

$$\begin{aligned} v_A(S_A) &= v_A(\overline{Y_A}) + v_A(\underline{Y_B}) \geq v_A(\overline{Y_A}) + v_A(\underline{Y_A}) \\ &= v_A(Y_A) \geq \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} v_B(S_B) &= v_B(\overline{Y_B}) + v_B(\underline{Y_A}) \geq v_B(\overline{Y_B}) + v_B(\underline{Y_B}) \\ &= v_B(Y_B) \geq \frac{1}{2}. \end{aligned}$$

Thus, the social welfare maximizing allocation is also envy-free, which violates the assumption that the price of fairness is strictly greater than one. This establishes Equation (1).

It follows from Equation (1) that

$$v_A(S_A) = v_A(\overline{Y_A}) + v_A(\underline{Y_B}) > v_A(\overline{Y_A}) + v_A(\underline{Y_A}) = v_A(Y_A) \geq \frac{1}{2}.$$

Since we are assuming the price of fairness is strictly greater than one, this implies $v_B(S_B) < 1/2$. □

B Proof of Lemma 3

The statement about the value of $\mathtt{SW}(\mathcal{Z}^2)$ is trivial. For envy-freeness, it is clear Bob is not envious of Alice's share, since he will have at least half his total (remaining) value. We need to show that Alice is not envious.

Case 1: $C_B^2 = 0$. As in Lemma 2, this case cannot occur if $c = 1$, so assume $c \in (0, 1)$. We show that Alice will not envy Bob when she receives all the cash. Assume for the sake of contradiction that this does not hold: $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) > v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A})$. Repeating the steps of Lemma 2, $\mathtt{OPT} \geq \frac{6}{4+c}$ implies $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) < \frac{3c}{2(4+c)}$. Furthermore,

$$\begin{aligned}\frac{3c}{2(4+c)} &> v_A(\underline{Y_A}) + v_A(\overline{Y_B}) > v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) \\ &\geq v_B(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) \\ &= v_B(\underline{Y_B}) + v_B(\overline{Y_A}) - (1-c) \cdot v_B(\overline{Y_A}) \\ &\geq \frac{1}{2} - (1-c) \cdot v_B(\overline{Y_A}).\end{aligned}$$

Thus, $v_B(\overline{Y_A}) > \frac{2-c}{(1-c)(4+c)}$. Again, this is strictly greater than 1/2 for all $c \in (0, 1)$. However, this contradicts the assumption that Bob was envy-free in $\mathcal{Y}$, since then $v_B(\underline{Y_B}) + v_B(\overline{Y_B}) < 1/2$. As a result, when $C_B^2 = 0$, Alice does not envy Bob.

Case 2: $C_B^2 > 0$. Alice's value is

$$\begin{aligned}&v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) - \left(\frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\overline{Y_A})\right) \\ &\quad + v_B(\overline{Y_B}) + v_B(\underline{Y_A}) \\ &= v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) - \left(\frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\overline{Y_A})\right) \\ &\quad + (1 - v_B(\overline{Y_A}) - v_B(\underline{Y_B})) \\ &= \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\overline{Y_A}) + v_A(\underline{Y_B}) - v_B(\underline{Y_B}) \\ &\geq \frac{1}{2}\left(1 - (1-c) \cdot v_B(\overline{Y_A})\right).\end{aligned}$$

This is half of Alice's total (remaining) value. □

C Proof of Lemma 4

Note that C_B^* is non-negative by definition of C_B^1 and C_B^2. We show that if $C_A^1 < 0$, then $\mathcal{Z} = \mathcal{Z}^2$, and if $C_A^2 < 0$, then $\mathcal{Z} = \mathcal{Z}^1$.

We examine two cases for contradiction: either

Case 1: Allocation 1 yields at least the social welfare of Allocation 2 but has insufficient cash, or

Case 2: Allocation 2 yields at least the social welfare of Allocation 1 but has insufficient cash.

Case 1: We have that there is insufficient cash; hence,

$$c \cdot v_B(\underline{Y_B}) < \frac{1}{2} - \frac{1-c}{2} \cdot v_B(\underline{Y_B}) - v_B(\underline{Y_A}) - v_B(\overline{Y_B}).$$

Note that when $c = 1$, we can rearrange to get $v_B(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) < 1/2$, which contradicts the envy-freeness of $\mathcal{Y}$, since then Bob would be envious. Thus, assume $c < 1$.

