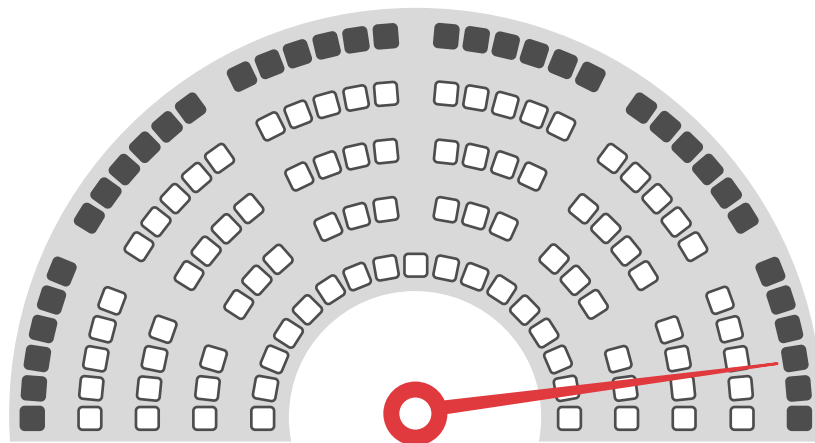




Optimized Democracy

Fall 2026 | Lecture 11

Quadratic Voting

Ariel Procaccia | Harvard University

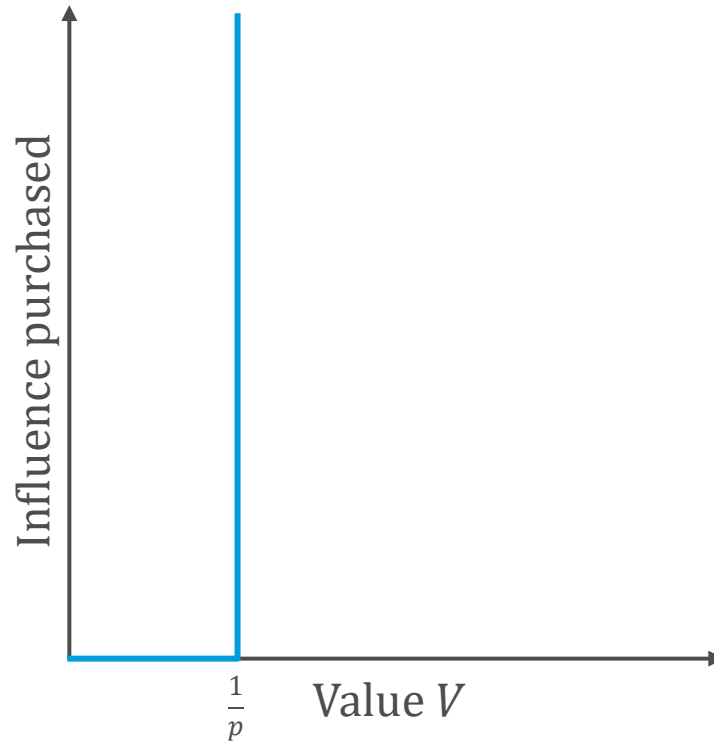
DISTORTION, REVISITED

- We saw that standard voting rules can't (even approximately) optimize utilitarian social welfare
- Today, we revise the setting by allowing voters to invest “money” to purchase votes, thereby expressing the intensity of their preferences
- The setting is related to mechanism design (e.g., Vickrey-Clarke-Groves Mechanism) but the proposed solution is more practical

ONE DOLLAR, ONE VOTE

- Given the funding cuts, Harvard will buy a ping pong table only if faculty members contribute \$400
- Let's say (although this technically doesn't make sense) that every dollar I donate increases the probability the table is funded by p , and my own value is V
- If $pV \geq 1$, I contribute (a lot), and if $pV < 1$, I don't contribute

