Optimized Democracy

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Apportionment in the 19th Century

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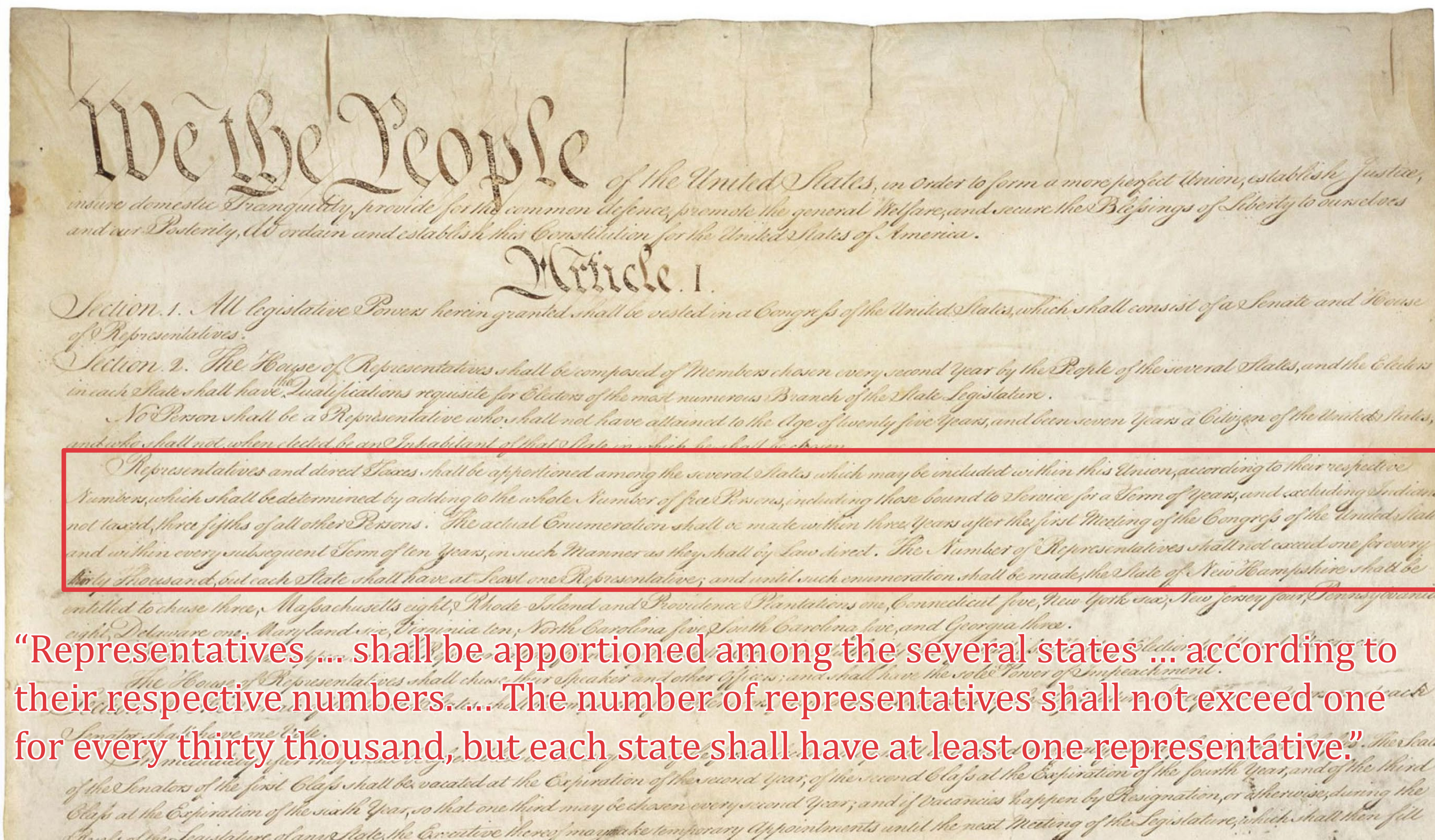
THE CONSTITUTION



We the People of the United States, in Order to form a more perfect Union, establish Justice, insure domestic Tranquility, provide for the common defence, promote the general Welfare, and secure the Blessings of Liberty to ourselves and our Posterity, do ordain and establish this Constitution for the United States of America.

Article. I.

Section. 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section. 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of that State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative; and until such enumeration shall be made, the State of New Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode-Island and Providence Plantations one, Connecticut five, New-York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

The House of Representatives shall chuse their Speaker and other Officers; and shall have the sole Power of Impeachment.

of the Senators of the first Class shall be vacated at the Expiration of the second Year, of the second Class at the Expiration of the fourth Year, and of the third Class at the Expiration of the sixth Year, so that one third may be chosen every second Year; and if Vacancies happen by Resignation, or otherwise, during the

“Representatives … shall be apportioned among the several states … according to their respective numbers. … The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative.”