Substituting $v_B(\underline{Y_B}) = 1 - (v_B(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}))$ in the above inequality, and then rearranging, we obtain the inequality

$$v_B(\overline{Y_A}) > \frac{c + (1-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c}. \qquad (2)$$

Additionally, we assumed $\mathtt{SW}(\mathcal{Z}^1) \geq \mathtt{SW}(\mathcal{Z}^2)$, so $v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) \geq v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B})$, or $v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) \geq v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A})$. Again substituting for $v_B(\underline{Y_B})$ and rearranging, we obtain an upper bound for $v_B(\overline{Y_A})$, which we then combine with the lower bound from (2):

$$\begin{aligned}&\frac{c + v_A(\overline{Y_A}) - v_A(\underline{Y_B}) - c(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{2c} \\ &\geq v_B(\overline{Y_A}) > \frac{c + (1-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c}.\end{aligned}$$

Solving for $v_A(\underline{Y_B})$ and simplifying,

$$\begin{aligned}v_A(\underline{Y_B}) &< \frac{c(1-c)}{1+c} + v_A(\overline{Y_A}) \\ &\quad - \frac{c(3-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c}. \qquad (3)\end{aligned}$$

Then we use the inequality $\mathtt{OPT} > \frac{6}{4+c} \cdot \mathtt{OPT}_{\text{EF}}$, along with (3) and the envy-free condition $v_B(\underline{Y_B}) + v_B(\overline{Y_B}) \geq 1/2$:

$$\begin{aligned}&\frac{c(1-c)}{1+c} + 2v_A(\overline{Y_A}) \\ &\quad - \frac{c(3-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c} + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) \\ &> v_A(\overline{Y_A}) + v_A(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) \\ &> \frac{6}{4+c} \cdot (v_A(\overline{Y_A}) + v_A(\underline{Y_A}) + v_B(\underline{Y_B}) + v_B(\overline{Y_B})) \\ &\geq \frac{6}{4+c} \cdot v_A(\overline{Y_A}) + \frac{3}{4+c}.\end{aligned}$$

We solve for $v_A(\overline{Y_A})$ and simplify:

$$\begin{aligned}
&\frac{2(1+c)}{4+c} \cdot v_A(\overline{Y_A}) \\
&> \frac{3}{4+c} - \frac{c(1-c)}{1+c} - \frac{(1-c)^2}{1+c}(v_B(\underline{Y_A}) + v_B(\overline{Y_B})) \\
&> \frac{3}{4+c} - \frac{c(1-c)}{1+c} - \frac{(1-c)^2}{2(1+c)} \\
&= \frac{(1+c)(2+c)}{2(4+c)}.
\end{aligned}$$

Cross-multiplying, we obtain

$$v_A(\overline{Y_A}) > \frac{1}{2} + \frac{c}{4}. \tag{4}$$

Thus, we have $v_A(\overline{Y_A})$ is strictly greater than 1/2.

When $v_A(\overline{Y_A}) > 1/2$, $v_B(\underline{Y_A}) = v_A(\underline{Y_A}) = 0$, as mentioned previously. Then we obtain an upper bound for $v_B(\underline{Y_B})$ from substituting $v_A(\overline{Y_A}) + v_A(\underline{Y_B}) \leq 1$ and $v_B(\overline{Y_B}) = 1 - v_B(\overline{Y_A}) - v_B(\underline{Y_B})$:

$$\begin{aligned}
&1 + 1 - v_B(\overline{Y_A}) - v_B(\underline{Y_B}) \\
&\geq v_A(\overline{Y_A}) + v_A(\underline{Y_B}) + v_B(\overline{Y_B}) \\
&> \frac{6}{4+c}(v_A(\overline{Y_A}) + v_B(\underline{Y_B}) + v_B(\overline{Y_B})) \\
&= \frac{6}{4+c}(v_A(\overline{Y_A}) + 1 - v_B(\overline{Y_A})),
\end{aligned}$$

which yields

$$\begin{aligned}
&v_B(\underline{Y_B}) \\
&< \frac{1}{4+c}\left(2(1+c) + (2-c)v_B(\overline{Y_A}) - 6v_A(\overline{Y_A})\right).
\end{aligned} \tag{5}$$

