Optimized Democracy

Spring 2022 | Lecture 17
Redistricting As Cake-Cutting
Jamie Tucker-Foltz | Harvard University

REDISTRICTING

Representatives... shall be apportioned among the several States which may be included within this Union, according to their respective Numbers... within every subsequent Term of ten Years

REDISTRICTING

The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State by the Legislature thereof...
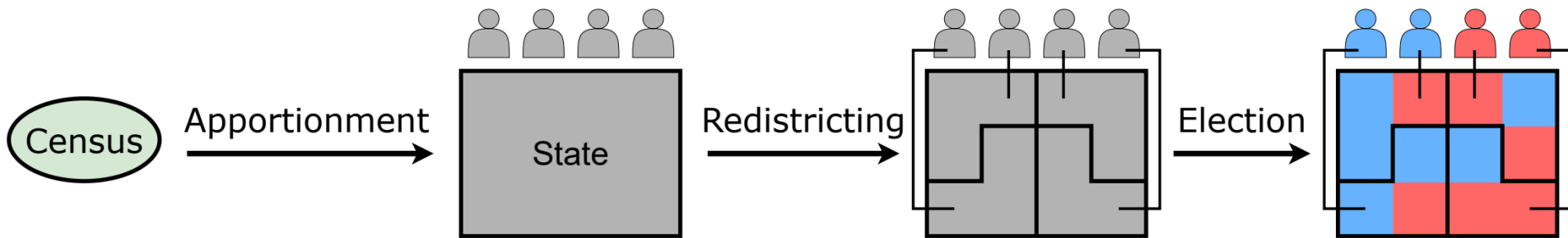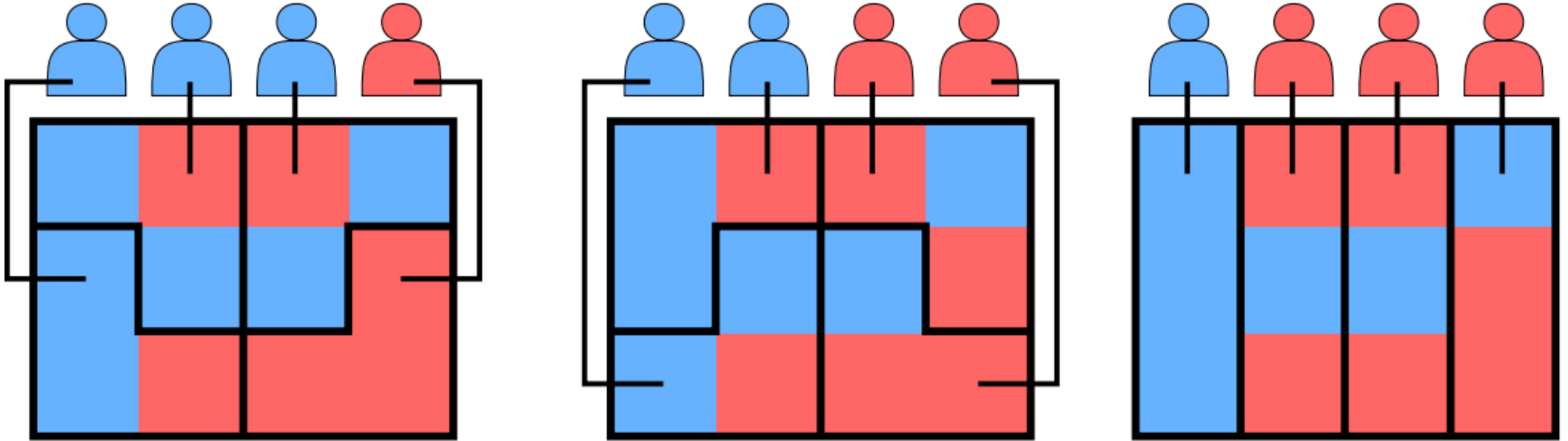
GERRYMANDERING

gerrymander • *verb* • to divide or arrange (a territorial unit) into election districts in a way that gives one political party an unfair advantage

– Merriam-Webster

GERRYMANDERING

Ideas to prevent it:

- Have an independent commission draw fair districts
- Use an interactive protocol with participation from both parties
- Statistically prove a map is gerrymandered
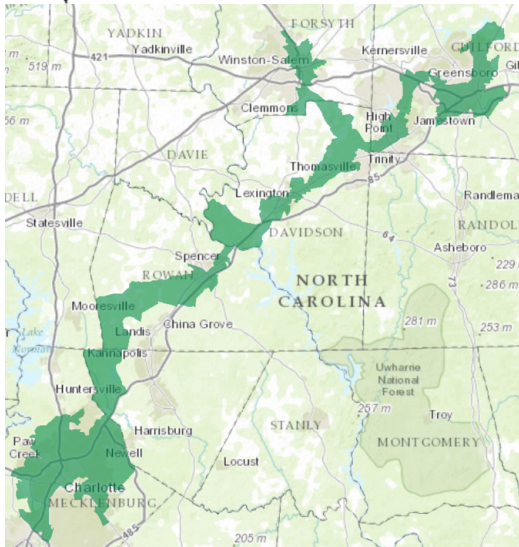
ABSTRACT MODEL

- State S, with set of feasible districts $\mathcal{D} \subseteq 2^S$

- Set of parties $N := \{1, 2, \ldots, n\}$ (today $n := 2$)

- Population measure $\mu : \mathcal{D} \to \mathbb{R}_{\geq 0}$

- For each $j \in N$, distribution function $v^j : \mathcal{D} \to \mathbb{R}_{\geq 0}$

- Target number of districts $m \in \mathbb{Z}_{>0}$

A *partition* is a set P of m disjoint* districts covering S, each of equal measure. The utility of party j is

$$u^j(P) := \left| \{ D \in P \mid \forall i \neq j, \ v^j(D) >^* v^i(D) \} \right|.$$

