

Optimized Democracy

Spring 2022 | Lecture 17

Redistricting As Cake-Cutting

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REDISTRICTING

We the People

of the United States, in Order to form a more perfect Union, establish Justice, insure domestic Tranquillity, provide for the common Defence, promote the general Welfare, and secure the Blessings of Liberty to ourselves and our Posterity, do ordain and establish this Constitution for the United States of America.

Article. I.

Section. 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section. 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of that State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative; and until such Enumeration shall be made, the State of New Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode Island and Providence Plantations one, Connecticut five, New York six, New Jersey four, Pennsylvania

Representatives... shall be apportioned among the several States which may be included within this Union, according to their respective Numbers... within every subsequent Term of ten Years

When vacancies happen in the Representation from any State, the Executive Authority thereof shall issue Writs of Election to fill such Vacancies. The House of Representatives shall chuse their Speaker and other Officers; and shall have the sole Power of Impeachment.

Section. 3. The Senate of the United States shall be composed of two Senators from each State, chosen by the Electors in each State for six Years, and each Senator shall have one Vote.

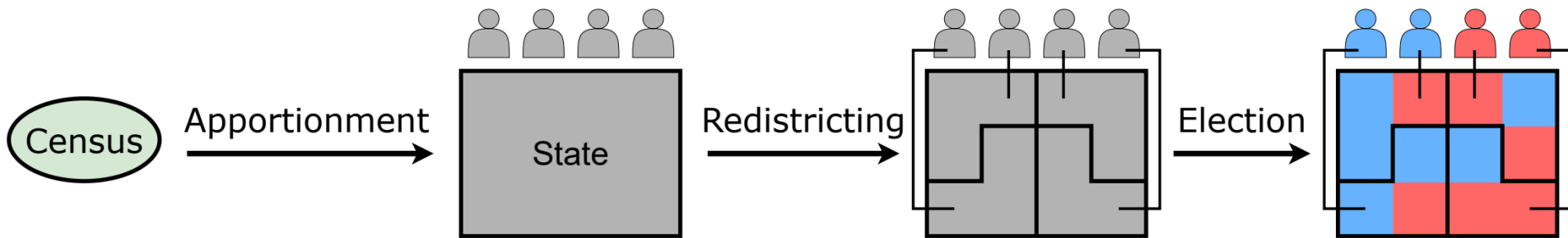
REDISTRICTING

The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State by the Legislature thereof...

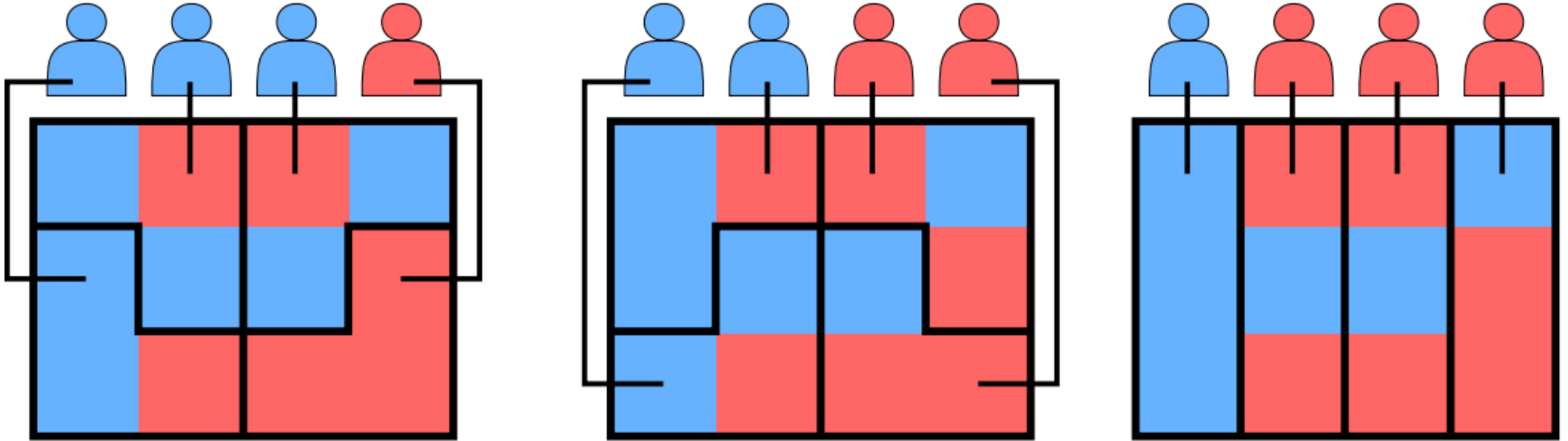
Section. 1. The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State by the Legislature thereof, but the Congress may at any time by Law make or alter such Regulations, except as to the Places of choosing Senators.

The Congress shall assemble at least once in every Year, and such Meetings shall be on the first Monday in December, unless they shall by Law appoint a different Day.

Section. 5. Each House shall be the Judge of the Elections, Returns and Qualifications of its own Members, and a Majority of each shall constitute a



GERRYMANDERING



gerrymander • *verb* • to divide or arrange (a territorial unit) into election districts in a way that gives one political party an unfair advantage

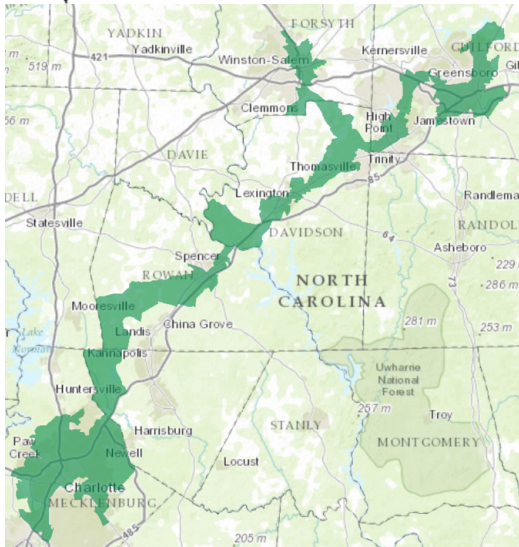
– Merriam-Webster

GERRYMANDERING



Ideas to prevent it:

- Have an independent commission draw fair districts
- Use an interactive protocol with participation from both parties
- Statistically prove a map is gerrymandered



ABSTRACT MODEL

- State S , with set of feasible districts $\mathcal{D} \subseteq 2^S$
- Set of parties $N := \{1, 2, \dots, n\}$ (today $n := 2$)
- Population measure $\mu : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$
- For each $j \in N$, distribution function $v^j : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$
- Target number of districts $m \in \mathbb{Z}_{>0}$

A *partition* is a set P of m disjoint* districts covering S , each of equal measure. The utility of party j is

$$u^j(P) := |\{D \in P \mid \forall i \neq j, v^j(D) >^* v^i(D)\}|.$$

DISCRETE GRAPH MODEL

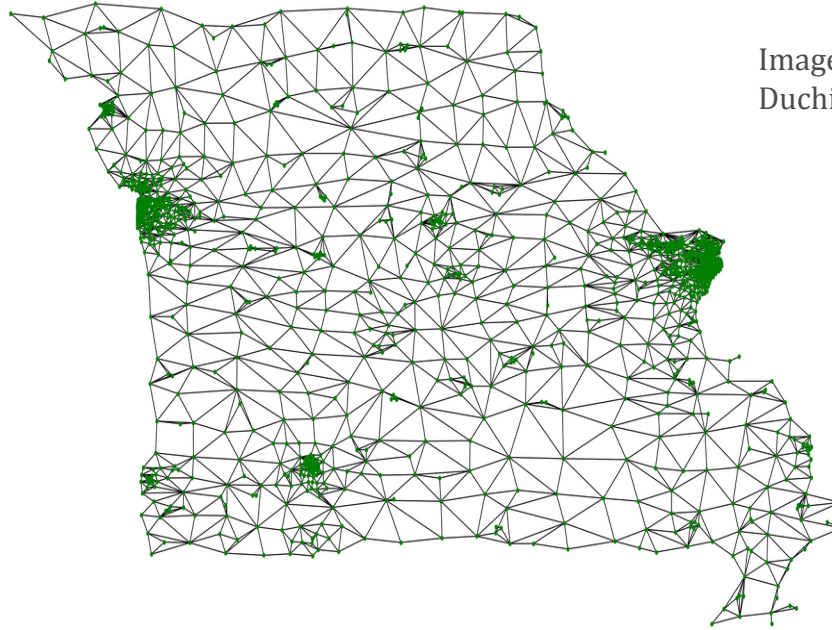
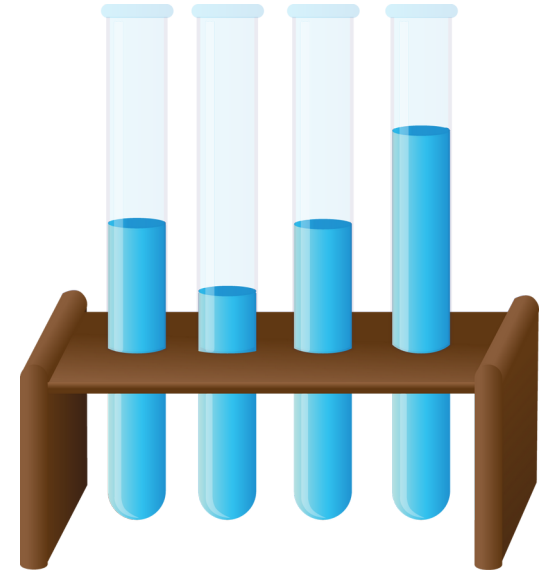