THE MODEL

- Set of states $N = \{1, \dots, n\}$
- K seats to be allocated
- Each state has population p_i, and the total population is $P = \sum_{i=1}^{n} p_i$
- The standard quota of state i is $q_i = \frac{p_i}{P} \cdot K$
- The upper quota of i is $\lceil q_i \rceil$, and the lower quota is $\lfloor q_i \rfloor$
- Let k_i be the number of seats allocated to i

ROUNDING STANDARD QUOTAS

- The problem is that the standard quotas are fractional
- Simply rounding the standard quotas to the nearest integers may give seat allocations that don't add up to K

State	p_i	q_i	k_i
1	506	50.6	51
2	307	30.7	31
3	187	18.7	19
Total	1,000	100	101

Alexander Hamilton

1755–1804

First secretary of the treasury, co-author of the Federalist Papers. Also known for his role in the eponymous musical.

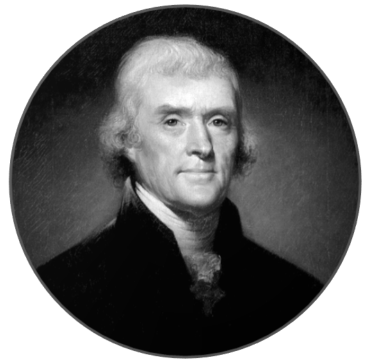
HAMILTON’S METHOD

- Hamilton’s Method allocates each state its lower quota and then allocates the remaining seats one at a time to the state with the largest residue $r_i = q_i - \lfloor q_i \rfloor$
- Congress presented a bill on March 26, 1792 that would apportion seats according to Hamilton’s Method

HAMILTON'S METHOD

State	p_i	$p_i/30{,}000$	k_i
Connecticut	236,841	7.895	8
Delaware	55,540	1.851	2
Georgia	70,835	2.361	2
Kentucky	68,705	2.290	2
Maryland	278,514	9.284	9
Massachusetts	475,327	15.844	16
New Hampshire	141,822	4.727	5
New Jersey	179,570	5.986	6
New York	331,589	11.053	11
North Carolina	353,523	11.784	12
Pennsylvania	432,879	14.419	14
Rhode Island	68,446	2.282	2
South Carolina	206,236	6.875	7
Vermont	85,533	2.851	3
Virginia	630,560	21.019	21
Total	3,615,920	120.821	120

Based on the census of 1790; 120 seats to be allocated.



Thomas Jefferson

1743–1826

Third president of the United States, first secretary of state. Also known for his supporting role in Hamilton.

JEFFERSON'S METHOD

- Jefferson's Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} \lfloor p_i/D \rfloor = K$, where $\hat{q}_i = p_i/D$ is the modified quota
  - Each state is allocated $k_i = \lfloor \hat{q}_i \rfloor$
- Washington was persuaded to veto the bill enacting Hamilton's Method
- Congress adopted Jefferson's Method on April 10, 1792
- It was used until 1830

JEFFERSON'S METHOD: EXAMPLE

- Jefferson's Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} \lfloor p_i/D \rfloor = K$, where $\hat{q}_i = p_i/D$ is the modified quota
  - Each state is allocated $k_i = \lfloor \hat{q}_i \rfloor$
- Suppose there are three states with populations $p_1 = 150$, $p_2 = 320$, and $p_3 = 530$, and $K = 10$

Poll

What is the allocation given by Jefferson's Method for the above instance?

- (2,3,5)
- (1,4,5)
- (2,2,6)
- (1,3,6)

JEFFERSON IS WELL-DEFINED

- **Theorem:** If D and D' are two different divisors yielding Jefferson apportionments $k_1, \ldots k_n$ and $k'_1, \ldots, k'_n$ then $k_i = k'_i$ for all $i \in N$
- **Proof:**
  - Assume w.l.o.g. that $D \leq D'$, then $p_i/D \geq p_i/D'$ for all $i \in N$
  - We conclude that $k_i \geq k'_i$ for all $i \in N$
  - It also holds that $\sum_{i \in N} k_i = K = \sum_{i \in N} k'_i$
  - It can't be the case that $k_i > k'_i$ for some $i \in N$ $\blacksquare$

JEFFERSON'S LARGE-STATE BIAS

		$D = 100{,}000$		$D = 97{,}000$	
State	p_i	$\hat{q}_i$	k_i	$\hat{q}_i$	k_i
1	2,620,000	26.20	26	27.01	27
2	168,000	1.68	1	1.73	1
…	…	…	…		
Total	10,000,000	…	99	…	100

- State 1 gets the additional seat despite initially having the smaller residue
- When the divisor is reduced, each seat requires 3,000 fewer citizens, and state 1 gains for each of its 26 seats
- State 1 needs 97,037 citizens per seat whereas state 2 needs 168,000

John Adams

1735–1826

Second president of the United States, first vice president. Also known for being mocked by King George III.