DISCRETE GRAPH MODEL
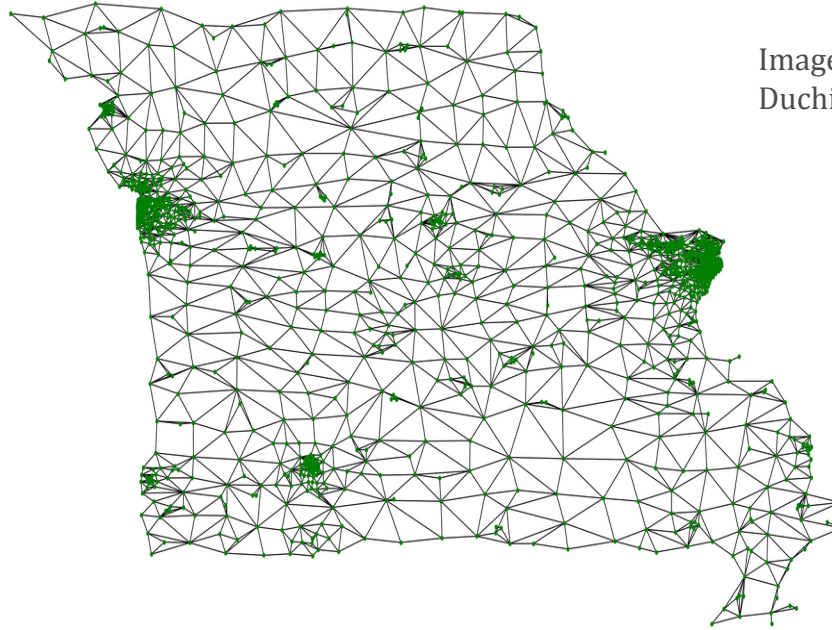
Image credit: DeFord,
Duchin, Solomon, 2020

- Graph G of indivisible census blocks

- $S := V(G)$

- $\mathcal{D} := \{D \subseteq S \,|\, \text{induced subgraph of } D \text{ is connected}\}$

- $\mu(D) := \sum_{b \in D}(\text{population of } b)$

- $v^j(D) := \sum_{b \in D}(\text{number of } j\text{-voters in } b)$

GEOMETRY-FREE MODEL

Continuous model of "placing voters in buckets" with no constraints.
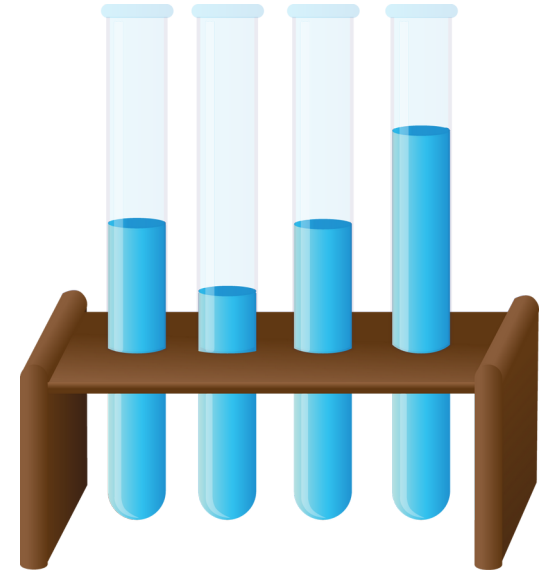
- $S := \bigcup_{j \in N} V_j$
  $V_j := (j, [0, p_j])$
  $\sum_{j \in N} p_j = 1$

- $\mathcal{D} := \{\bigcup_{j \in N}(j, [a_j, b_j]) \mid \forall j \in N, 0 \leq a_j \leq b_j \leq p_j\}$

- $\mu(D) := \sum_{j \in N}(b_j - a_j)$

- $v^j(D) := b_j - a_j$

PROPORTIONALITY

$$\text{For all } j \in N, u^j(P) \geq \left\lfloor m \cdot \frac{v^j(S)}{\sum_{i \in N} v^i(S)} \right\rfloor.$$
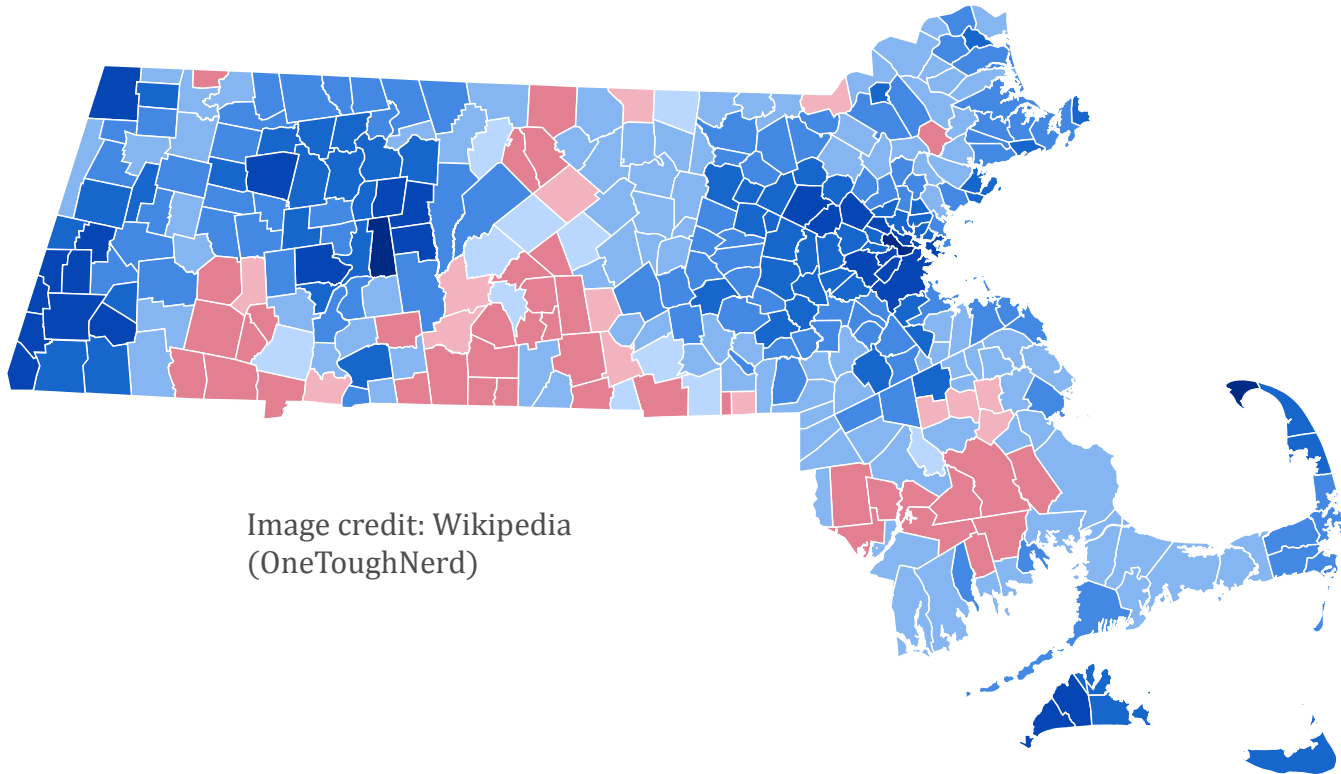
Image credit: Wikipedia
(OneToughNerd)

