# Optimized Democracy (Fall 2025)
# Problem Set #4

Due: 11/12/2025 11:59pm ET

**Instructions:**

- You may discuss the problems with classmates but please write down solutions completely on your own.
- The solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You must not use AI in any way.
- Please *type up* your solution and submit to Gradescope.

**Problems:**

1. Computing the Banzhaf power index in weighted voting games is generally hard, but there are many special cases where it is easy. For example:

   **[50 points]** Consider a weighted voting game where the $n$ players have $k$ distinct weights. Show that if $k$ is bounded by a constant, then the Banzhaf power index of a given player can be computed in polynomial time in $n$.

2. In the "apportionment in the 20th century" lecture, Slides 20–21, we discussed a randomized apportionment method due to Grimmett. The method starts by randomly permuting the states, and while this is irrelevant to ensuring that each state receive $q_i$ seats in expectation, there's a feeling that it might break strange correlations between states. In this problem, we'll see one sense in which such correlations persist.

   Let $\mathbf{r}, \mathbf{r}' \in [0,1)^n$ be two residue vectors, summing up to the same integer $R$. (Recall that $R = K - \sum_{i=1}^{n} \lfloor q_i \rfloor$.) Let $T$ be a set of $R$ states such that $r'_i \geq r_i$ for all $i \in T$ and $r_i \geq r'_i$ for all $i \notin T$. We say that *selection monotonicity* is satisfied if it always holds that the probability that precisely the states in $T$ get rounded up under $\mathbf{r}'$ is at least as high as under $\mathbf{r}$.

   **[50 points]** Show that Grimmett's randomized apportionment method (including its initial random permutation) is *not* selection monotonic.

   **Note:** There's an ingenious randomized apportionment method which, like Grimmett's method, gives each state $q_i$ seats in expectation, and is also selection monotonic.