Then using $v_B(\overline{Y_A}) + v_B(\underline{Y_B}) + v_B(\overline{Y_B}) = 1$, the insufficient cash condition can be rewritten as $v_B(\overline{Y_A}) + \frac{1-c}{2} \cdot v_B(\underline{Y_B}) > 1/2$, which yields a lower bound on $v_B(\underline{Y_B})$:

$$v_B(\underline{Y_B}) > \frac{1}{1-c}\left(1 - 2v_B(\overline{Y_A})\right). \tag{6}$$

Combining the lower bound (6) and upper bound (5) on $v_B(\underline{Y_B})$, we can simplify to obtain

$$\begin{aligned}
\frac{6}{4+c} \cdot v_A(\overline{Y_A}) &< \frac{2(1+c)(1-c) - 4 - c}{(1-c)(4+c)} \\
&\quad + \frac{2(4+c) + (1-c)(2-c)}{(1-c)(4+c)} \cdot v_B(\overline{Y_A}) \\
&= \frac{-2 - c - 2c^2}{(1-c)(4+c)} \\
&\quad + \frac{10 - c + c^2}{(1-c)(4+c)} \cdot v_B(\overline{Y_A}) \\
&< \frac{-2 - c - 2c^2}{(1-c)(4+c)} \\
&\quad + \frac{5 - \frac{1}{2}c + \frac{1}{2}c^2}{(1-c)(4+c)} \\
&= \frac{3 - \frac{3}{2}c - \frac{3}{2}c^2}{(1-c)(4+c)}.
\end{aligned}$$

Cross-multiplying and simplifying, we obtain

$$v_A(\overline{Y_A}) < \frac{2 - c - c^2}{4(1-c)} = \frac{(1-c)(2+c)}{4(1-c)} = \frac{1}{2} + \frac{c}{4}. \tag{7}$$

Thus, for all $c > 0$, (4) and (7) yield a contradiction.

Case 2: We have that there is insufficient cash; hence,

$$c \cdot v_B(\overline{Y_A}) < \frac{1}{2} - \frac{1-c}{2} \cdot v_B(\overline{Y_A}) - v_B(\underline{Y_A}) - v_B(\overline{Y_B}).$$

As before, we can assume that $c < 1$, since otherwise we would obtain $v_B(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) < 1/2$, which implies $v_B(\underline{Y_B}) > 1/2$. Since $v_A(\underline{Y_B}) \geq v_B(\underline{Y_B})$, this contradicts the envy-freeness of Alice.

Substituting $v_B(\overline{Y_A}) = 1 - (v_B(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}))$ in the above inequality then rearranging we obtain the inequality

$$v_B(\underline{Y_B}) > \frac{c + (1-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c}. \tag{8}$$

Additionally, we assumed $\mathtt{SW}(\mathcal{Z}^2) \geq \mathtt{SW}(\mathcal{Z}^1)$, so $v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) \geq v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B})$, or $v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) \geq v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B})$. Again substituting for $v_B(\overline{Y_A})$ and rearranging, we obtain an upper bound for $v_B(\underline{Y_B})$, which we then combine with the lower bound from (8):

$$\begin{aligned}
&\frac{c + v_A(\underline{Y_B}) - v_A(\overline{Y_A}) - c(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{2c} \\
&> v_B(\underline{Y_B}) > \frac{c + (1-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c}.
\end{aligned}$$

Solving for $v_A(\overline{Y_A})$,

$$\begin{aligned}
v_A(\overline{Y_A}) &< \frac{c(1-c)}{1+c} + v_A(\underline{Y_B}) \\
&\quad - \frac{c(3-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c}.
\end{aligned} \tag{9}$$

Then we use the inequality $\mathtt{OPT} > \frac{6}{4+c} \cdot \mathtt{OPT}_{\text{EF}} \geq \frac{6}{4+c}$, along with (9).