GEOMETRIC TARGET

For all $j \in N$, let $P_{\max}^j$ be a partition maximizing u^j and let $P_{\min}^j$ be a partition minimizing u^j. Then

$$u^j(P) \geq \left\lfloor \frac{u^j(P_{\max}^j) + u^j(P_{\min}^j)}{2} \right\rfloor.$$

GEOMETRIC TARGET

Theorem: In the geometry-free model, a partition satisfies proportionality if and only if it satisfies the geometric target (up to ties).

Proof:
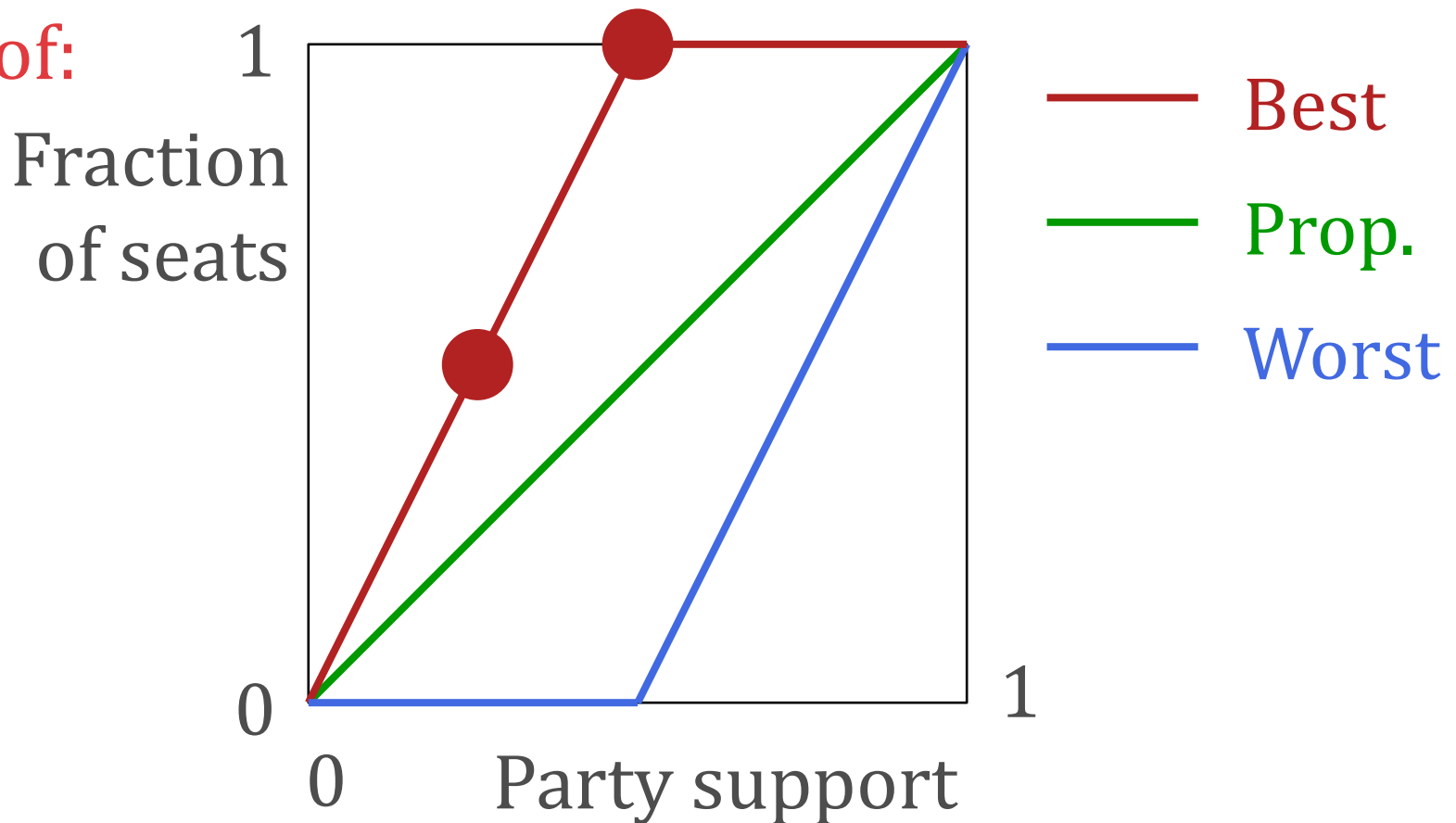
LRY PROTOCOL

Interactive protocol by Landau, Reid, and Yershov that uses a neutral administrator.

1. Administrator presents both parties with a series of bipartitions $(L_1, R_1), (L_2, R_2), \ldots, (L_{m-1}, R_{m-1})$ of S, such that each $L_i \subseteq L_{i+1}$.