ADAMS' METHOD

- Adams' Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} \lceil \hat{q}_i \rceil = K$
  - Each state is allocated $k_i = \lceil \hat{q}_i \rceil$
- The large states were against the proposal
- Adams' Method was considered by Congress but never adopted

ADAMS' SMALL-STATE BIAS

		$D = 100{,}000$		$D = 104{,}000$	
State	p_i	$\hat{q}_i$	k_i	$\hat{q}_i$	k_i
1	2,668,000	26.68	27	25.65	26
2	120,000	1.20	2	1.15	2
…	…	…	…		
Total	10,000,000	…	101	…	100

- State 1 loses a seat despite initially having the larger residue
- When the divisor is increased, each seat requires 4,000 more citizens, and state 1 loses for each of its 27 seats
- State 1 needs 102,615 citizens per seat whereas state 2 needs 60,000

WEBSTER'S METHOD

- Webster's Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} [\hat{q}_i] = K$
  - Each state is allocated $k_i = [\hat{q}_i]$
- This method isn't biased towards small or large states
- Webster's Method was adopted by Congress in 1842

WEBSTER IS “UNBIASED”

State	p_i	$\hat{q}_i$	k_i	Ratio
1	304,000	30.4	30	10,133
2	26,000	2.6	3	8,667
Total	330,000	33	33	

Small state is better off

($D = 10{,}000$ in both examples)

Large state is better off

State	p_i	$\hat{q}_i$	k_i	Ratio
1	296,000	29.6	30	9,867
2	34,000	3.4	3	11,333
Total	330,000	33	33	

HISTORICAL INTERLUDE

- In 1850, Senator Samuel Vinton (independently?) proposed a method that is identical to Hamilton's
- Vinton's (Hamilton's) Method was finally adopted by Congress that year
- The House increased from 233 seats to 234, a size on which the allocations from Hamilton's Method and Webster's Method coincided
- The size of the House increased to 241 in 1860 and to 292 in 1870

ALABAMA PARADOX

Under Hamilton's Method, adding seats can decrease a state's allocation!

		$K = 10$		$K = 11$	
State	p_i	q_i	k_i	q_i	k_i
1	6	4.286	4	4.714	5
2	6	4.286	4	4.714	5
3	2	1.429	2	1.571	1
Total	14	10	10	11	11

A method that avoids this paradox is called house monotonic

ALABAMA PARADOX

- The Alabama Paradox was discovered in 1880 by C. W. Seaton, the chief clerk of the Census Office
- Using the 1880 census results, he calculated allocations according to Hamilton’s Method for all House sizes between 275 and 350
- When he went from 299 to 300, Alabama lost a seat!
- Congress decided to go with 325 seats, on which Hamilton’s Method and Webster’s Method agreed
- In 1890 there were no issues, but in 1900 the Alabama Paradox reappeared with Colorado and Maine taking the place of Alabama

POPULATION PARADOX

Under Hamilton's Method, a state whose population grew can lose a seat to a state whose population shrank

	Before			After		
State	p_i	q_i	k_i	p_i	q_i	k_i
1	145	1.45	2	147	1.55	1
2	340	3.40	3	338	3.56	4
3	515	5.15	5	465	4.89	5
Total	1000	10	10	950	10	10

A method that avoids this paradox is called population monotonic

POPULATION PARADOX

- In 1900, the populations of Virginia and Maine were 1,854,184 and 694,466, respectively
- In the following year Virginia's population grew by 19,767 (+1.06%) while Maine's increased by 4,649 (+0.7%)
- Hamilton's Method would have allocated an additional seat to Maine at the expense of Virginia

OKLAHOMA PARADOX

Under Hamilton's Method, adding a state and increasing the size of the house accordingly can change the allocation of existing states

	Before			After		
State	p_i	q_i	k_i	p_i	q_i	k_i
1	145	1.45	2	145	1.50	1
2	340	3.40	3	340	3.51	4
3	515	5.15	5	515	5.31	5
4	—	—	—	260	2.68	3
Total	1000	10	10	1260	13	13

OKLAHOMA PARADOX

- When Oklahoma became a state in 1907, it was awarded 5 representatives and the size of the House increased by 5
- But if the allocation was recomputed according to Hamilton’s method (which was used at the time) and the same 1900 census data, New York would have had to transfer a seat to Maine

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