ONE DOLLAR, ONE VOTE

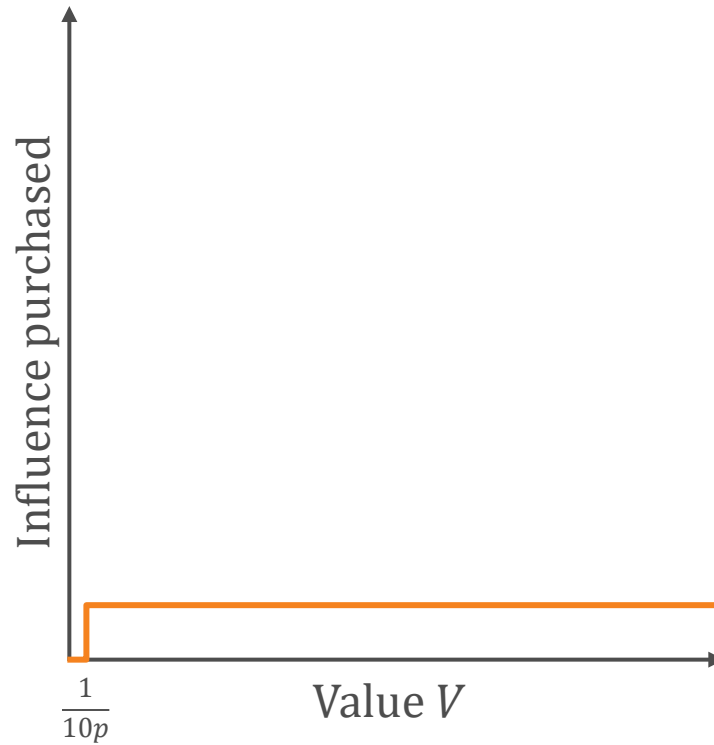


Under “one dollar, one vote,” I would want to purchase as much influence as possible when $pV \geq 1$, and the table is funded if someone really wants it

ONE PERSON, ONE VOTE

- Now imagine that Harvard will buy the ping pong table only if 400 faculty members vote in favor
- It costs me \$0.1 (in effort) to cast a vote
- I'll cast a vote if $pV \geq 0.1$, but I can't purchase more influence (or the cost of purchasing more is ∞)

ONE PERSON, ONE VOTE

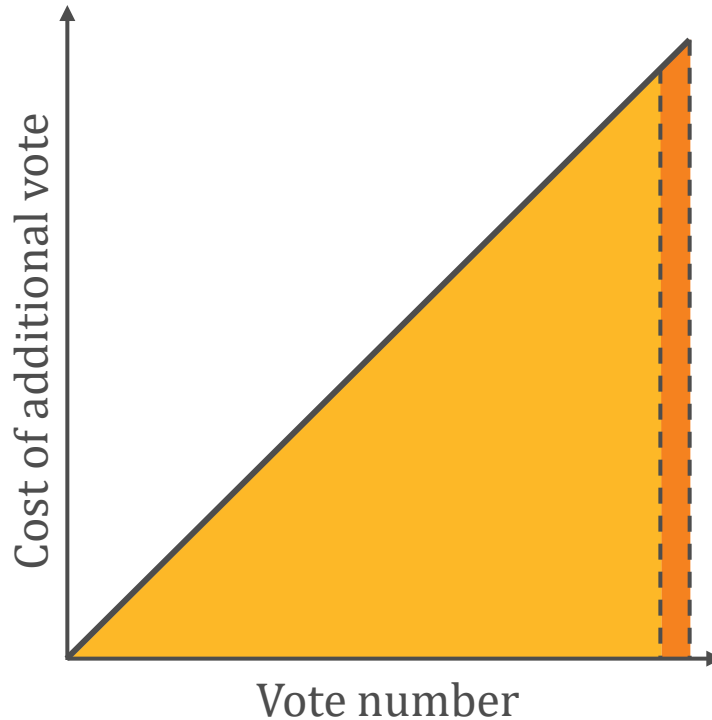


Under “one person, one vote,” I would want to purchase one unit of influence when $pV \geq 0.1$, and the table is funded if many people want it at least a little