Image credits:
Landau and Su, 2015
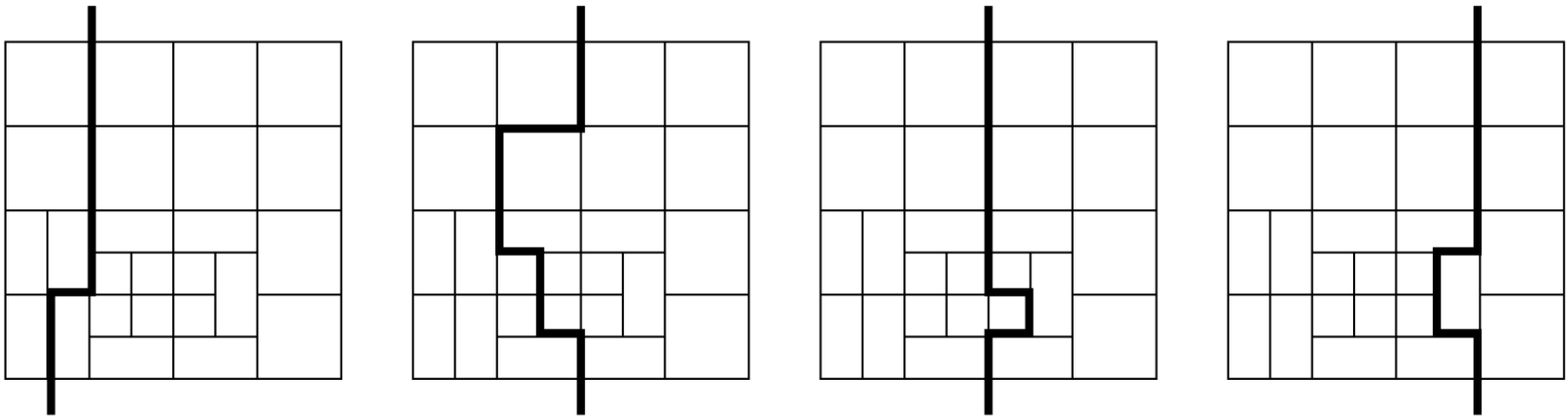
LRY PROTOCOL

2. For each $i \in [m]$, each party is asked, "Would you rather redistrict L_i, with the other party redistricting R_i, or vice versa?"
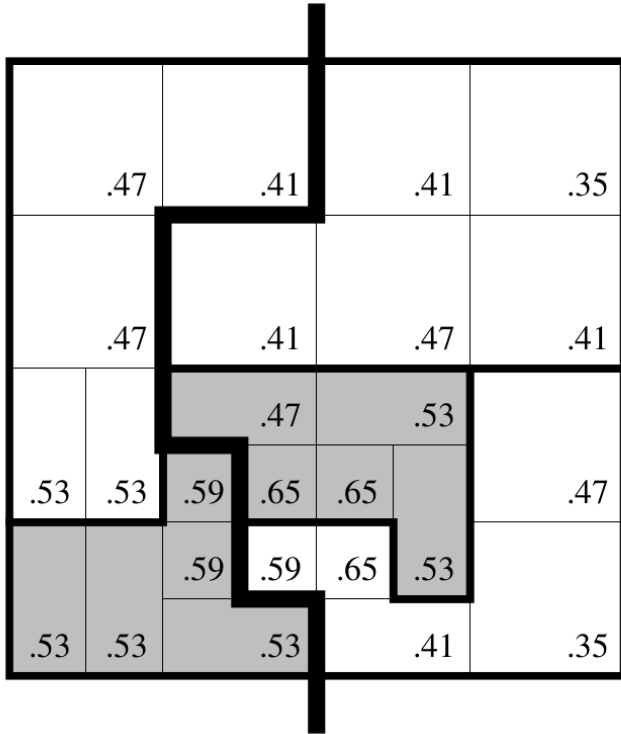
LRY PROTOCOL

3. Try to find an i such that one party prefers redistricting L_i and the other prefers redistricting R_i. If no such i exists, randomly select an outcome at the cross-over point.

Theorem (Good Choice Property): Restricting the feasible set of partitions to respect a given split, a party's preferred choice satisfies its geometric target.

Proof:
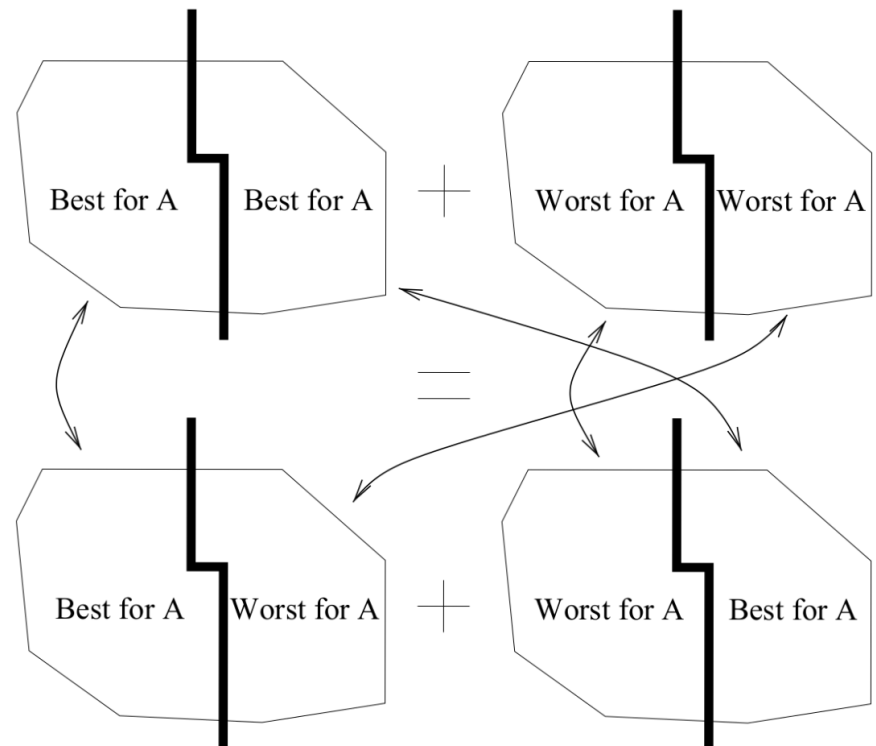
LRY PROTOCOL

Pros:

- Realistically implementable

- Simple party participation

- Guaranteed to be within 2 of prop. / geometric target in geometry-free model

Cons:

- Relies heavily on neutrality of the administrator

- Can be arbitrarily far from geometric target in grid-based model

CUT AND FREEZE

By Pegden and Procaccia: partition, freeze, and re-partition until all districts are frozen.
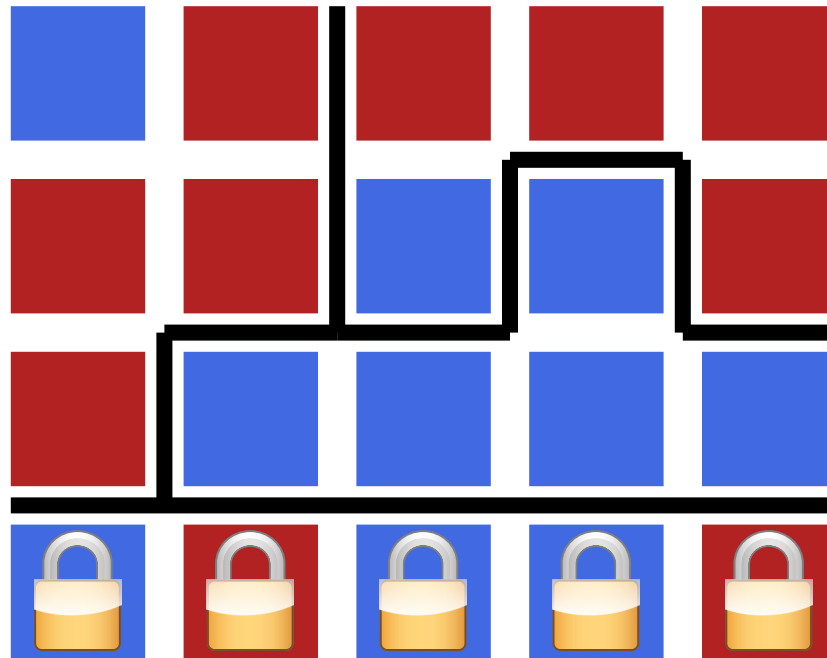
CUT AND FREEZE

Theorem: In the geometry-free model, under optimal play, each party can guarantee a number of seats as in the following graphs.
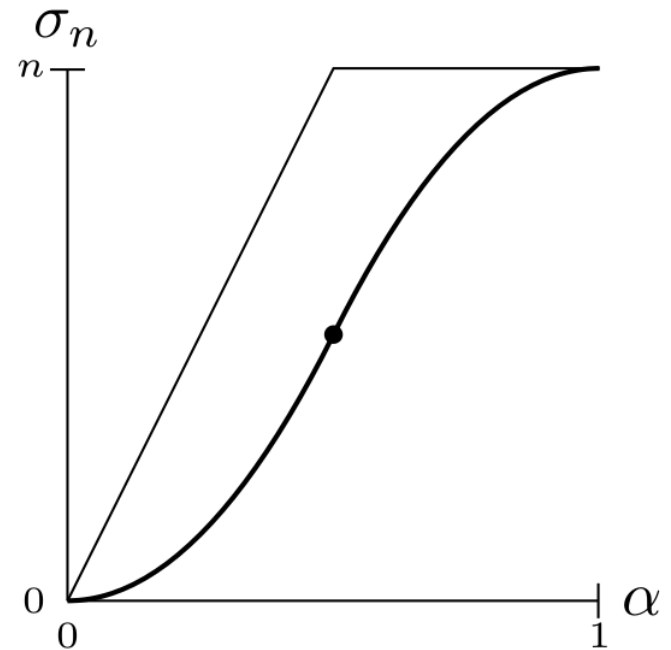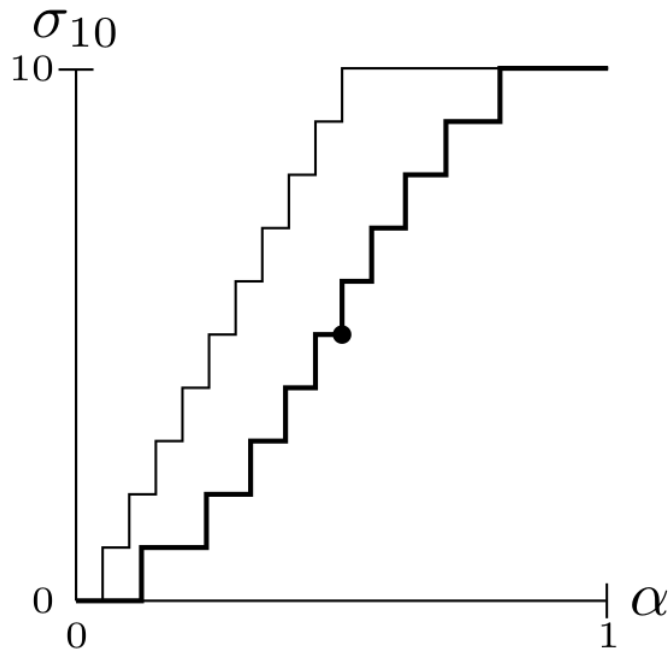
CUT AND FREEZE

Pros:

- Realistically implementable
- Approximate proportionality in geometry-free model
- Hard to pack specific groups into one district

Cons:

- Requires complicated strategies
- Requires several rounds of interaction

STATE-CUTTING MODEL 1

Cake-cutting analogue introduced by Benade, Procaccia, and T-F.