$$\begin{aligned}
&\frac{c(1-c)}{1+c} + 2v_A(\underline{Y_B}) \\
&\quad - \frac{c(3-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B}))}{1+c} + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) \\
&> v_A(\overline{Y_A}) + v_A(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) \\
&> \frac{6}{4+c}.
\end{aligned}$$

Solving for $v_A(\underline{Y_B})$ and simplifying,

$$\begin{aligned}
&2v_A(\underline{Y_B}) \\
&> \frac{6}{4+c} - \frac{c(1-c)}{1+c} - \frac{(1-c)^2}{1+c}(v_B(\underline{Y_A}) + v_B(\overline{Y_B})) \\
&> \frac{6}{4+c} - \frac{c(1-c)}{1+c} - \frac{(1-c)^2}{2(1+c)} \\
&= \frac{8 + 3c + c^2}{2(4+c)}.
\end{aligned}$$

Hence, we obtain

$$v_A(\underline{Y_B}) > \frac{8+3c+c^2}{4(4+c)}. \tag{10}$$

This quantity is strictly greater than 1/2 for all $c > 0$, which is a contradiction since then $v_A(\overline{Y_A}) + v_A(\underline{Y_A}) < 1/2$, contradicting envy-freeness. □

D Proof of Theorem 3

The problem is obviously in **NP**. To show **NP**-hardness, we reduce from the PARTITION problem. We show how to reduce an arbitrary PARTITION instance, given n numbers, to an instance of MAX-EFS(c). Specifically, we show that if the MAX-EFS(c) instance has a solution of value at least $1+\epsilon/2$, then there exists a partition of the given instance, and if there does not exist a solution to the MAX-EFS(c) instance with value at least $1+\epsilon/2$, then there is no partition.

Let the input to the PARTITION instance be $a_1, \ldots, a_n$, and assume without loss of generality (by normalizing) that $\sum_{i=1}^n a_i = 1$. Then, we define a MAX-EFS(c) instance with $n+1$ goods. For $i \in \{1, \ldots, n\}$, we let $v_A(i) = a_i$ and $v_B(i) = (1-\epsilon)a_i$ for a small $\epsilon > 0$. For the last good, $v_A(n+1) = 0$ and $v_B(n+1) = \epsilon$.

If there is a perfect partition of $a_1, \ldots, a_n$, we can allocate one side of the partition to Alice, the other side to Bob and the $n+1$-st good to Bob. This gives us an allocation where Alice gets value 1/2 and Bob gets value $(1+\epsilon)/2$. Since both payers receive at least 1/2, this allocation is envy free and the social welfare of this allocation is $1+\epsilon/2$.

It remains to be shown that if no perfect partition exists then there is no envy-free allocation with social welfare at least $1+\epsilon/2$. Note that without selling any goods there is no envy-free allocation. This is true because if Alice cannot receive value exactly 1/2, and if Alice receives value $1/2 + x$ for some $x > 0$, then Bob gets $(1-\epsilon)(1/2-x)+\epsilon$. For a sufficiently small ϵ, Bob's value will be less than 1/2, so he will be envious of Alice. Therefore any envy-free allocation in this set of instances will have at least one good sold and split between the two players.

Consider an arbitrary envy-free allocation of the $n+1$ goods. We can split the first n goods into three sets X_A, X_B, and X_S, which are the goods allocated to Alice, those allocated to Bob, and the goods that are sold and split between the players, respectively. Due to envy-freeness, $v_B(X_B \cup X_S) \geq 1/2$, which means that $v_A(X_B \cup X_S) > 1/2$; otherwise, we could just allocate $X_B \cup X_S$ to Bob and X_A to Alice and obtain an envy-free allocation without selling. Then, we can upper-bound the social welfare of this allocation as follows:

$$\begin{aligned}
&v_A(X_A) + v_B(X_B) + c \cdot v_B(X_S) + \epsilon \\
&\quad \leq v_A(X_A) + v_B(X_B) + v_B(X_S) + \epsilon \\
&\quad = v_A(X_A) + (1-\epsilon)v_A(X_B \cup X_S) + \epsilon \\
&\quad = v_A(X_A \cup X_B \cup X_S) + \epsilon(1 - v_A(X_B \cup X_S)) \\
&\quad = 1 + \epsilon(1 - v_A(X_B \cup X_S)) \\
&\quad < 1 + \epsilon\left(1 - \frac{1}{2}\right) = 1 + \frac{\epsilon}{2}.
\end{aligned}$$