QUADRATIC PAYMENTS

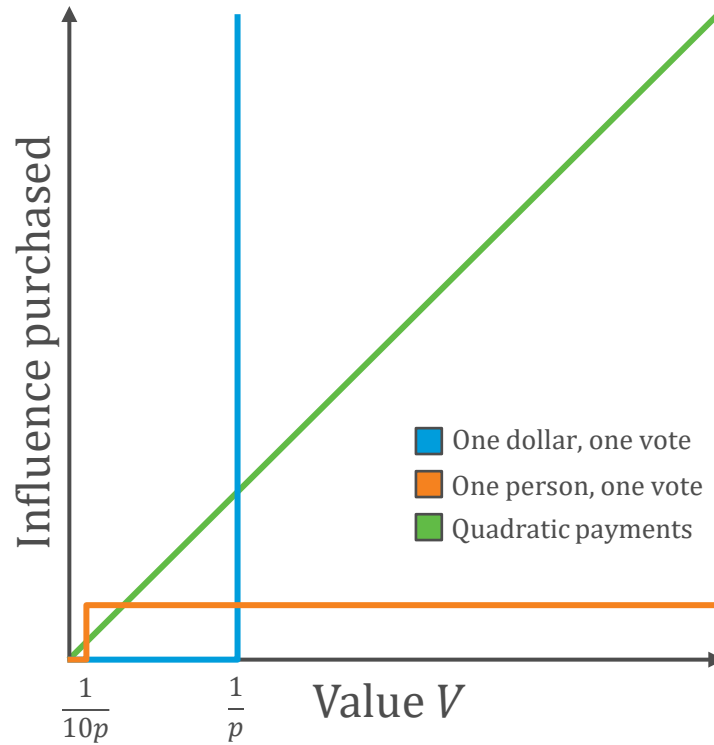
- If my value is V , I'm willing to pay up to $V/100$ for an additional 1% chance of funding the table
- If your value is $V/2$, you're willing to pay $V/200$ for the same unit of influence
- What's a scheme that would ensure that you would buy half as many votes as me?
- **Key idea:** My k th vote (or unit of influence) costs $\$k$

QUADRATIC PAYMENTS



Cost of k th vote (orange) is k , but overall cost (yellow and orange) is $k(k + 1)/2 = \Theta(k^2)$

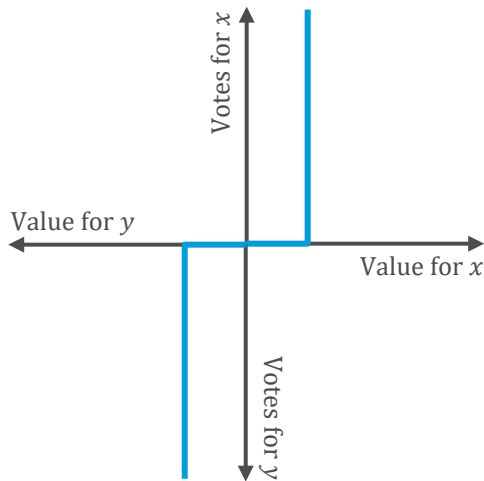
QUADRATIC PAYMENTS



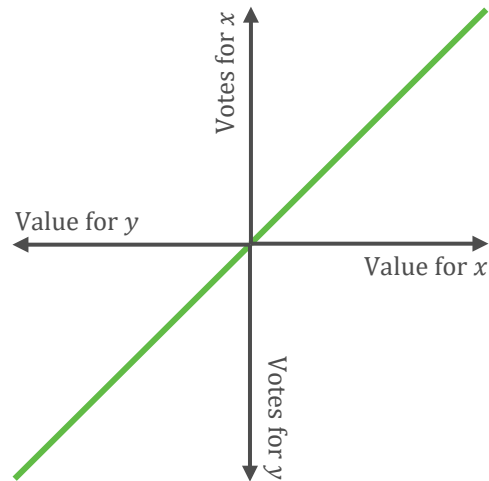
Under quadratic payments, the influence I purchase is proportional to my value

QUADRATIC VOTING: BINARY CASE

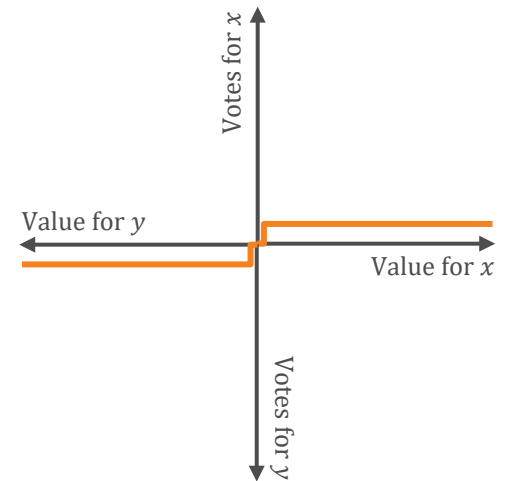
- Suppose there are two alternatives, x and y
- As before, my k th vote in favor of an alternative costs k
- Casting k votes costs k^2