Image credit: DeFord,
Duchin, Solomon, 2020

- Graph G of indivisible census blocks
- $S := V(G)$
- $\mathcal{D} := \{D \subseteq S \mid \text{induced subgraph of } D \text{ is connected}\}$
- $\mu(D) := \sum_{b \in D} (\text{population of } b)$
- $v^j(D) := \sum_{b \in D} (\text{number of } j\text{-voters in } b)$

GEOMETRY-FREE MODEL

Continuous model of
“placing voters in buckets”
with no constraints.



- $S := \bigcup_{j \in N} V_j$
 $V_j := (j, [0, p_j])$
 $\sum_{j \in N} p_j = 1$
- $\mathcal{D} := \{ \bigcup_{j \in N} (j, [a_j, b_j]) \mid \forall j \in N, 0 \leq a_j \leq b_j \leq p_j \}$
- $\mu(D) := \sum_{j \in N} (b_j - a_j)$
- $v^j(D) := b_j - a_j$

PROPORTIONALITY

$$\text{For all } j \in N, u^j(P) \geq \left\lfloor m \cdot \frac{v^j(S)}{\sum_{i \in N} v^i(S)} \right\rfloor.$$

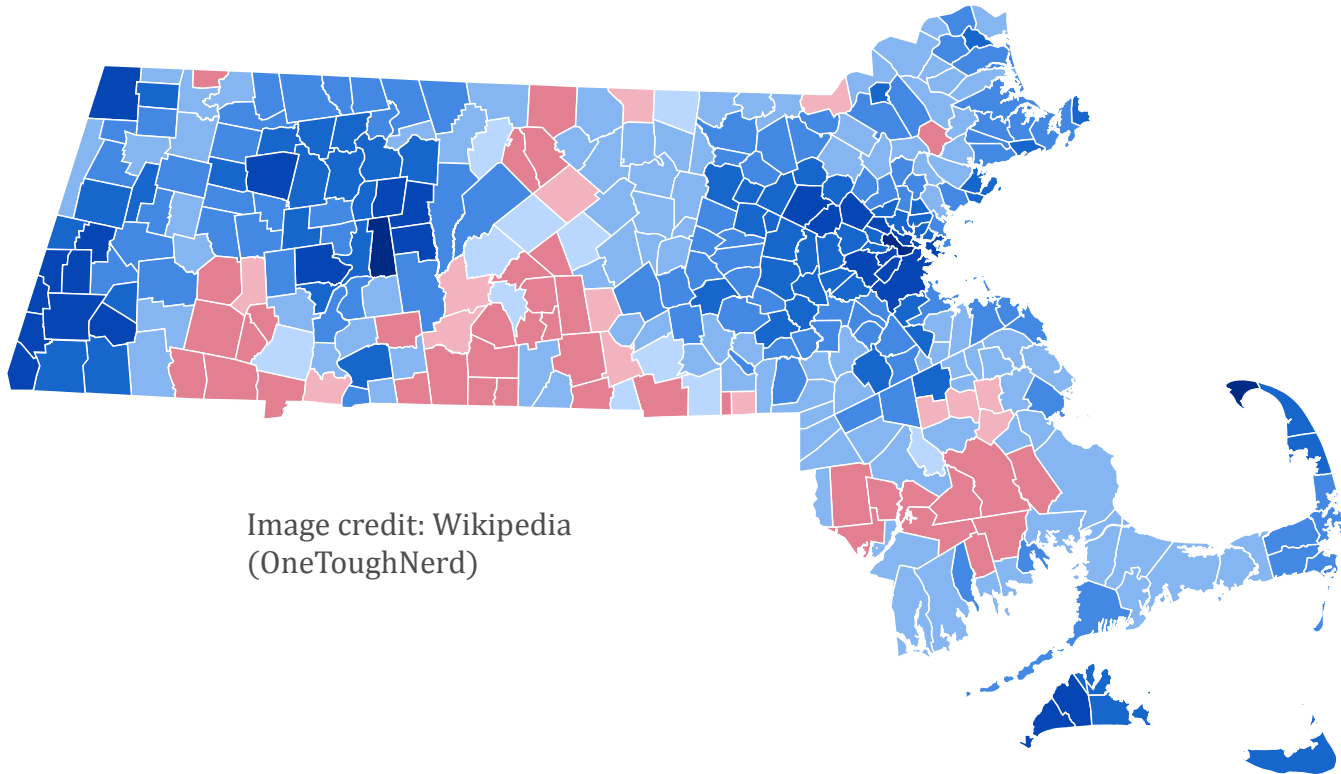


Image credit: Wikipedia
(OneToughNerd)

GEOMETRIC TARGET

For all $j \in N$, let P_{\max}^j be a partition maximizing u^j and let P_{\min}^j be a partition minimizing u^j . Then

$$u^j(P) \geq \left\lfloor \frac{u^j(P_{\max}^j) + u^j(P_{\min}^j)}{2} \right\rfloor.$$

Poll

In the geometry-free model, for the minority party, which is easier for a given partition to satisfy?

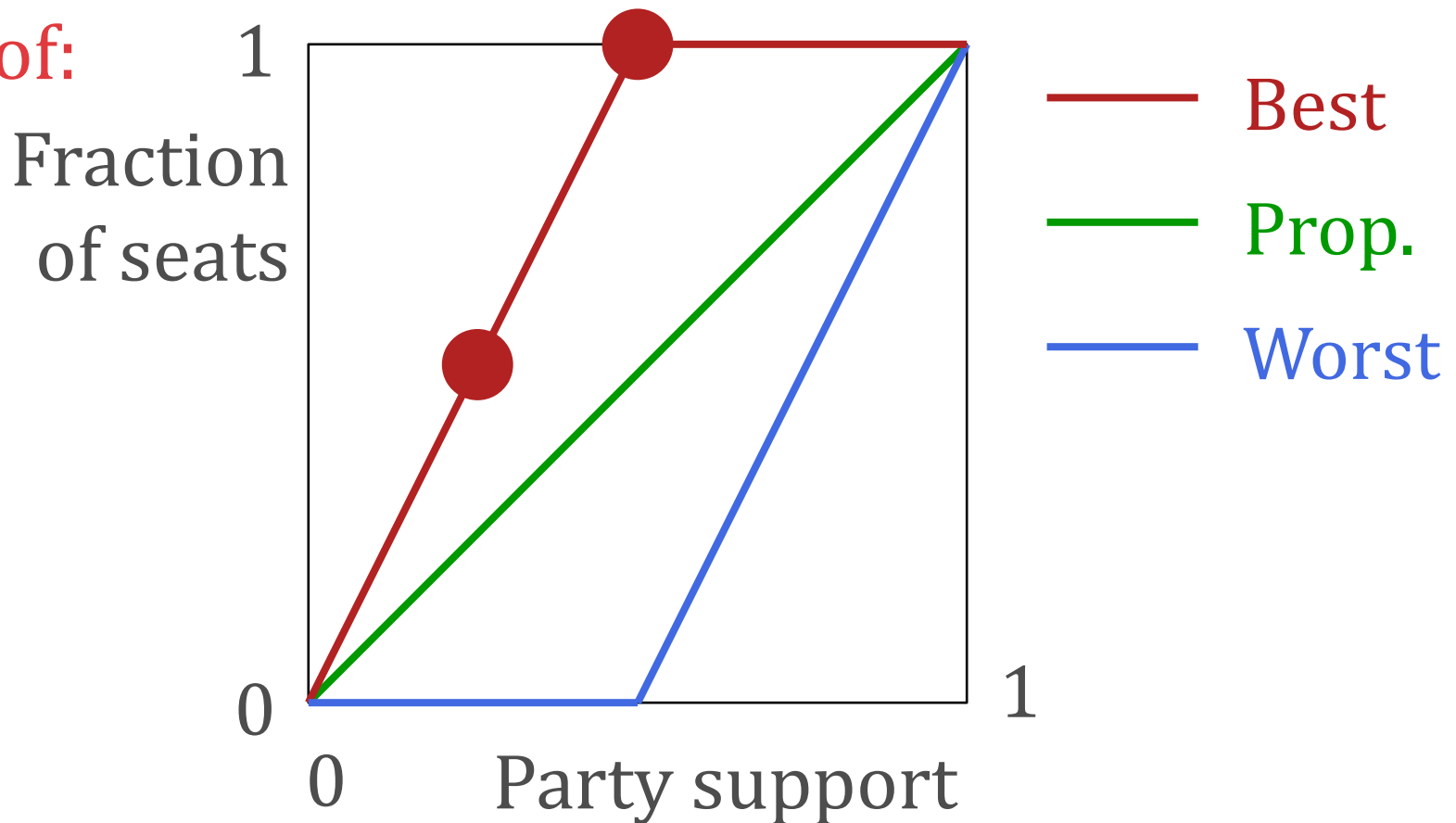
- Proportionality
- Geometric target
- Equivalent
- Incomparable