- $S := [0, 1]$
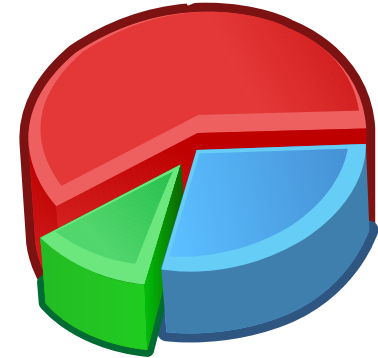
- $\mathcal{D} := \{\text{finite unions of closed intervals}\}$

- $\mu :=$ Lebesgue measure

- $v^j(D) := \int_D f^j(D)$ where, for all $x \in S$,

$$\sum_{j \in N} f^j(x) = 1$$

STATE-CUTTING PROTOCOL 1

1. Ask each party j to construct an optimal partition P_j.

2. Construct a sequence of partitions from P_1 to P_2, each differing from the previous one on at most two districts.

3. Select an intermediate partition that satisfies the geometric targets of both parties.

STATE-CUTTING PROTOCOL 1

How to achieve step 2? Bubble sort!
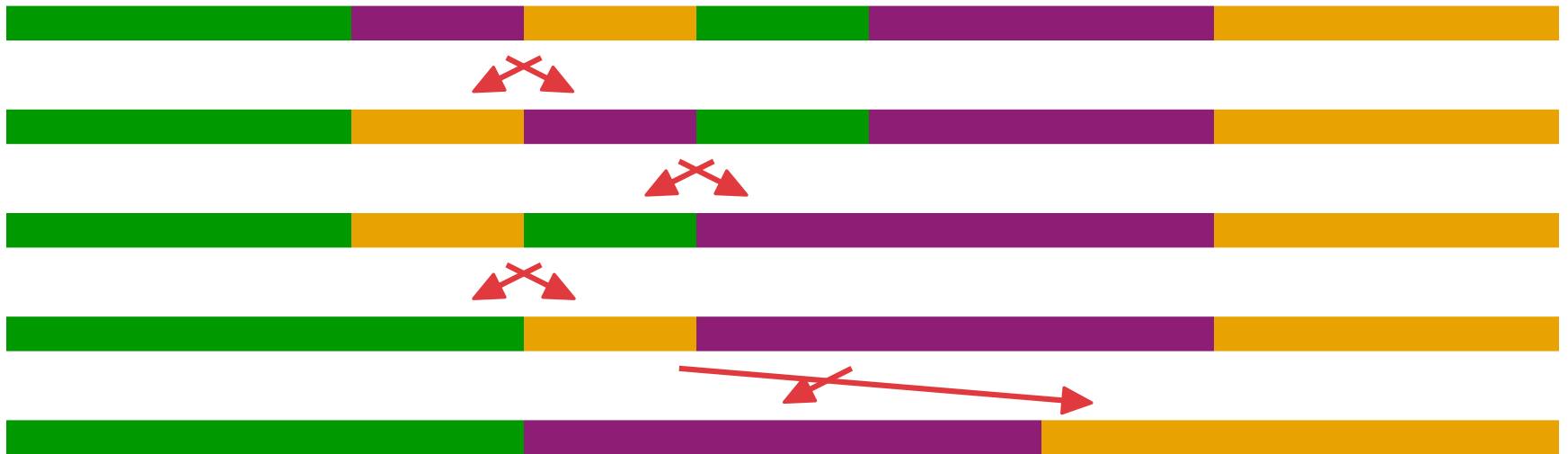
Can transition from P_1 to P_2 via the simplest possible partition $\{[\frac{k-1}{m}, \frac{k}{m}] \mid k \in [m]\}$ (the bottom one). Each swap modifies only two districts.

STATE-CUTTING PROTOCOL 1

Theorem: If two partitions differ on at most two districts, the balance of power can differ by at most one.

Proof: Suppose P and P' differ on districts $D_1, D_2 \in P$ and $D_1', D_2' \in P'$. Suppose party 1 has a majority in D_1 and D_2, but a minority in D_1' and D_2'. Then:

$$\frac{1}{m} < v^1(D_1) + v^1(D_2) = v^1(D_1 \cup D_2)$$

$$= v^1(D_1' \cup D_2') = v^1(D_1') + v^1(D_2') < \frac{1}{m}$$

Contradiction.

STATE-CUTTING MODEL 2

Now parties are allowed to disagree over the distribution of voters!

- $S := [0, 1]$
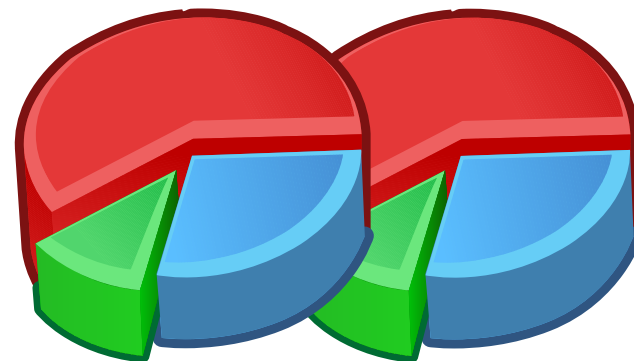
- $\mathcal{D} := \{\text{finite unions of closed intervals}\}$