□

E Proof of Theorem 4

In this section, we use a social welfare maximizing allocation $\mathcal{S} = \{S_A, S_B\}$. If $\mathcal{S}$ is envy free, there is nothing to show, so assume without loss of generality that $v_B(S_B) < 1/2$. It will be useful to define $S^= \triangleq \{i \in S_A : v_A(i) = v_B(i)\}$

We also define a social welfare maximizing envy-free allocation with selling: $\mathcal{E} = \{E_A, E_B\}$ where $E_A = G_A \cup C''_A$ and $E_B = G_B \cup C''_B$. Let $P \in \{A, B\}$. G_P refers to the full goods in E_P and C''_P is cash in E_P. Let $\underline{C''_A}$ refer to the cash allocated to Alice from the sale of goods that Bob values strictly more than Alice, and $\overline{C''_A}$ will be Alice's remaining cash. We analogously define $\underline{C''_B}$ and $\overline{C''_B}$. We can then define $\underline{E_P} \triangleq \underline{G_P} \cup \underline{C''_P}$ and $\overline{E_P} \triangleq \overline{G_P} \cup \overline{C''_P}$. We assume that of all social welfare maximizing envy-free allocations with selling, $\mathcal{E}$ is one that minimizes $v_A(\underline{E_A})$.

Lemma 5. *$v_A(\underline{E_A}) = 0$.*

Proof. We show that if $v_A(\underline{E_A}) > 0$, then Alice and Bob can exchange goods without decreasing social welfare but decreasing $v_A(\underline{E_A})$. Note that if $\texttt{OPT} = \texttt{OPT}_{\text{EFS}}$, then there exists an envy-free allocation (with selling) such that the only goods sold are valued equally by both players, and no player receives a good that he or she values strictly less. Thus, when $\texttt{OPT} = \texttt{OPT}_{\text{EFS}}$, the existence of such an allocation implies $v_A(\underline{E_A}) = 0$. Hence, suppose $\texttt{OPT} > \texttt{OPT}_{\text{EFS}}$, and consider two cases:

Case 1: $v_B(S_B) \geq v_B(\overline{E_B})$. When $S^= = \emptyset$, Case 1 applies, because $\overline{G_B}$ is a subset of S_B and $\overline{C''_B}$ is cash generated from the sale of a subset of $S_B \setminus \overline{G_B}$.

First, note that $v_B(\underline{E_B}) > 0$. Otherwise $v_B(S_B) \geq v_B(\overline{E_B}) = v_B(E_B) \geq 1/2$, by envy-freeness of $\mathcal{E}$. Since we assumed $1/2 > v_B(S_B)$, this is a contradiction.

We suppose for the sake of contradiction that $v_A(\underline{E_A}) > 0$. Without loss of generality, we assume that $\underline{E_A}$ and $\underline{E_B}$ are all cash, since the players lose no value from selling any goods in $\underline{E_A}$ or $\underline{E_B}$.

Thus, if Alice gives Bob $\min\{v_A(\underline{E_A}), v_B(\underline{E_B})\}$ (a strictly positive quantity) of cash from $\underline{E_A}$, in exchange for the same amount from $\underline{E_B}$, we obtain an alternative social welfare maximizing envy-free allocation in which $v_A(\underline{E_A})$ has decreased, a contradiction.

Case 2: $v_B(S_B) < v_B(\overline{E_B})$. We show that in this case, there exists a different social welfare maximizing allocation that is either envy-free or satisfies the condition of Case 1.

As argued above, $S^= \neq \emptyset$, since otherwise we would be in Case 1. If $v_A(S_A \setminus S^=) \leq v_A(S_B \cup S^=)$, then the social welfare maximizing allocation can be converted into an envy-free allocation by selling $S^=$ and giving $1/2 - v_A(S_A \setminus S^=)$ to Alice (there is enough cash because $v_A(S_A) \geq 1/2$). Bob will also be envy free since his value for Alice's allocation will be at most 1/2. Then $\mathtt{OPT} = \mathtt{OPT}_{\text{EFS}}$, which implies $v_A(\underline{E_A}) = 0$.