One dollar, one vote



Quadratic voting



One person, one vote

QUADRATIC VOTING: GENERAL CASE

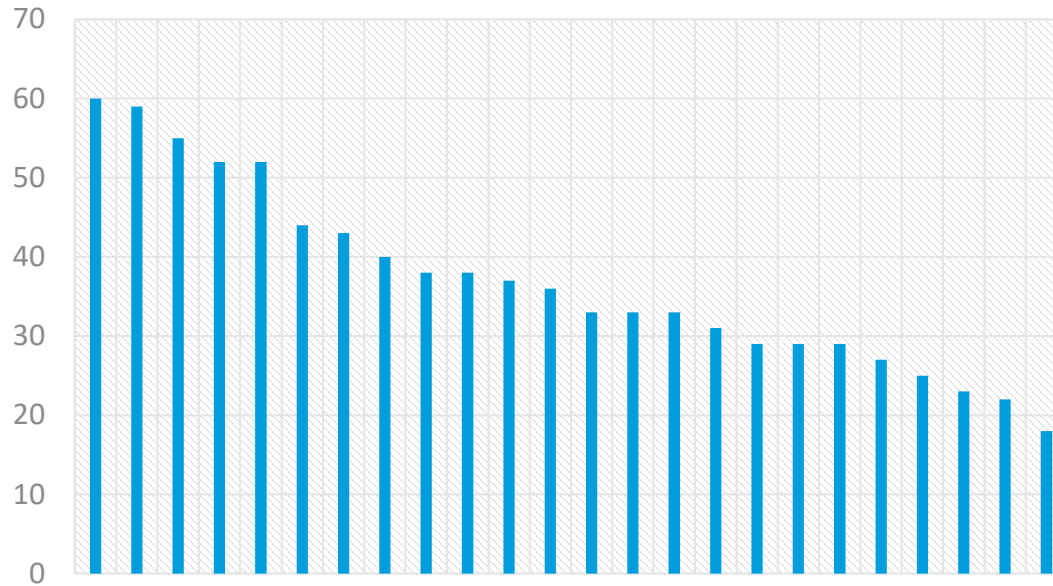
- The idea behind quadratic voting easily extends to the case of m alternatives
- If a voter casts k votes for an alternative, they pay k^2
- Notably, it is not a voter's total number of votes that's squared, but their votes for each alternative

DEMO OF QUADRATIC VOTING



<https://qv.geek.sg/vote?election=77707475-b7a0-4f92-862d-0e3ae87226c5>

QUADRATIC VOTING IN COLORADO



60 Equal Pay For Equal Work Act
 59 Demographic Notes For Certain Legislative Bills
 55 CDPHE Maternal Mortality Review Committee
 52 Youth Mental Health Ed & Suicide Prevention
 52 Grants For Property Tax Rent & Heat
 44 Mental Health Parity Insurance Medicaid
 43 Health Care Cost Savings Act Of 2019
 40 Increase Student Aid Application Completion Rates
 38 School Nurse Grant Program
 38 Comprehensive Human Sexuality Education
 37 GreenHouse Gas Pollution Impact In Fiscal Notes
 36 School Incentives To Use Colorado Food & Producers

33 Expand Child Nutrition School Lunch Protection Act
 33 Sexual Assault While In Custody Or Detained
 33 Office Of Public Guardianship Operation Conditions State
 31 Court Administrator Reminder Program Increase Tax
 29 Credit Allocation Affordable Housing Modify Innovative
 29 System Enhancement Regulate Student Education Loan Servicers
 29 Motor Vehicle Income Tax Credits Wildfire Mitigation
 27 Wildland-Urban Interface Areas Child Care Expenses Tax
 25 Credit Low-income Families Investment In Primary Care
 23 To Reduce Health Cost CO Child Abuse Response And
 22 Evaluation Network Child & Youth Behavioral Health
 18 Colorado Resiliency Office Reauthorization Funding