- $\mu :=$ Lebesgue measure

- $v_i^j(D) := \int_D f_i^j(D)$ where, for all $x \in S$ and $i \in N$,

$$\sum_{j \in N} f_i^j(x) = 1$$

STATE-CUTTING MODEL 2
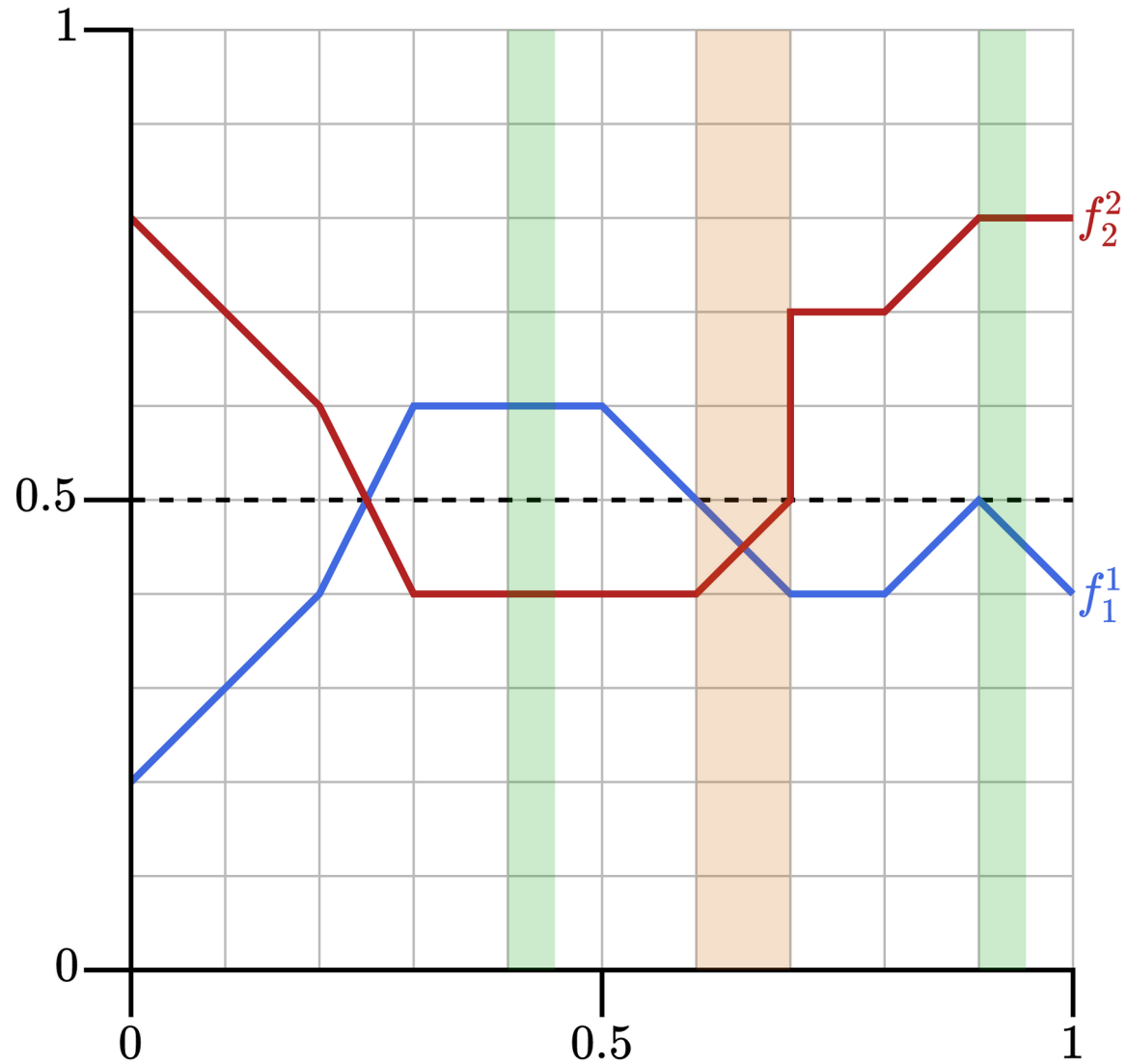
STATE-CUTTING PROTOCOL 2

Theorem: Even when parties disagree, there always exists a partition satisfying the geometric targets of both parties:

$$u_i^i(P) \geq \left| \frac{\min\limits_{P'} u_i^i(P') + \max\limits_{P'} u_i^i(P')}{2} \right|$$

STATE-CUTTING PROTOCOL 2

1. Each party i computes a maximal set $X_i \subseteq S$ such that $m\mu(X_i) \in \mathbb{Z}$ and $v_i^i(X_i) = \frac{\mu(X_i)}{2}$.

2. Let i be the party with the larger X_i set, and let j be the other party.

3. Party j divides X_j into two pieces of equal size and equal party support according to j.

4. Party i chooses a piece for j to redistrict.

5. Party i redistricts the rest of S.

STATE-CUTTING PROTOCOL 2
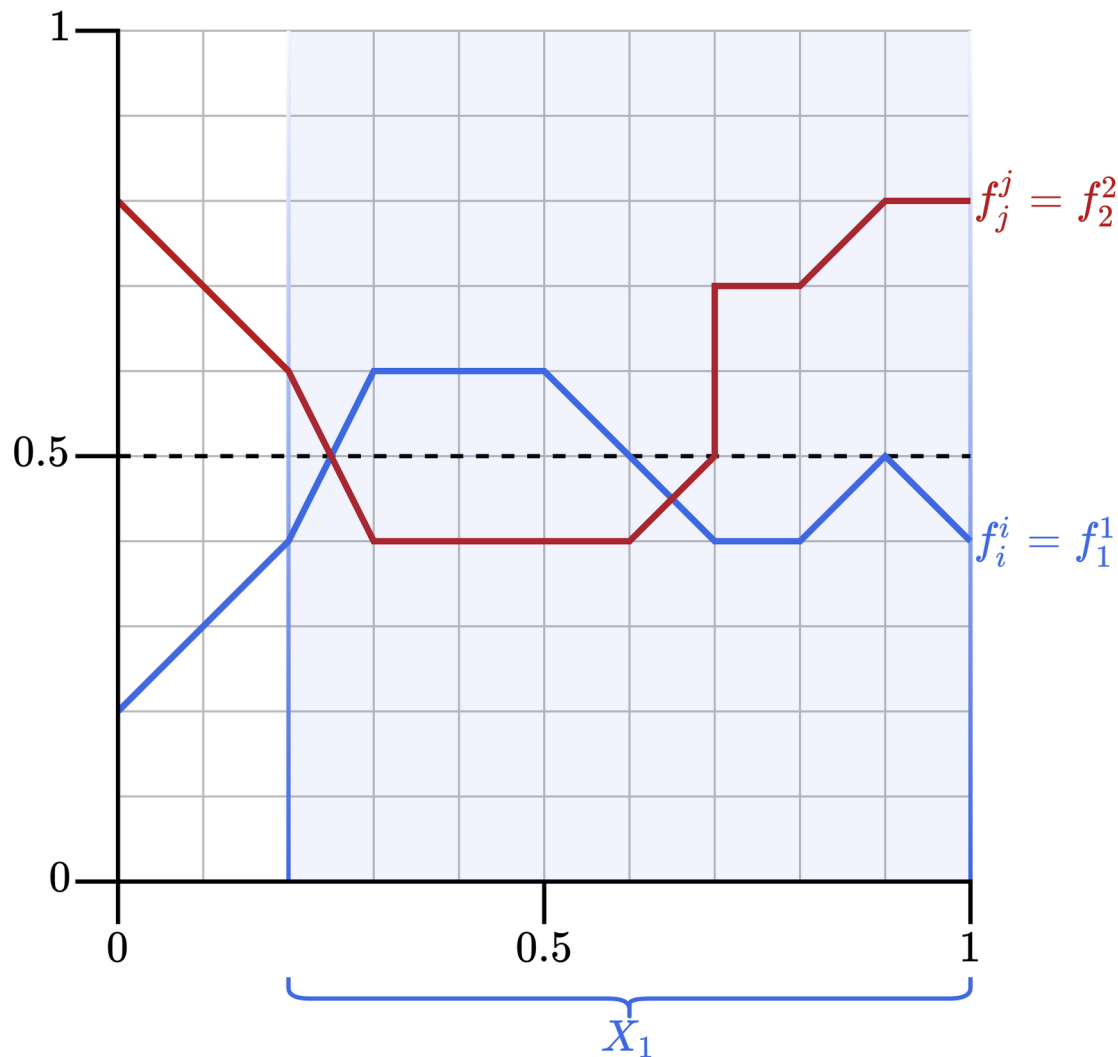
Best partition:
Divide X_1 into 8
equal districts (same
average support),
barely winning all.