GEOMETRIC TARGET

Theorem: In the geometry-free model, a partition satisfies proportionality if and only if it satisfies the geometric target (up to ties).

Proof:

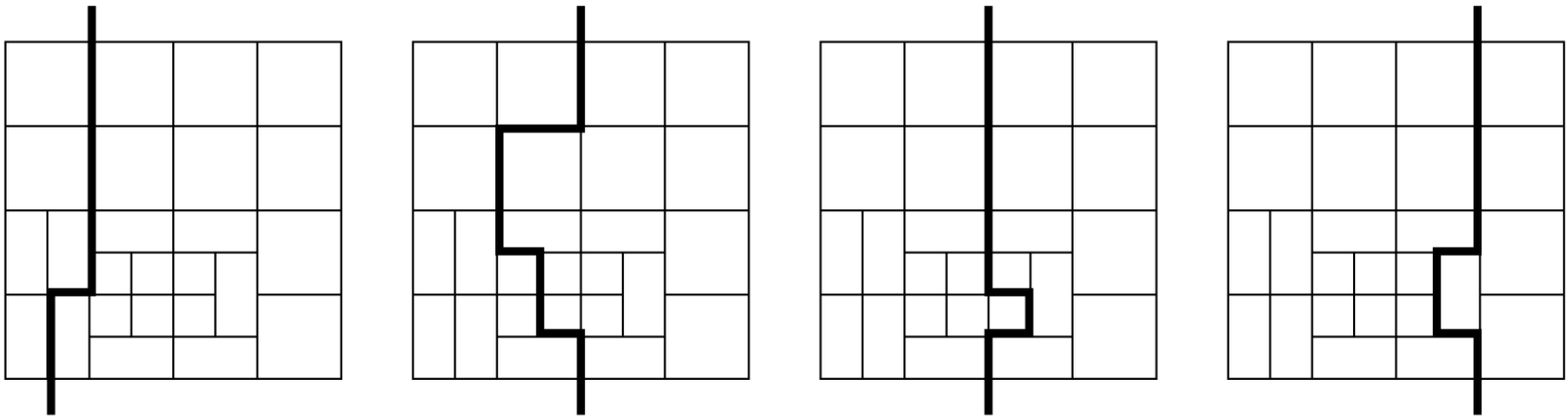


LRY PROTOCOL

Interactive protocol by Landau, Reid, and Yershov that uses a neutral administrator.

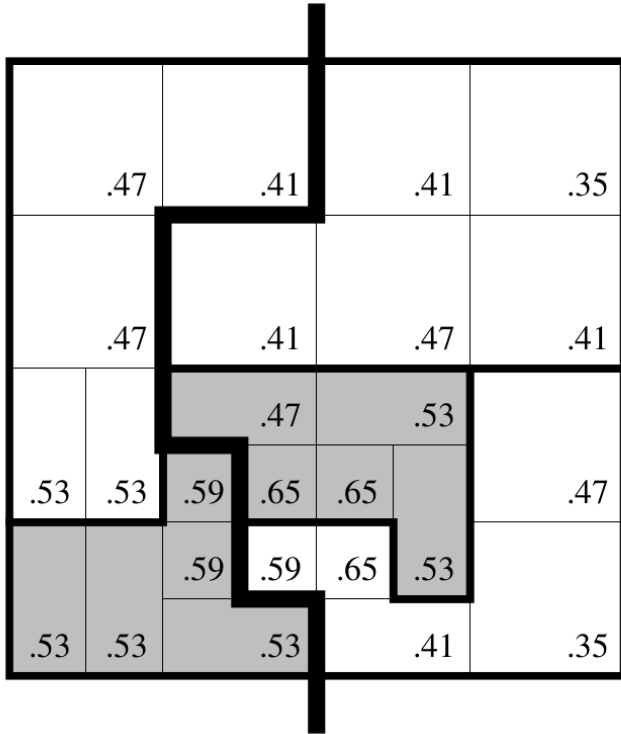
1. Administrator presents both parties with a series of bipartitions $(L_1, R_1), (L_2, R_2), \dots, (L_{m-1}, R_{m-1})$ of S , such that each $L_i \subseteq L_{i+1}$.

Image credits:
Landau and Su, 2015



LRY PROTOCOL

2. For each $i \in [m]$, each party is asked, “Would you rather redistrict L_i , with the other party redistricting R_i , or vice versa?”

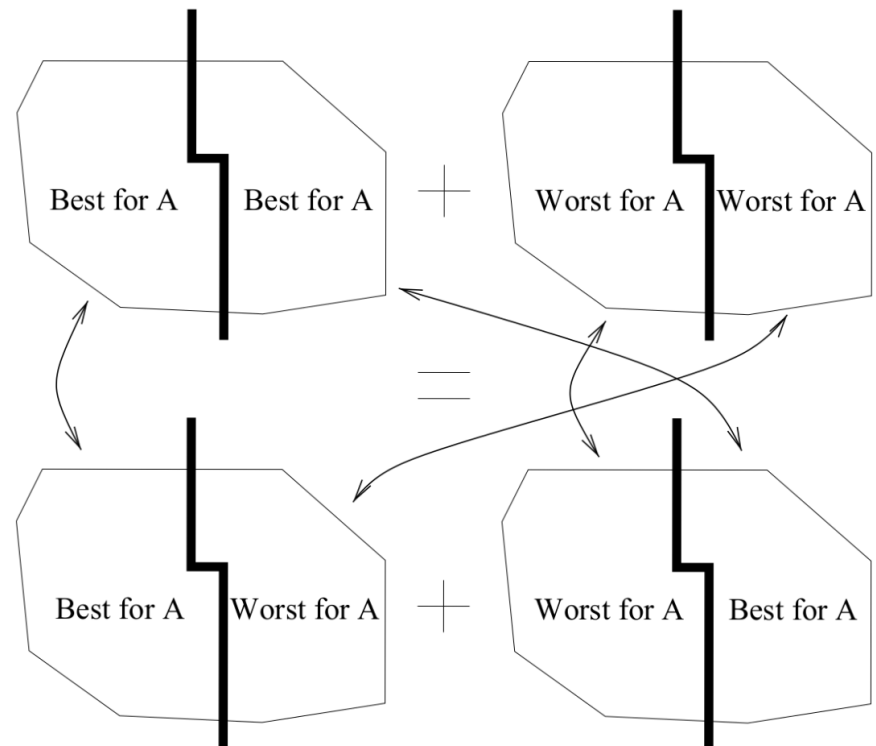


LRY PROTOCOL

3. Try to find an i such that one party prefers redistricting L_i and the other prefers redistricting R_i . If no such i exists, randomly select an outcome at the cross-over point.

Theorem (Good Choice Property): Restricting the feasible set of partitions to respect a given split, a party's preferred choice satisfies its geometric target.