A FORMAL MODEL

- Set of voters $N = \{1, \dots, n\}$
- Choice between two alternatives a and b
- The utility of voter i for a is u_i and for b is $-u_i$
- Each voter chooses a number of votes v_i , either positive (for a) or negative (for b)
- a is implemented if and only if $\sum_i v_i \geq 0$
- Each voter pays $c(v_i)$, according to a **vote pricing rule** c
- Voters agree on the **marginal pivotality** of votes, denoted by p

A FORMAL MODEL

- Voters choose v_i to maximize $2u_i v_i p - c(v_i)$
- A vote pricing rule is **robustly optimal** if for every p , N and utilities $\mathbf{u}$, each voter chooses v_i^* such that $\text{sgn}(\sum_{i \in N} v_i^*) = \text{sgn}(\sum_{i \in N} u_i)$
- **Theorem:** A differentiable, strictly convex, even, and strictly increasing (in $|v|$) vote pricing rule is robustly optimal if and only if it is a quadratic function

PROOF OF THEOREM (IF)

- Assume that $c(x) = kx^2 + C$
- Voters maximize $2u_i v_i p - k(v_i)^2 - C$
- This function is maximized when its derivative is 0:

$$2u_i p - 2k v_i^* = 0 \Rightarrow v_i^* = \frac{p}{k} \cdot u_i$$

- Therefore,

$$\text{sgn} \left(\sum_{i \in N} v_i^* \right) = \text{sgn} \left(\sum_{i \in N} \frac{p}{k} \cdot u_i \right) = \text{sgn} \left(\sum_{i \in N} u_i \right)$$

PROOF OF THEOREM (IF)

- Assume that $c(x) = kx^2 + C$
- Voters maximize $2u_i v_i p - k(v_i)^2 - C$
- This function is maximized when its derivative is 0:

$$2u_i p - 2k v_i^* = 0 \Rightarrow v_i^* = \frac{p}{k} \cdot u_i$$

Poll

To optimize the social welfare function $\sum_{i \in N} \sqrt[3]{u_i}$, which vote pricing rule should we choose?

