Optimized Democracy

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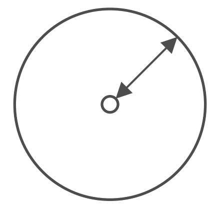
The Axiomatic Approach

Ariel Procaccia | Harvard University

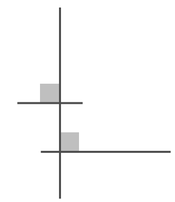
AXIOMS OF EUCLIDEAN GEOMETRY

To draw a straight line from any point to any point

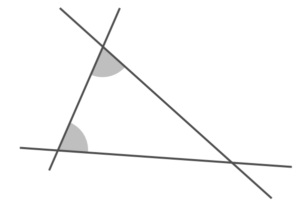
To produce a finite straight line continuously in a straight line



To describe a circle with any center and distance



That all right angles are equal to one another



That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles

Social choice theory similarly tries to analyze group decision making through an axiomatic lens. Another point of similarity is that, as we shall see (on Wednesday), some axioms are much more intuitive than others.

INDEPENDENCE OF CLONES

A subset of alternatives S is called clones in a given preference profile if no voter ranks any alternative $x \notin S$ between two alternatives in S

1	2	3	4	5	6
b	a	c	d	c	d
a	b	d	b	d	c
c	c	b	a	b	a
d	d	a	c	a	b

a and b are clones

c and d are not clones

INDEPENDENCE OF CLONES

- A voting rule is independent of clones if, when adding another clone x to a set of clones S:
  - If the winner was in S, it is in $S \cup \{x\}$
  - If the winner was $y \notin S$, it is still y

Poll 1

Which rule is independent of clones?

- Plurality
- Borda count
- IRV
- None of the above

INSTANT RUNOFF VOTING, REVISITED

Theorem: IRV is independent of clones

Intuition by example:

1	2	3	4	5	6
a	a	c	d	c	d
c	c	d	a	d	c
d	d	a	c	a	a

→

1	2	3	4	5	6
a	a	c		c	
c	c		a		c
		a	c	a	a

1	2	3	4	5	6
b	a	c	d	c	d
a	b	d	b	d	c
c	c	b	a	b	a
d	d	a	c	a	b

→

1	2	3	4	5	6
	a	c	d	c	d
a		d		d	c
c	c		a		a
d	d	a	c	a	

→

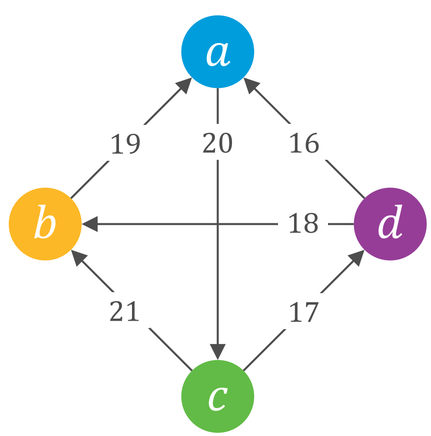
1	2	3	4	5	6
	a	c		c	
a					c
c	c		a		a
		a	c	a	

THE SCHULZE METHOD

- Let $P(x, y)$ denote the number of voters who prefer x to y
- A path from x to y of strength p is a sequence of alternatives $x = a_1, \dots, a_k = y$ such that for all $i = 1, \dots, k - 1$, $P(a_i, a_{i+1}) > P(a_{i+1}, a_i)$ and $P(a_i, a_{i+1}) \geq p$
- Let $S(x, y)$ be the strength of the strongest path from x to y — it’s 0 if there’s no path

THE SCHULZE METHOD: EXAMPLE

5 voters	2 voters	3 voters	4 voters	3 voters	3 voters	1 voter	5 voters	4 voters
a	*a*	*a*	*b*	*c*	*c*	*d*	*d*	*d*
c	*c*	*d*	*a*	*b*	*d*	*a*	*b*	*c*
b	*d*	*c*	*c*	*d*	*b*	*c*	*a*	*b*
d	*b*	*b*	*d*	*a*	*a*	*b*	*c*	*a*