Consequently, suppose $v_A(S_A \setminus S^=) > v_A(S_B \cup S^=)$. We transfer $S^=$ to Bob. When $v_B(S_A \setminus S^=) \leq v_B(S_B \cup S^=)$, $\{S_A \setminus S^=, S_B \cup S^=\}$ is an envy-free allocation with the same social welfare as $\mathcal{S}$, again implying $v_A(\underline{E_A}) = 0$. Else, $v_B(S_A \setminus S^=) > v_B(S_B \cup S^=)$. In this case, $\{S_A \setminus S^=, S_B \cup S^=\}$ is a different social welfare maximizing allocation without selling. Bob still envies Alice, but Alice is not allocated any goods that she and Bob value equally. As a result, we can proceed with the proof by applying Case 1 to this new allocation. □

We consider the following protocol. Start with the social welfare maximizing allocation $\mathcal{S} = \{S_A, S_B\}$. If this is an envy free allocation then we are done. Thus, without loss of generality, assume Bob is envious of Alice. If there is any good $i \in S_B$ such that $v_A(i) = v_B(i)$, then we update $\mathcal{S}$ by transferring all such i to S_A. We then solve a Min-Knapsack instance, where the goods are those allocated to Alice in the social welfare maximizing allocation. For a good $i \in S_A$, the value of i in the Min-Knapsack instance is $v_B(i)$. The cost of i is $v_A(i) - v_B(i)$. The capacity requirement of the knapsack is that it contains at least $1/2 - v_B(S_B)$. We sell the goods returned in the solution to the Min-Knapsack instance, then give $1/2 - v_B(S_B)$ in cash to Bob and the rest to Alice.

Lemma 6. *The protocol above, when the knapsack problem is solved optimally, provides the social welfare maximizing envy free allocation.*

Proof. By Lemma 5, we know that there is a social welfare maximizing envy free allocation in which Alice is not allocated any good that Bob values more than Alice. Indeed, since $v_B(\underline{E_A}) > 0 = v_A(\underline{E_A})$, we would be able to strictly improve social welfare without harming envy freeness by giving any goods in $\underline{E_A}$ to Bob.

Then the allocation $\mathcal{E}$ is within the set of allocations reachable by the protocol outlined above. The minimum cost knapsack problem described in the protocol searches this space to find the allocation with the greatest social welfare (by looking to minimize how much social welfare is lost be enforcing envy-freeness) subject to a constraint that ensures that Bob is no longer envious. In the allocation provided by the protocol, Bob values Alice's allocation at exactly 1/2. Alice's allocation is entirely cash and goods that Alice values at least as much as Bob, so she must value this allocation at at least 1/2. Therefore, Alice is not envious of Bob so the allocation is feasible. Hence, the protocol provides the social welfare maximizing envy free allocation. □

We now claim that the protocol above, when the knapsack problem is solved within a factor of $1 + \epsilon$ from the optimal solution for any $\epsilon > 0$, provides an $(1 - \epsilon)$-approximation of the social welfare maximizing envy free allocation.

Indeed, let $\mathtt{KNAP}$ be the optimal solution to the minimum knapsack problem defined in the protocol. Then by Lemma 5 and Lemma 6, $\mathtt{OPT}_{\text{EFS}} = \mathtt{OPT} - \mathtt{KNAP}$. Note that $\mathtt{KNAP} \leq 1 \leq \mathtt{OPT}_{\text{EFS}}$. Then, if we solve the minimum knapsack problem within a factor $(1 + \epsilon)$, we get an envy free solution with the following social welfare:

$$\begin{aligned} \mathtt{OPT} - (1+\epsilon)\mathtt{KNAP} &= \mathtt{OPT} - \mathtt{KNAP} - \epsilon\mathtt{KNAP} \\ &= \mathtt{OPT}_{\text{EFS}} - \epsilon\mathtt{KNAP} \\ &\geq \mathtt{OPT}_{\text{EFS}} - \epsilon\, \mathtt{OPT}_{\text{EFS}} \\ &= (1-\epsilon)\, \mathtt{OPT}_{\text{EFS}}\,. \end{aligned}$$ □