Proof:



LRY PROTOCOL

Pros:

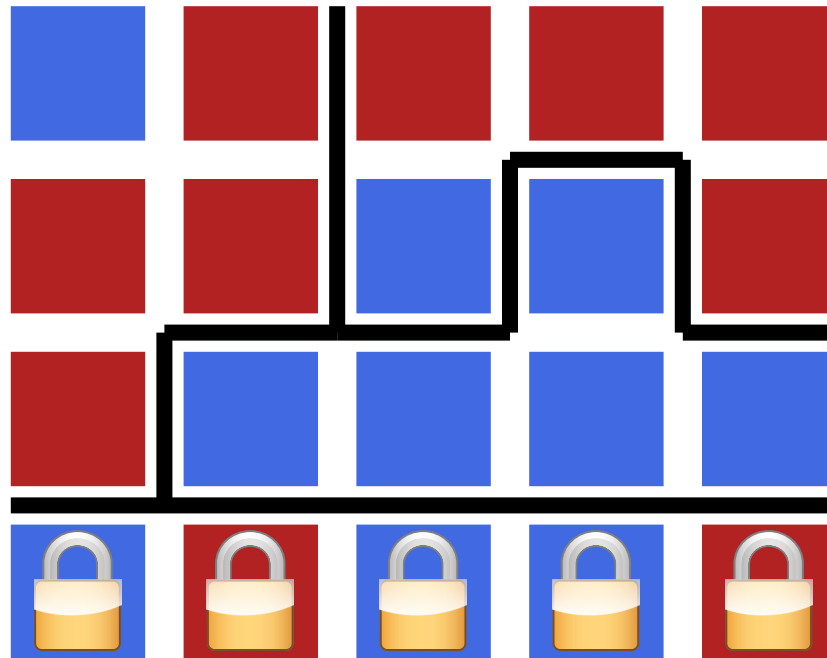
- Realistically implementable
- Simple party participation
- Guaranteed to be within 2 of prop. / geometric target in geometry-free model

Cons:

- Relies heavily on neutrality of the administrator
- Can be arbitrarily far from geometric target in grid-based model

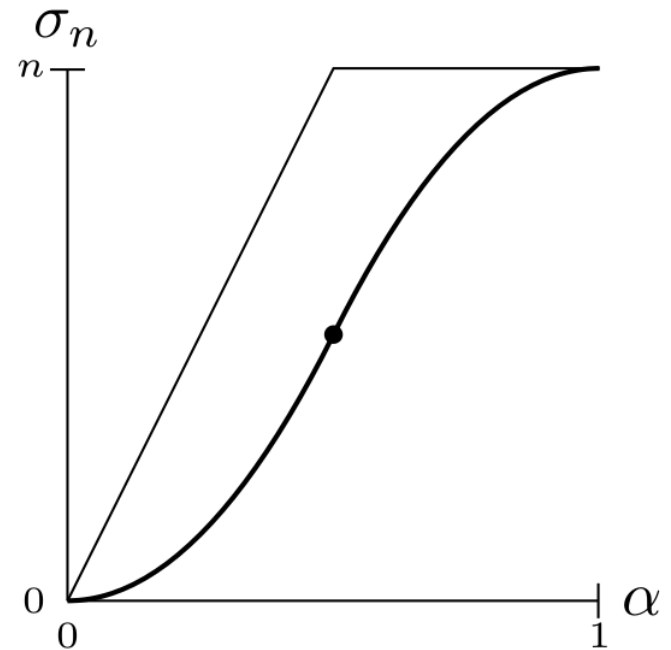
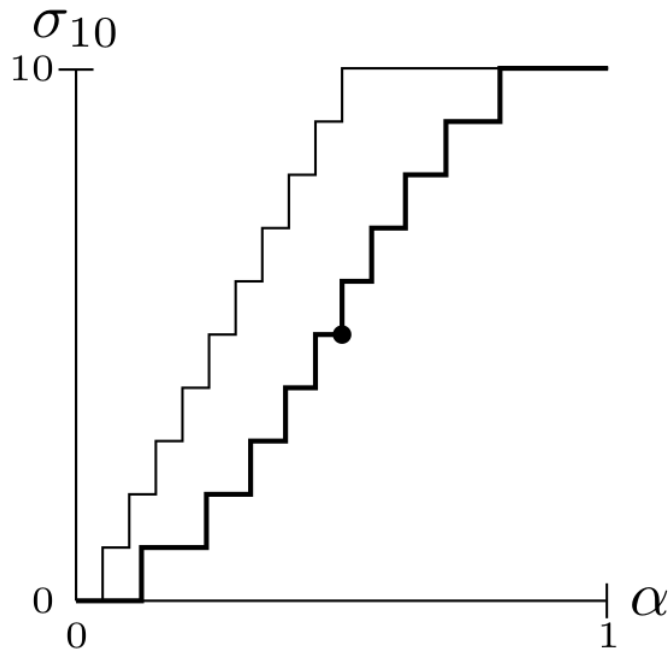
CUT AND FREEZE

By Pegden and Procaccia: partition, freeze, and re-partition until all districts are frozen.



CUT AND FREEZE

Theorem: In the geometry-free model, under optimal play, each party can guarantee a number of seats as in the following graphs.



CUT AND FREEZE

Pros:

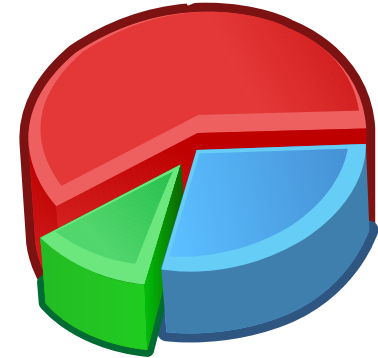
- Realistically implementable
- Approximate proportionality in geometry-free model
- Hard to pack specific groups into one district

Cons:

- Requires complicated strategies
- Requires several rounds of interaction

STATE-CUTTING MODEL 1

Cake-cutting analogue
introduced by Benade,
Procaccia, and T-F.



- $S := [0, 1]$
- $\mathcal{D} := \{\text{finite unions of closed intervals}\}$
- $\mu := \text{Lebesgue measure}$
- $v^j(D) := \int_D f^j(x) dx$ where, for all $x \in S$,

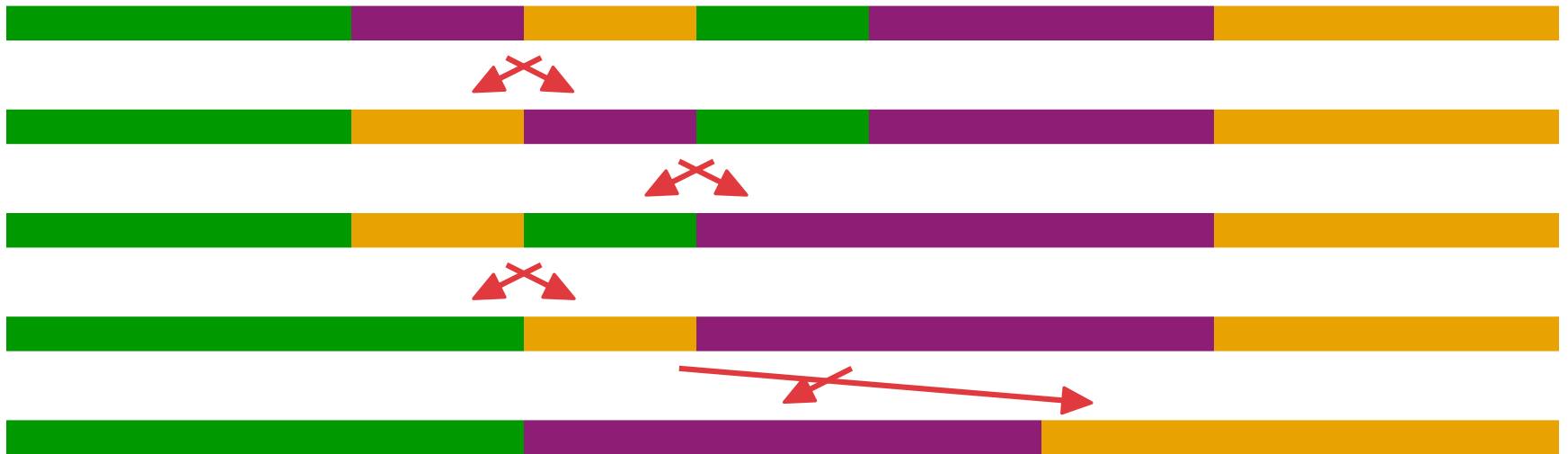
$$\sum_{j \in N} f^j(x) = 1$$

STATE-CUTTING PROTOCOL 1

1. Ask each party j to construct an optimal partition P_j .
2. Construct a sequence of partitions from P_1 to P_2 , each differing from the previous one on at most two districts.
3. Select an intermediate partition that satisfies the geometric targets of both parties.

