Optimized Democracy (Fall 2025)
Problem Set #3
— Solutions —

Due: 10/29/2025 11:59pm ET

Instructions:

- You may discuss the problems with classmates but please write down solutions completely on your own.
- The solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You must not use AI in any way.
- Please *type up* your solution and submit to Gradescope.

Problems:

1. Recall the epistemic liquid democracy model, which was introduced and analyzed in the "liquid democracy" lecture, slides 5–12. We saw that local delegation mechanisms cannot satisfy do no harm (DNH) and positive gain (PG).

   Now consider the following non-local delegation mechanism. For each voter $i = 1, \ldots, n$, if $A_G(i) \neq \emptyset$ (i approves other voters), the mechanism determines the lowest j such that i approves j, that is, $j = \min A_G(i)$. Then i delegates to j if and only if j has not already delegated their vote and there is no other voter who has already delegated to i or j. Intuitively, under this mechanism, some delegations could take place but no voter would ever have a weight of more than 2.

   For simplicity, let us fix $\alpha = 0.1$ for this problem, that is, i approves j if and only if $(i, j) \in E$ and $p_j > p_i + 0.1$.

   (a) [**5 points**] Show that the above mechanism satisfies PG.

   Solution: We choose $\gamma = 1$. Let there be one voter with competence 1, and two voters with competence 0. The graph G includes one edge from one of the competence 0 voters to the competence 1 voter. It holds that $P_D(G) = 0$ and $P_M(G) = 1$, hence $\text{gain}(M, G) = 1$.

(b) **[15 points]** Show that the above mechanism does *not* satisfy DNH.

Note and hint: This is surprising because the mechanism prevents the problem of voters amassing large weight. It turns out that the mechanism does satisfy DNH with an additional assumption: for all $i \in N$, $p_i \in [\beta, 1-\beta]$ for $\beta > 0$. Here you are asked to give a family of counterexamples that would necessarily have to violate this assumption.

Solution: The formal definition of not satisfying DNH is the following: There exists some $\epsilon > 0$ such that for infinitely many n, there is a graph G_n on n vertices such that $\text{gain}(M, G_n) < -\epsilon$. We choose $\epsilon = 0.1$. We construct a graph for all odd $n = 2\ell + 1$ with $\ell \geq 1$. We will have ℓ voters with competence 1, $\ell - 1$ with competence 0, one with competence 0.4 and one with competence 0.6. In G_n, there will be an edge from the 0.4 voter to the 0.6 voter. Note that $\ell + 1$ correct votes are needed for a majority. Hence, $P_D(G_n)$ is simply the probability that at least one of the 0.4 and 0.6 competence voters vote correctly, which is $1 - (1 - 0.4)(1 - 0.6) = 0.76$. On the other hand, under the mechanism, the 0.4 voter delegates to the 0.6 voter, so the outcome is only the correct one exactly when the 0.6 voter is correct, i.e., $P_M(G_n) = 0.6$. Therefore, for all such n, $\text{gain}(M, G_n) = -0.16 < -0.1$, as needed.

2. In class we discussed notions of proportionality for approval-based elections like extended justified representation (EJR), which only guarantees that one voter in each "deserving" coalition is satisfied. In this problem our goal is to provide guarantees that hold *on average*.

   Recall that in the approval-based committee elections settings we have a set N of n voters and a target committee size k, where each voter $i \in N$ approves a set of alternatives $\alpha_i \subseteq A$. Let $q := \frac{n}{k}$. We say that a set of $S \subseteq N$ of voters is *ℓ-cohesive* if $|S| \geq \ell \cdot q$ and $|\bigcap_{i \in S} \alpha_i| \geq \ell$. As in class, we write $u_i(W) = |W \cap \alpha_i|$.

(a) **[10 points]** Assume that q is an integer. Suppose that a committee $W \subseteq A$, $|W| = k$, satisfies extended justified representation (EJR), so for each $1 \leq \ell \leq k$ and every ℓ-cohesive group S, there exists $i \in S$ with $u_i(W) \geq \ell$. Now let S be an ℓ-cohesive group with $|S| = \ell \cdot q$. Prove that

$$\sum_{i \in S} \frac{1}{|S|} u_i(W) \geq \frac{\ell - 1}{2},$$

that is, S obtains high average utility.

Note: With more work it can be shown that it is possible to achieve average utility at least $\ell - 1$ for ℓ-cohesive groups.