- $|v|$
- $|v|^{3/2}$
- $|v|^3$
- $|v|^4$



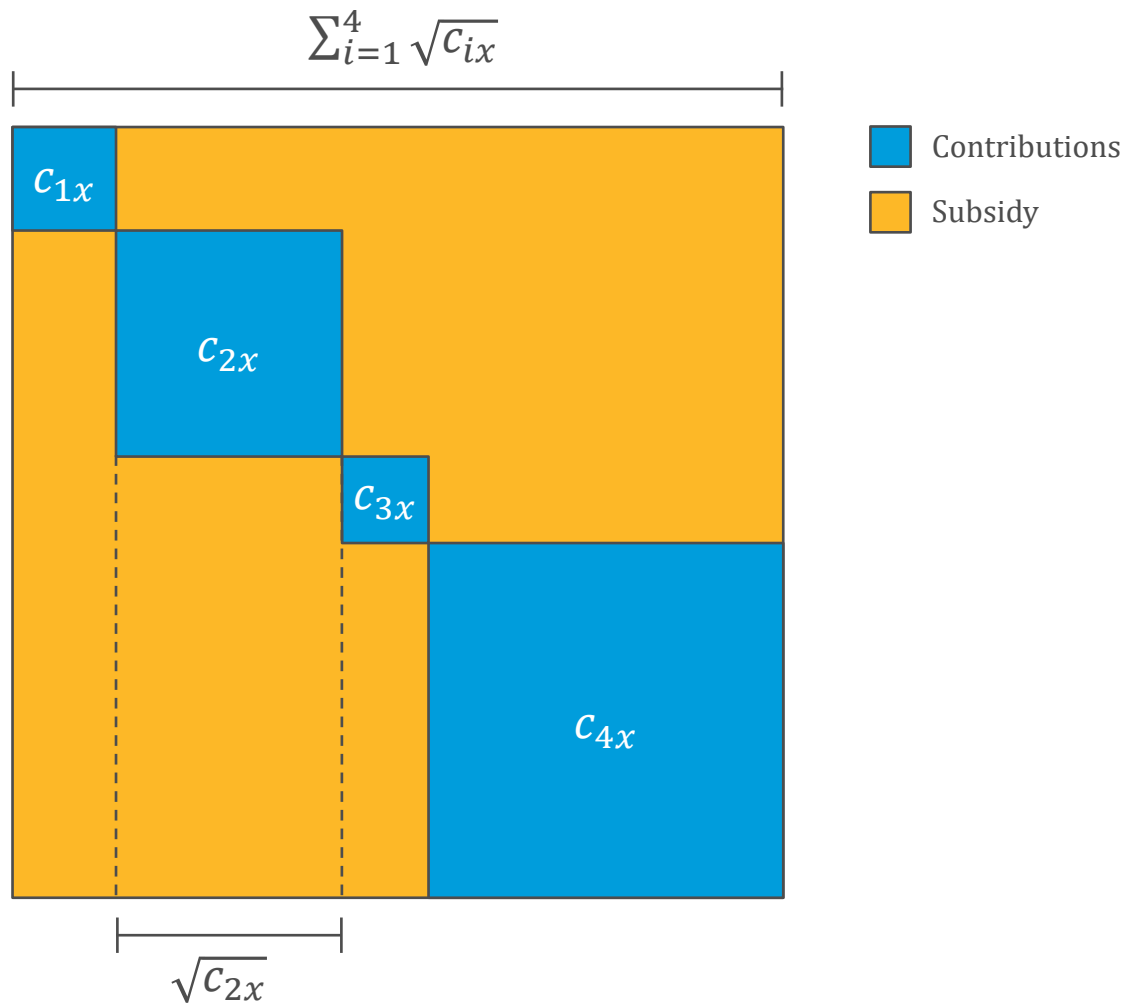
QUADRATIC FUNDING

- Anyone can propose public goods (alternatives)
- Let c_{ix} be the contribution of player $i \in N$ to alternative x , then the payment to x is

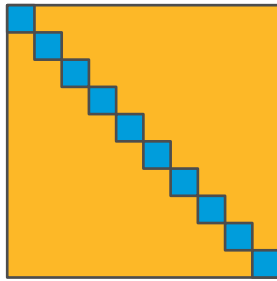
$$\left(\sum_{i=1}^n \sqrt{c_{ix}} \right)^2$$

- If $n > 1$, this payment is greater than $\sum_i c_{ix}$ and the difference comes from a subsidy pool

QUADRATIC FUNDING



QUADRATIC FUNDING



$$c_{ix} = 1 \text{ for } i = 1, \dots, 10$$

Subsidy = 90



$$c_{1x} = 10$$

Subsidy = 0

Many small contributions receive a much larger subsidy than few large contributions!

A MODEL FOR PUBLIC FUNDING

- As usual, the players are $N = \{1, \dots, n\}$ and the alternatives are A
- Let $u_{ix}(f_x)$ be the utility player i receives from alternative x being funded at level f_x
- Each player $i \in N$ contributes c_{ix} to x
- The utility of i is $\sum_{x \in A} (u_{ix}(f_x) - c_{ix})$
- A **mechanism** takes as input c_{ix} for all $i \in N$ and $x \in A$, and outputs f_x for all $x \in A$

WELFARE AND OPTIMALITY

- The **social welfare** is

$$\sum_{x \in A} \left(\sum_{i \in N} u_{ix}(f_x) - f_x \right)$$

- Note that the social welfare includes the deficit

$$\sum_{x \in A} \left(f_x - \sum_{i \in N} c_{ix} \right)$$

- **Theorem:** Quadratic funding maximizes social welfare

BIBLIOGRAPHY

V. Buterin. **Quadratic Payments: A Primer.** Blog post, 2019.

S. Lalley and E. G. Weyl. **Quadratic Voting: How Mechanism Design Can Radicalize Democracy.** American Economic Association Papers and Proceedings, 2018.

V. Buterin, Z. Hitzig and E. G. Weyl. **A Flexible Design for Funding Public Projects.** Management Science, 2019.