STATE-CUTTING PROTOCOL 1

How to achieve step 2? Bubble sort!



Can transition from P_1 to P_2 via the simplest possible partition $\{[\frac{k-1}{m}, \frac{k}{m}] \mid k \in [m]\}$ (the bottom one). Each swap modifies only two districts.

STATE-CUTTING PROTOCOL 1

Theorem: If two partitions differ on at most two districts, the balance of power can differ by at most one.

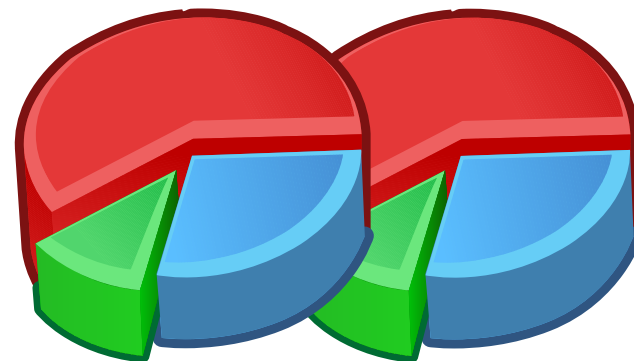
Proof: Suppose P and P' differ on districts $D_1, D_2 \in P$ and $D'_1, D'_2 \in P'$. Suppose party 1 has a majority in D_1 and D_2 , but a minority in D'_1 and D'_2 . Then:

$$\begin{aligned}\frac{1}{m} &< v^1(D_1) + v^1(D_2) = v^1(D_1 \cup D_2) \\ &= v^1(D'_1 \cup D'_2) = v^1(D'_1) + v^1(D'_2) < \frac{1}{m}\end{aligned}$$

Contradiction.

STATE-CUTTING MODEL 2

Now parties are allowed to disagree over the distribution of voters!

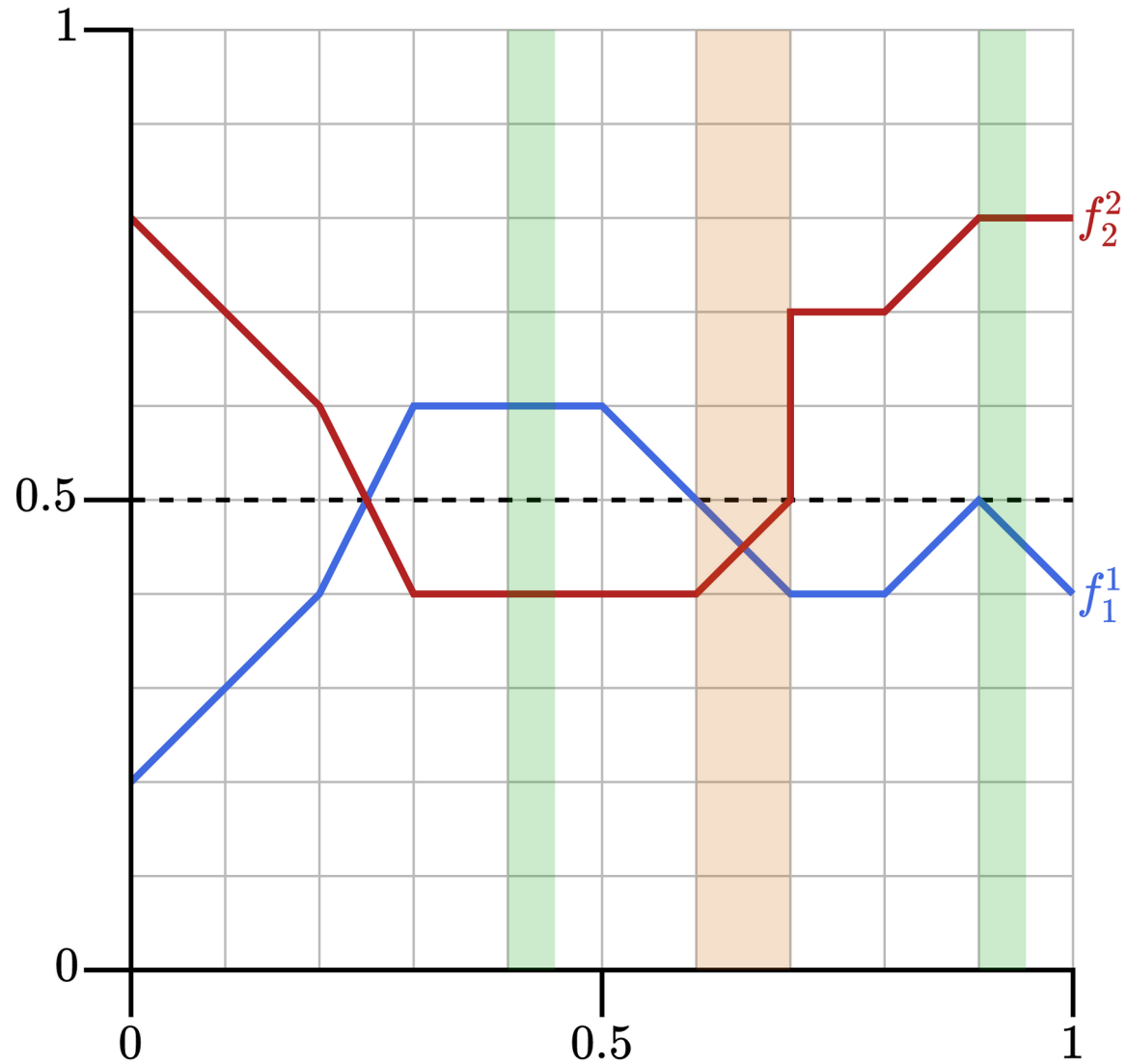


- $S := [0, 1]$
- $\mathcal{D} := \{\text{finite unions of closed intervals}\}$
- $\mu := \text{Lebesgue measure}$

- $v_i^j(D) := \int_D f_i^j(x) dx$ where, for all $x \in S$ and $i \in N$,

$$\sum_{j \in N} f_i^j(x) = 1$$

STATE-CUTTING MODEL 2



STATE-CUTTING PROTOCOL 2

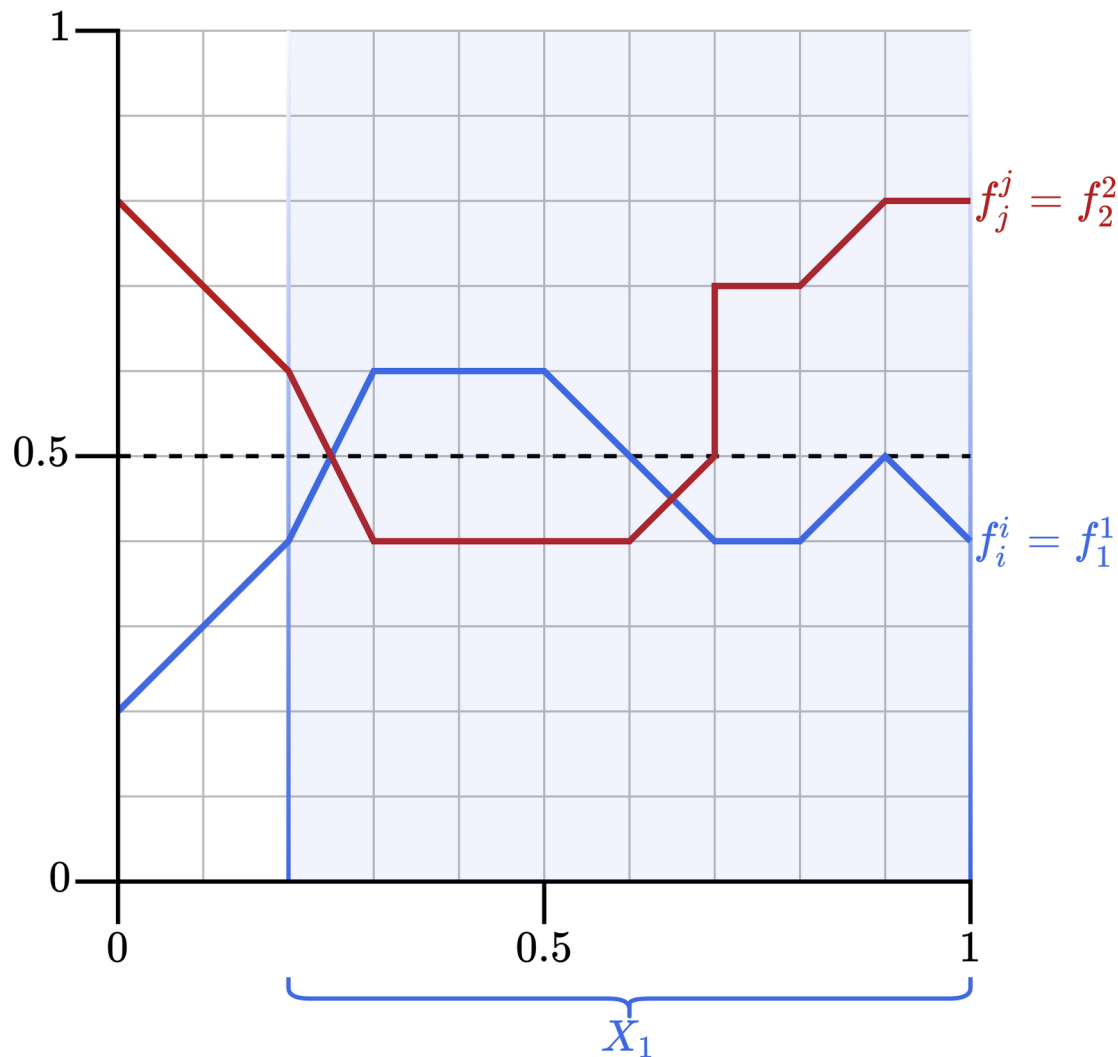
Theorem: Even when parties disagree, there always exists a partition satisfying the geometric targets of both parties:

$$u_i^i(P) \geq \left\lfloor \frac{\min_{P'} u_i^i(P') + \max_{P'} u_i^i(P')}{2} \right\rfloor$$

STATE-CUTTING PROTOCOL 2

1. Each party i computes a maximal set $X_i \subseteq S$ such that $m\mu(X_i) \in \mathbb{Z}$ and $v_i^i(X_i) = \frac{\mu(X_i)}{2}$.
2. Let i be the party with the larger X_i set, and let j be the other party.
3. Party j divides X_j into two pieces of equal size and equal party support according to j .
4. Party i chooses a piece for j to redistrict.
5. Party i redistricts the rest of S .

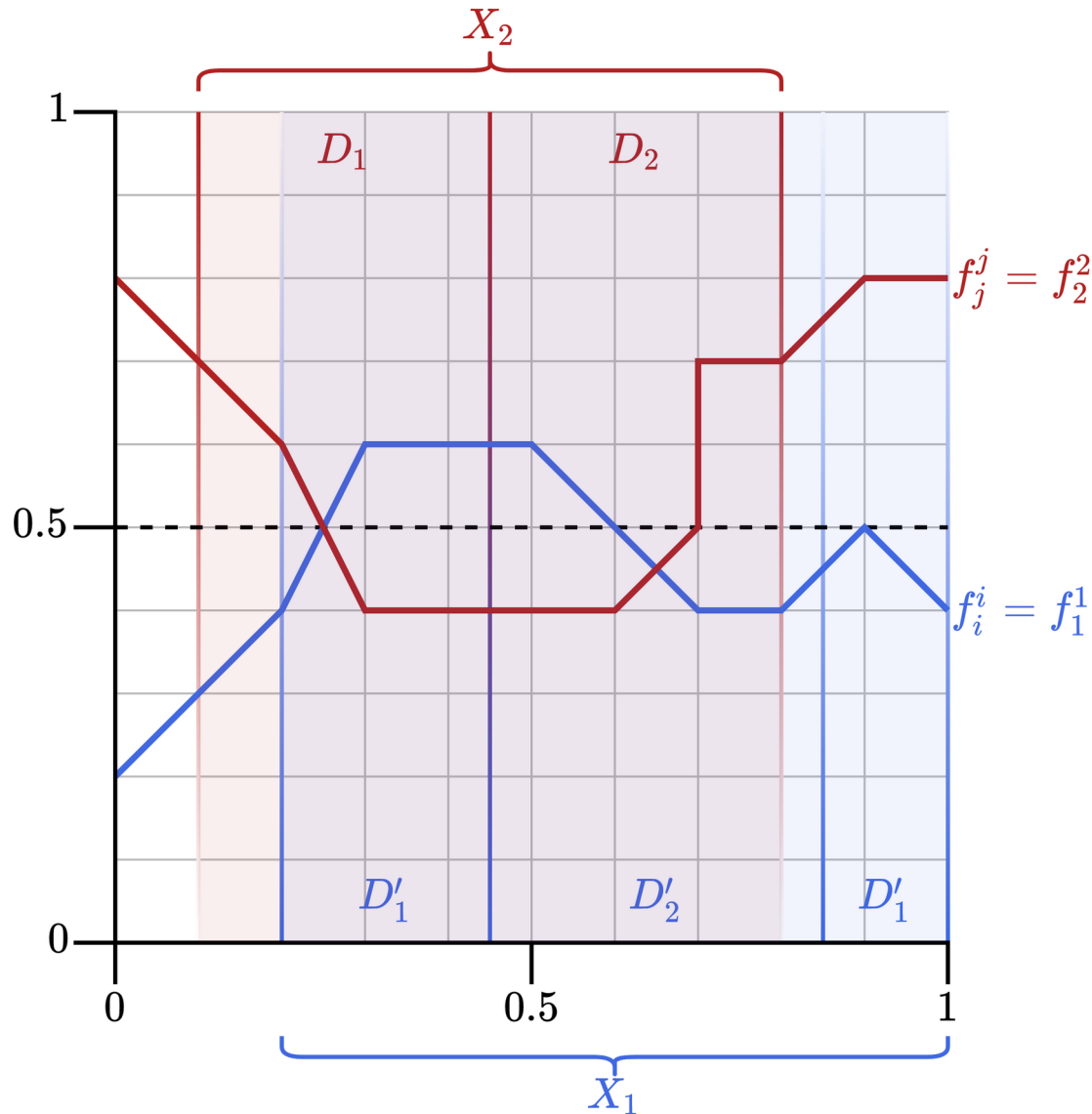
STATE-CUTTING PROTOCOL 2



Best partition:
Divide X_1 into 8
equal districts (same
average support),
barely winning all.

Worst partition:
Divide $[0, 1]$ into 10
equal districts,
losing all. GT =
 $\lfloor 8/2 \rfloor = 4$.

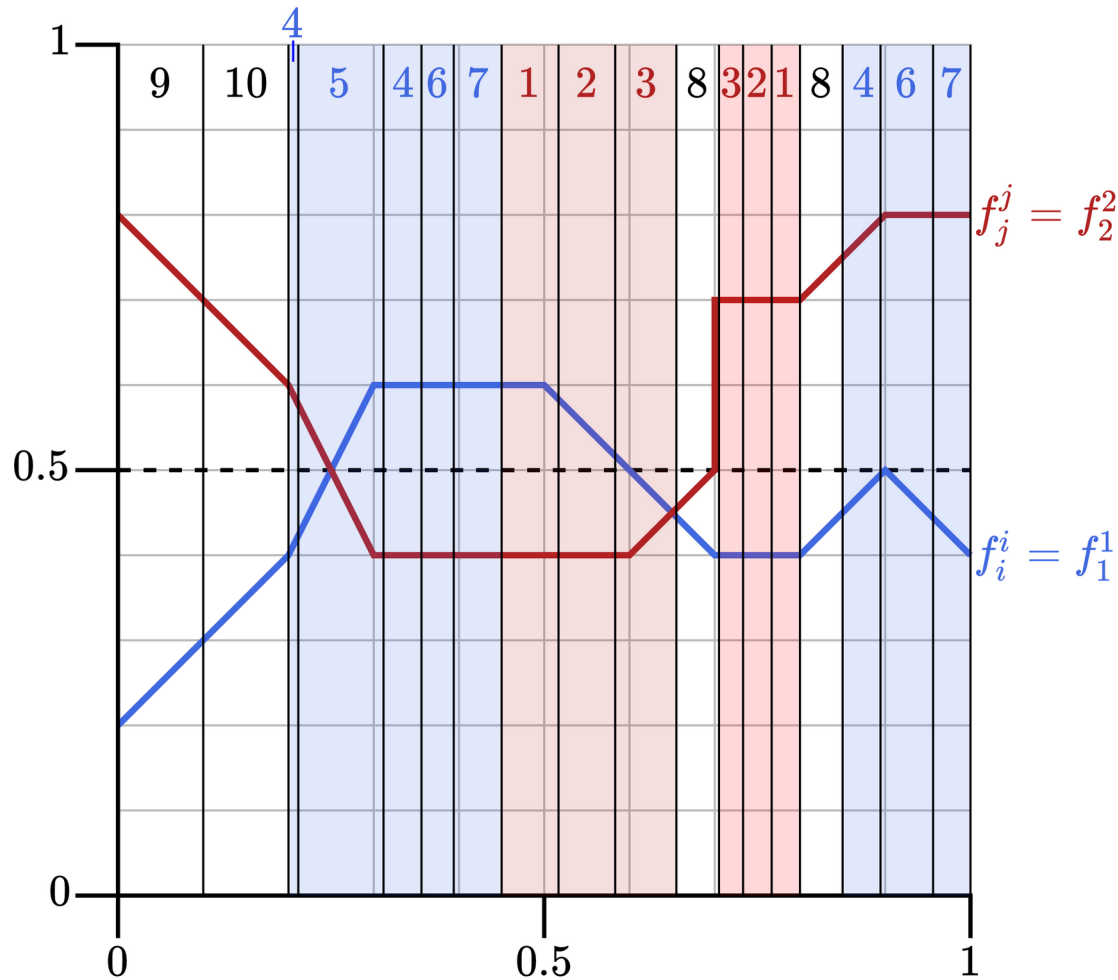
STATE-CUTTING PROTOCOL 2



Best partition:
Divide $[0, 1]$ into 10
equal districts,
winning all.