Solution: Let S be an ℓ-cohesive group. Label $S = \{1, \ldots, \ell q\}$ so that $u_1(W) \geq \cdots \geq u_{\ell q}(W)$. Write $S_j = S \setminus \{1, \ldots, j-1\}$. Note that $|\bigcap_{i \in S_j} \alpha_i| \geq \ell$ for each j. It follows that for $j = 1, \ldots, q$, S_j is an $(\ell - 1)$-cohesive group. Applying EJR to each of them, we find that there exists $i \in S_j$ with $u_i(W) \geq \ell - 1$. By our labeling of voters in S, in fact $i = j$ works. Hence we have $u_j(W) \geq \ell - 1$ for $j = 1, \ldots, q$. Similarly, we find that $u_j(W) \geq \ell - 2$ for $j = q+1, \ldots, 2q$. Continuing in this way, we see that

$$\sum_{i \in S} \frac{1}{|S|} u_i(W) \geq \frac{1}{|S|} \cdot q \cdot (1 + 2 + \cdots + (\ell - 1)) = \frac{1}{\ell q} \cdot q \cdot \ell \cdot \frac{\ell - 1}{2} = \frac{\ell - 1}{2}.$$

(b) **[15 pt]** Prove that for all $\varepsilon > 0$, there exists an election such that, no matter which committee is chosen, there is a 1-cohesive group S that has average utility at most ε. More formally, there is a set N of n voters, a set A of m alternatives, target committee size k, and approval set $\alpha_i \subseteq A$ for each $i \in N$, such that the following holds. For all committees $W \subseteq A$ with $|W| = k$, there is a set of voters S with $|S| \geq n/k$ and $|\bigcap_{i \in S} \alpha_i| \geq 1$ such that,

$$\sum_{i \in S} \frac{1}{|S|} u_i(W) \leq \varepsilon.$$

Note: With more work it can be shown that there exist elections such that no matter which committee is chosen, there is an ℓ-cohesive group with average utility at most $\ell - 1 + \varepsilon$. This means the lower bound mentioned in the note in part (a) is tight.

Hint: Construct a family of instances with $m = k + 1$ alternatives where each voter approves either one or two alternatives.

Solution: Fix an $\varepsilon > 0$. Choose k sufficiently large such that $2/(k+1) < \varepsilon$; letting $k = \lceil 2/\varepsilon \rceil$ will do. We will construct an instance for choosing a committee of size k with $m = k + 1$ alternatives and $n = k(k+1)$ voters. Call the alternatives $a_1, \ldots, a_m$. The preferences will be as follows: for each alternative a_j, there will be $k - 1$ voters whose approval set is $\{a_j\}$. This accounts for $(k+1)(k-1)$ of the voters. The remaining m voters will each like a pair of "neighboring" alternatives $\{a_j, a_{j+1}\}$ for each $j \in [m]$ (where, by convention, we let $a_{m+1} = a_1$, so last voter has preference $\{a_m, a_1\}$).

Suppose a committee W of size k is chosen. Let a_j be the alternative not in W. Note that there are $k + 1$ voters that approve a_j, the $k - 1$ with approval set $\{a_j\}$, the one with approval set $\{a_{j-1}, a_j\}$, and the one with approval set $\{a_j, a_{j+1}\}$. Since there are $k(k+1)$ voters in total, these voters form a 1-cohesive group. Further, all voters in the group only approving a_j are receiving utility 0, while those approving a pair are receiving utility 1. Hence, the average utility of this 1-cohesive group is $2/(k+1) < \varepsilon$, as needed.

3. We proved ("committee elections," slide 17) that a committee satisfying extended justified representation (EJR) can be found through the PAV rule. But computing PAV is hard.

   To efficiently compute a committee satisfying EJR, then, we'll actually define a stronger notion and show that it can be easily achieved:

   A committee W of size k satisfies EJR+ if for all alternatives $x \in A \setminus W$, if $S \subseteq N$ is such that $|S| \geq \ell n/k$ and $x \in \bigcap_{i \in S} \alpha_i$, then there is $i \in S$ such that $u_i(W) \geq \ell$.

   (a) **[5 points]** Show that EJR+ implies EJR.

   Solution: Suppose for contradiction that W satisfies EJR+ but not EJR, i.e., there is $S \subseteq N$ is such that $|S| \geq \ell n/k$, $|\bigcap_{i \in S} \alpha_i| \geq \ell$, and $u_i(W) < \ell$ for all $i \in S$. It follows that there is $x \in A \setminus W$ such that $x \in \bigcap_{i \in S} \alpha_i$, as if all ℓ alternatives in $\bigcap_{i \in S} \alpha_i$ were in W, it would hold that $u_i(W) \geq \ell$ for all $i \in S$ (not just for one $i \in S$). We conclude that EJR+ is violated — a contradiction.

   (b) **[10 points]** Show that EJR does not imply EJR+.

Solution: Suppose there are 6 voters and 5 candidates a, b, c, d, e with the following approval:

$$1 : \{a, b\}, 2 : \{a, c\}, 3 : \{a, d\}, 4 : \{b\}, 5 : \{c\}, 6 : \{d\}.$$

Consider the committee $W = \{b, c, d, e\}$ without a.

We first verify EJR. For $\ell = 1$ it is trivial since every voter i has $u_i(W) \geq 1$. For $\ell \geq 2$, no group of voters shares two candidates in common. Thus, EJR holds for W.

However, for EJR+, consider $a \in A \setminus W$ and $S = \{1, 2, 3\}$. Consider $\ell = 2$. We have $|S| \geq 2n/k = 3$ and $a \in \bigcap_{i \in S} \alpha_i$. On the other hand, for all $i \in S$, $u_i(W) = 1 < \ell$. Thus W does not satisfy EJR+. We conclude EJR does not imply EJR+.