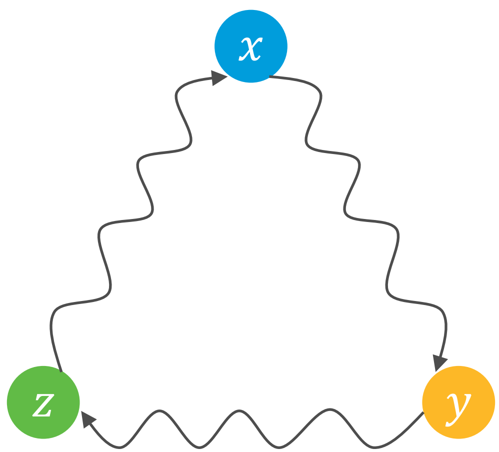
Pairwise comparisons

	a	*b*	*c*	*d*
a	–	20	20	17
b	19	–	19	17
c	19	21	–	17
d	18	18	18	–

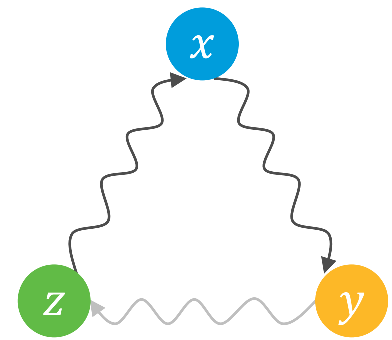
Strength of paths $S(x, y)$

THE SCHULZE METHOD

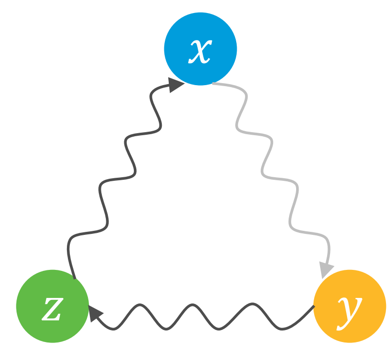
Lemma: If $S(x,y) > S(y,x)$ and $S(y,z) > S(z,y)$ then $S(x,z) > S(z,x)$



Assume that $S(z,x) \geq S(x,z)$
Note that $S(x,z) \geq \min\{S(x,y), S(y,z)\}$



$S(x,y) \geq S(y,z) \Rightarrow S(z,y) \geq S(y,z)$



$S(y,z) \geq S(x,y) \Rightarrow S(y,x) \geq S(x,y)$

THE SCHULZE METHOD

- Theorem: There exists a Schulze winner — an alternative $x^\star$ such that $S(x^\star, y) \geq S(y, x^\star)$ for all y
- Proof:
  - Draw an edge from x to y if $S(x, y) > S(y, x)$
  - By the lemma, the graph is acyclic
  - A finite, acyclic graph must have a source ■

Question

1	2	3
a	c	b
b	a	c
c	b	a

What can you say about Schulze winners in the Condorcet paradox example?

THE SCHULZE METHOD

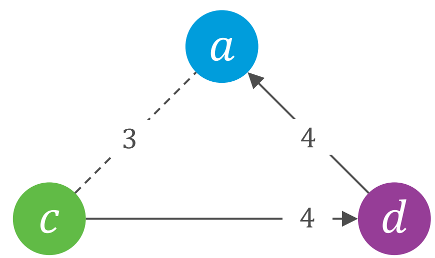
- **Theorem:** Schulze is Condorcet consistent
- **Proof:**
  - If $x^\star$ is a Condorcet winner, $P(x^\star, y) > n/2$ for all y
  - Therefore, $S(x^\star, y) > n/2$ for all y
  - Given y, any path to $x^\star$ has to use an edge $(z, x^\star)$ for some alternative z, but $P(z, x^\star) < n/2$
  - Therefore, $S(y, x^\star) = 0$ ■

THE SCHULZE METHOD

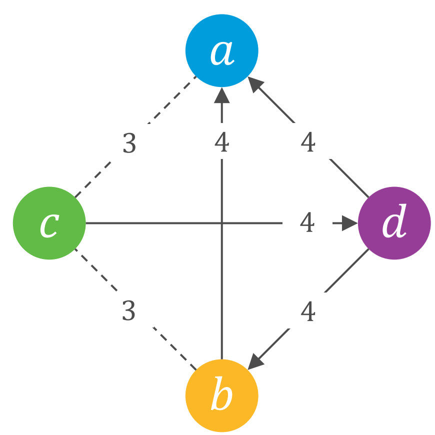
Theorem: Schulze is independent of clones

Intuition by example:

1	2	3	4	5	6
a	*a*	*c*	*d*	*c*	*d*
c	*c*	*d*	*a*	*d*	*c*
d	*d*	*a*	*c*	*a*	*a*



1	2	3	4	5	6
b	*a*	*c*	*d*	*c*	*d*
a	*b*	*d*	*b*	*d*	*c*
c	*c*	*b*	*a*	*b*	*a*
d	*d*	*a*	*c*	*a*	*b*