Worst partition:
Divide X_2 into 7
equal districts, barely
losing all. GT =
 $\lceil 7/2 \rceil + 3 = 6$.

STATE-CUTTING PROTOCOL 2



STATE-CUTTING PROTOCOLS

Pros:

- Guarantees geometric target in the state-cutting model
- Works even when parties disagree substantially over how voters are distributed

Cons:

- Protocols are both (somewhat) specific to the state-cutting model

THRESHOLD ELECTION

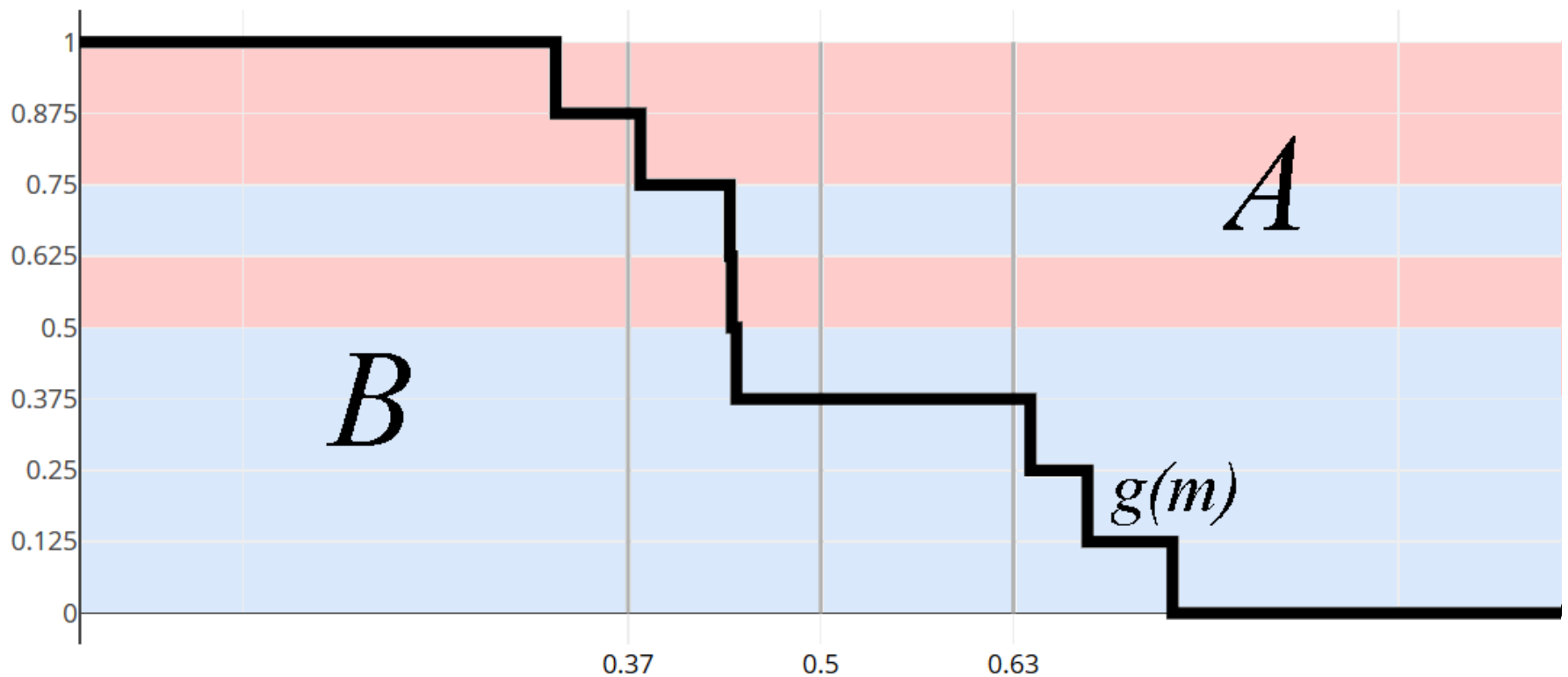
Simple asymmetric protocol with a non-standard election by T-F.

1. Party 1 constructs the partition.
2. Party 2 observes it and chooses $m \in [0.5, 1)$.
3. An “threshold election” is held in each district. If either party gets a majority strictly greater than m , they win. Otherwise, award the district randomly, with equal probabilities for each party.

THRESHOLD ELECTION

Example from 2012 Wisconsin election,
where each row represents a district.

A = Republican party, B = Democratic party.



THRESHOLD ELECTION

Pros:

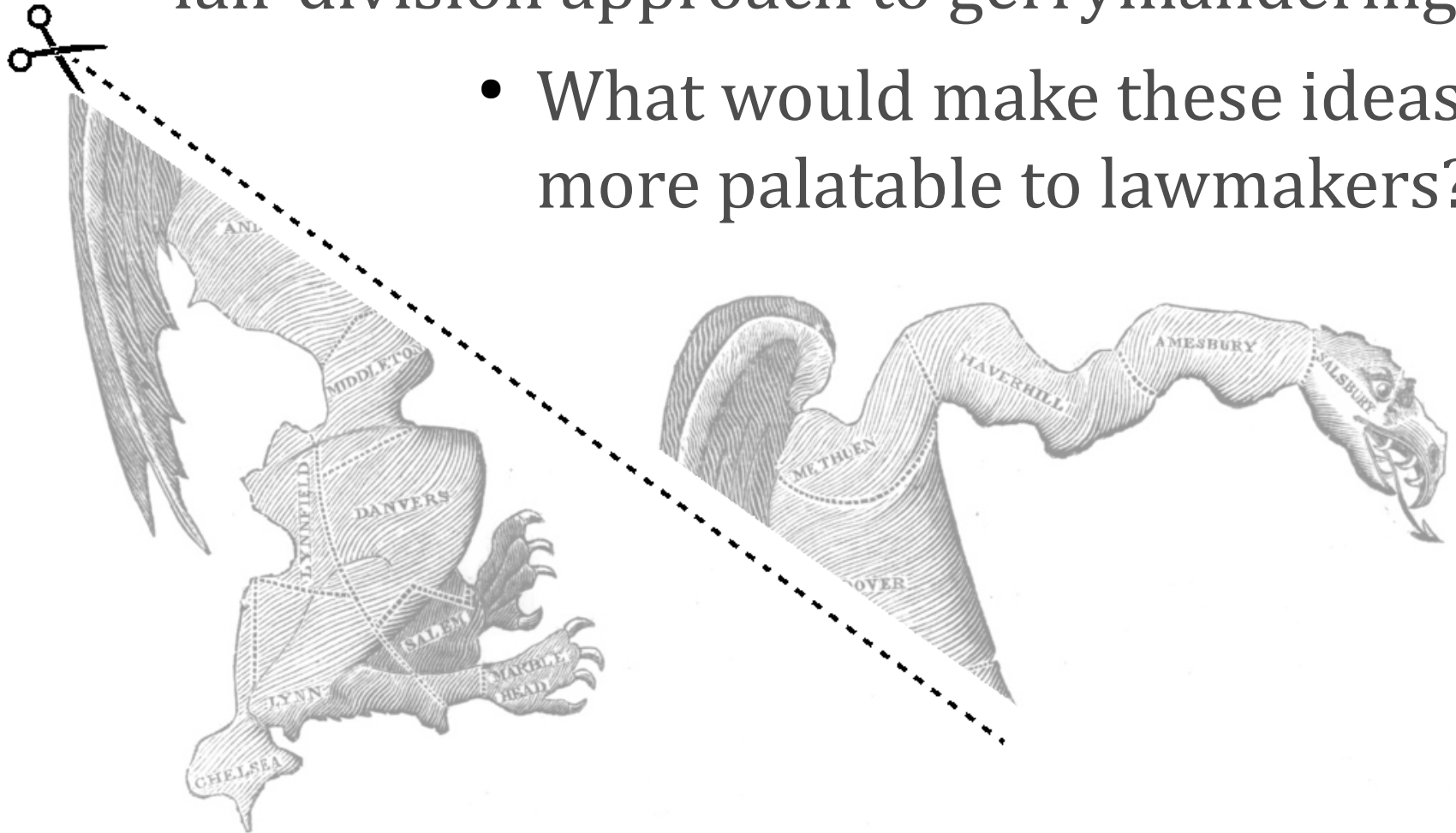
- Simple rules, simple strategies
- Guarantees expected prop. in geometry-free model
- Randomness not required at equilibrium