(c) **[15 points]** Consider the following greedy algorithm. The algorithm starts from $W = \emptyset$, and for $\ell = k, \ldots, 1$, it iteratively finds and adds all alternatives $x \in A \setminus W$ such that the number of voters i with $x \in \alpha_i$ and $u_i(W) < \ell$ is at least $\ell n/k$. Intuitively, the algorithm finds and adds all alternatives causing an EJR+ violation with respect to a given value of ℓ, which starts from $\ell = k$ and is iteratively decremented.

Prove that this algorithm outputs an EJR+ committee of size (at most) k.

Hint: The committee clearly satisfies EJR+, so the challenge is to show that at most k alternatives are added.

Solution: Satisfying EJR+ is clear: if there were such an x violating EJR+ with $\ell = \ell'$ at the end, this would have been detected at round ℓ' (since the utility of voters is monotonically increasing).

We now show that we add at most k alternatives. Assign each voter a budget of k/n, so the total budget is k, and the price of each alternative is 1 and it is equally divided between the voters responsible for adding it. It is sufficient to show that the algorithm does not overspend the budget.

Consider any alternative x added at round ℓ. Then, there exists $S_x \subseteq N$ such that $|S_x| \geq \ell n/k$, $x \in \bigcap_i \alpha_i$, and $u_i(W) < \ell$ for all $i \in S_x$. Consider any $i \in S_x$. We show that after charging the price of the alternative x, the remaining budget for voter i is still non-negative. Including the current alternative x, voter i has only been charged at some $\ell' \geq \ell$. Therefore, each time the price charged to voter i is at most $1/(\ell n/k) = k/(\ell n)$. Furthermore, after adding alternative x, $u_i(W) \leq \ell$, so they could have been charged at most ℓ times. We conclude the total price charged to voter i is at most $k/(\ell n) \cdot \ell = k/n$, which is exactly their budget.

4. In the context of sortition we discussed *allocation rules*, which receive a set of volunteers N, a panel size k, a set of features F, set of values V_f for each $f \in F$, and quotas $u_{f,v}, \ell_{f,v}$ for all $f \in F$ and $v \in V_f$; they output a distribution over panels that satisfy the given quotas if one exists. In particular, we saw that allocation rules like Leximin and Maximum Nash Welfare lead to seemingly fair selection probabilities. But do these rules satisfy appealing axiomatic properties? On a high level the answer is negative, but that is mostly due to strong, general impossibility results that hold in this domain; below you are asked to establish those negative results.

(a) **[15 points]** An allocation rule guarantees *population monotonicity* if, when additional volunteers are added to an instance — that is, there are two sets of volunteers N and N'

such that $N \subseteq N'$, and the panel size, features, values and quotas remain unchanged — the selection probabilities of all previously existing volunteers weakly decrease.

Prove that no allocation rule satisfies population monotonicity.

Solution: Fix an allocation rule A, and consider an instance with six agents, $k = 3$, and four features. We indicate an agent's feature membership as a four-element Boolean vector, where the ith entry of the vector indicates whether the agent exhibits feature i. Using this convention, let the agents' features be given as:

agent 1: $(1, 0, 0, 0)$

agent 2: $(0, 1, 0, 0)$

agent 3: $(1, 1, 0, 0)$

agent 4: $(0, 0, 1, 0)$

agent 5: $(0, 0, 0, 1)$

agent 6: $(0, 0, 1, 1)$

For each feature f, set the lower quota ℓ_f to 1 and the upper quota to 3 (i.e., there is effectively no upper quota). This instance has quota-compliant panels, for example the panel $\{\text{agent } 1, \text{agent } 2, \text{agent } 6\}$.

Consider the probability allocation of A on this instance. Since $k = 3$, agents 1, 2, 4, and 5 cannot all simultaneously have zero selection probability. WLOG, assume that agent 1 has positive selection probability.

Now, consider a modified instance in which agent 6 is removed. In this instance, one verifies that the only quota-compliant panel is $\{\text{agent } 3, \text{agent } 4, \text{agent } 5\}$, which means that A must select agent 1 with zero probability. This violates population monotonicity since adding back agent 6 would strictly increase the selection probability of agent 1.

(b) **[10 points]** An allocation rule guarantees *committee monotonicity* if, when an instance is modified by increasing the size of the panel — that is, there are two panel sizes k and k' such that $k' \geq k$, and the volunteers, features, values and quotas remain unchanged (and the instance remains feasible) — the selection probabilities of all volunteers weakly increase.

Prove that no allocation rule satisfies committee monotonicity.

Solution: Consider an instance with three agents and two features. Define the features of the agents using the vector notation from the previous part as:

agent 1: $(1, 0)$

agent 2: $(0, 1)$

agent 3: $(1, 1)$

If the lower and upper quotas for both features are set to 1, the only panel for $k = 1$ is $\{\text{agent } 3\}$, and the only panel for $k = 2$ is $\{\text{agent } 1, \text{agent } 2\}$. Thus, any allocation rule must strictly decrease agent 3's selection probability when going from $k = 1$ to $k = 2$.