THE SCHULZE METHOD: USAGE

Government
Silla, Spain

Political parties
Five Star Movement

Organizations
Debian

AXIOMATIC OVERLOAD

Table 2 Comparison of Election Methods

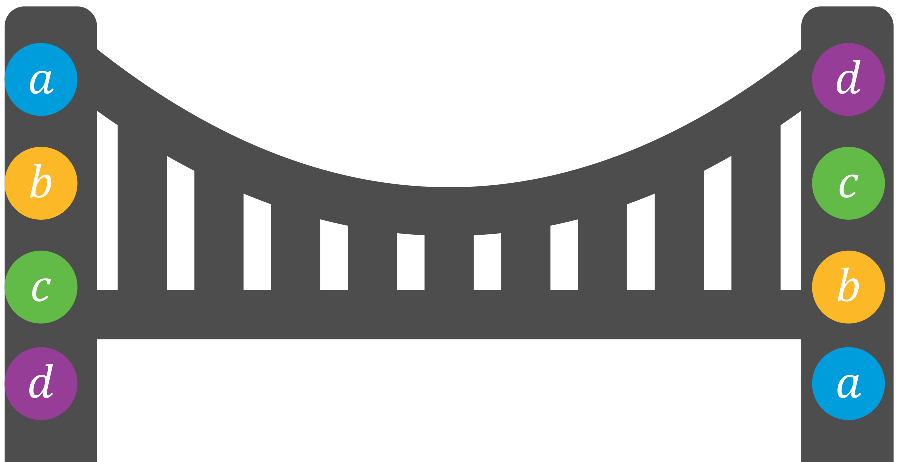
	Resolvability	Pareto	Reversal symmetry	Monotonicity	Independence of clones	Smith	Smith-IIA	Condorcet	Condorcet loser	Majority for solid coalitions	Majority	Majority loser	Participation	MinMax set	Prudence	Polynomial runtime
Baldwin	Y	Y	N	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
Black	Y	Y	Y	Y	N	N	N	Y	Y	N	Y	Y	N	N	N	Y
Borda	Y	Y	Y	Y	N	N	N	N	Y	N	N	Y	Y	N	N	Y
Bucklin	Y	Y	N	Y	N	N	N	N	N	Y	Y	Y	N	N	N	Y
Copeland	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	Y
Dodgson	Y	Y	N	N	N	N	N	Y	N	N	Y	N	N	N	N	N
Instant runoff	Y	Y	N	N	Y	N	N	N	Y	Y	Y	Y	N	N	N	Y
Kemeny–Young	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Nanson	Y	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
Plurality	Y	Y	N	Y	N	N	N	N	N	N	Y	N	Y	N	N	Y
Ranked pairs	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y
Schulze	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y
Simpson–Kramer	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	Y	Y
Slater	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Young	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	N	N

Y = compliance
N = violation

[Schulze, 2011]

PROPORTIONAL VETO PRINCIPLE

- Informally, a coalition consisting of p% of the voters should be able to veto an alternative that they all agree is in the bottom p% of alternatives
- Captures the idea of "bridging"



PROPORTIONAL VETO CORE

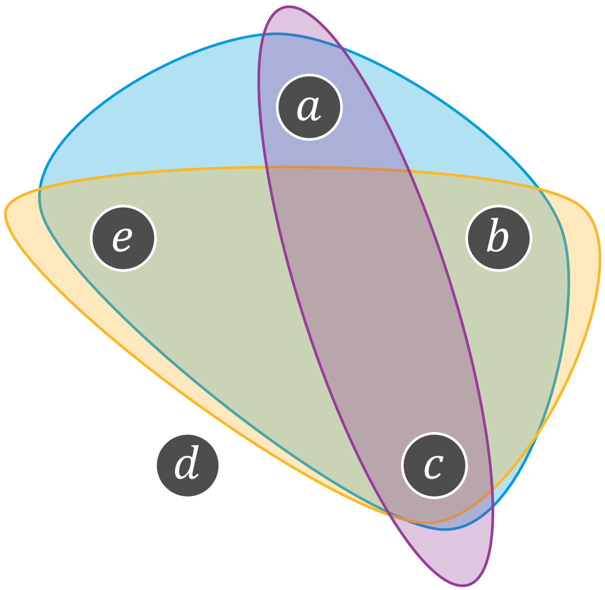
- Let there be n voters and m alternatives
- An alternative x is vetoed by a coalition of voters S if there's a subset of alternatives B that S all prefer to x and $m - |B| \leq \lceil m|S|/n \rceil - 1$
- The proportional veto core is the subset of alternatives that aren't vetoed by any coalition

PROPORTIONAL VETO CORE: EXAMPLE

1	2	3	4
e	b	d	a
b	e	b	c
c	c	e	d
a	d	c	e
d	a	a	b

A coalition of size k can veto $\left\lceil \frac{5k}{4} \right\rceil - 1 = k$ alternatives

d is vetoed by $S = \{1,2\}$ with $B = \{b, c, e\}$



Poll 2

Which candidate is in the proportional veto core?

- a
- b
- c
- e

VETO BY CONSUMPTION

- At every point in time, voters "eat" their least preferred remaining alternative "at the same rate"
- The last alternative to be eaten is the winner (with possible ties)

1	2	3	4
d	d	a	a
c	a	d	c
b	c	c	d
a	b	b	b

Alternative a

Alternative c

Alternative b

Alternative d

VETO BY CONSUMPTION

- Theorem: The veto-by-consumption outcome is in the proportional veto core
- Proof:
  - Suppose the rule elects x that is vetoed by S of size k
  - Then there exists B of size $m - \lceil mk/n \rceil + 1$ that S prefer to x
  - Since x is eaten last, the k voters in S only ate alternatives that are (weakly) less preferred than x, i.e., not in B
  - The number of alternatives not in B is at most $\lceil mk/n \rceil - 1 < mk/n$
  - Therefore, it takes the k voters less than m/n units of time to eat these alternatives, but the algorithm runs for exactly m/n units of time ■

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