Cons:

- Not realistically implementable
- Changes election system
- Need to modify substantially to get proportionality in other models

DISCUSSION

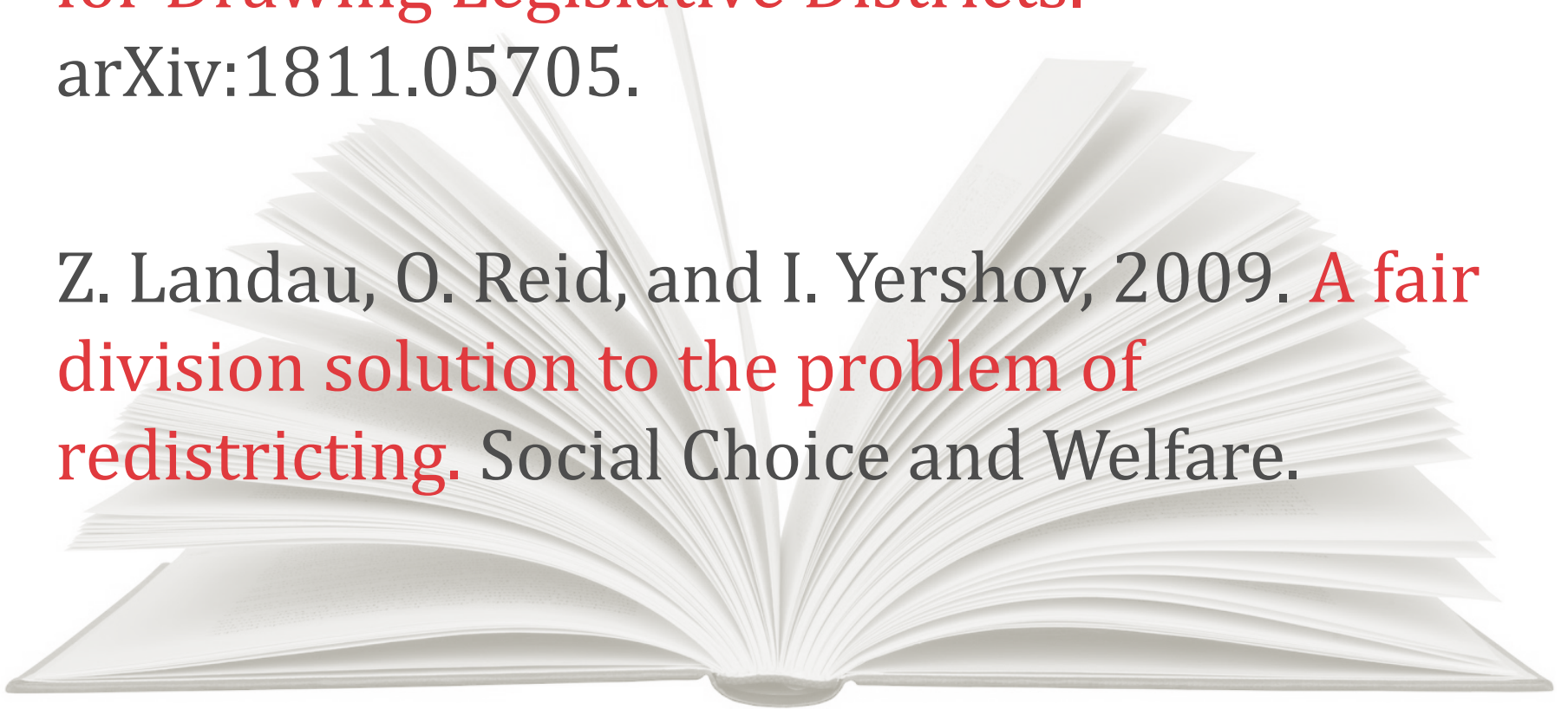
- What are the biggest weaknesses of the fair division approach to gerrymandering?
- What would make these ideas more palatable to lawmakers?



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2018. **An Analysis of a Fair Division Protocol
for Drawing Legislative Districts.**
arXiv:1811.05705.

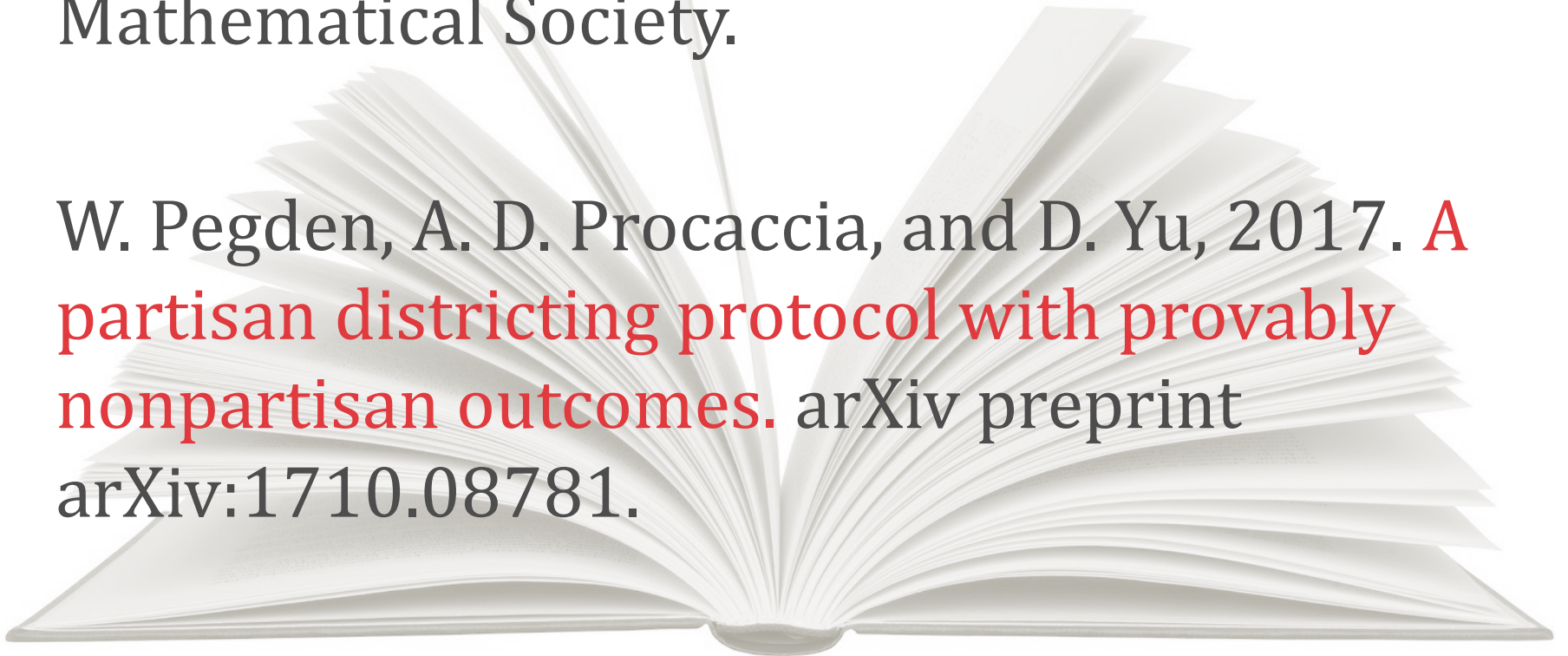
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W. Pegden, A. D. Procaccia, and D. Yu, 2017. **A partisan districting protocol with provably nonpartisan outcomes**. arXiv preprint arXiv:1710.08781.



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