Worst partition:
Divide $[0, 1]$ into 10
equal districts,
losing all. GT =
$\lfloor 8/2 \rfloor = 4$.

$f_j^j = f_2^2$

$f_i^i = f_1^1$

X_1

STATE-CUTTING PROTOCOL 2
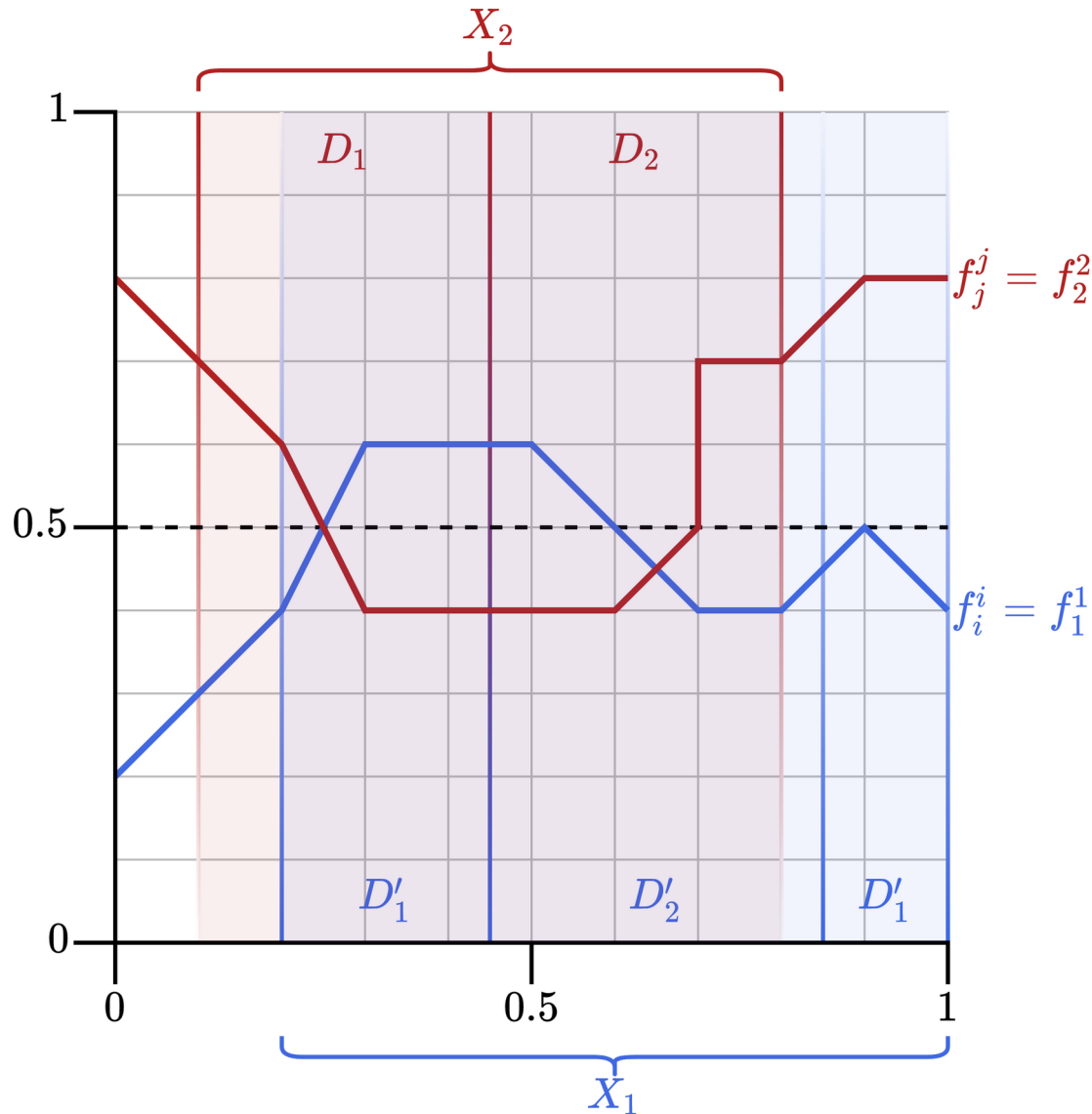
Best partition: Divide $[0, 1]$ into 10 equal districts, winning all.

Worst partition: Divide X_2 into 7 equal districts, barely losing all. GT = $\lfloor 7/2 \rfloor + 3 = 6$.

STATE-CUTTING PROTOCOL 2
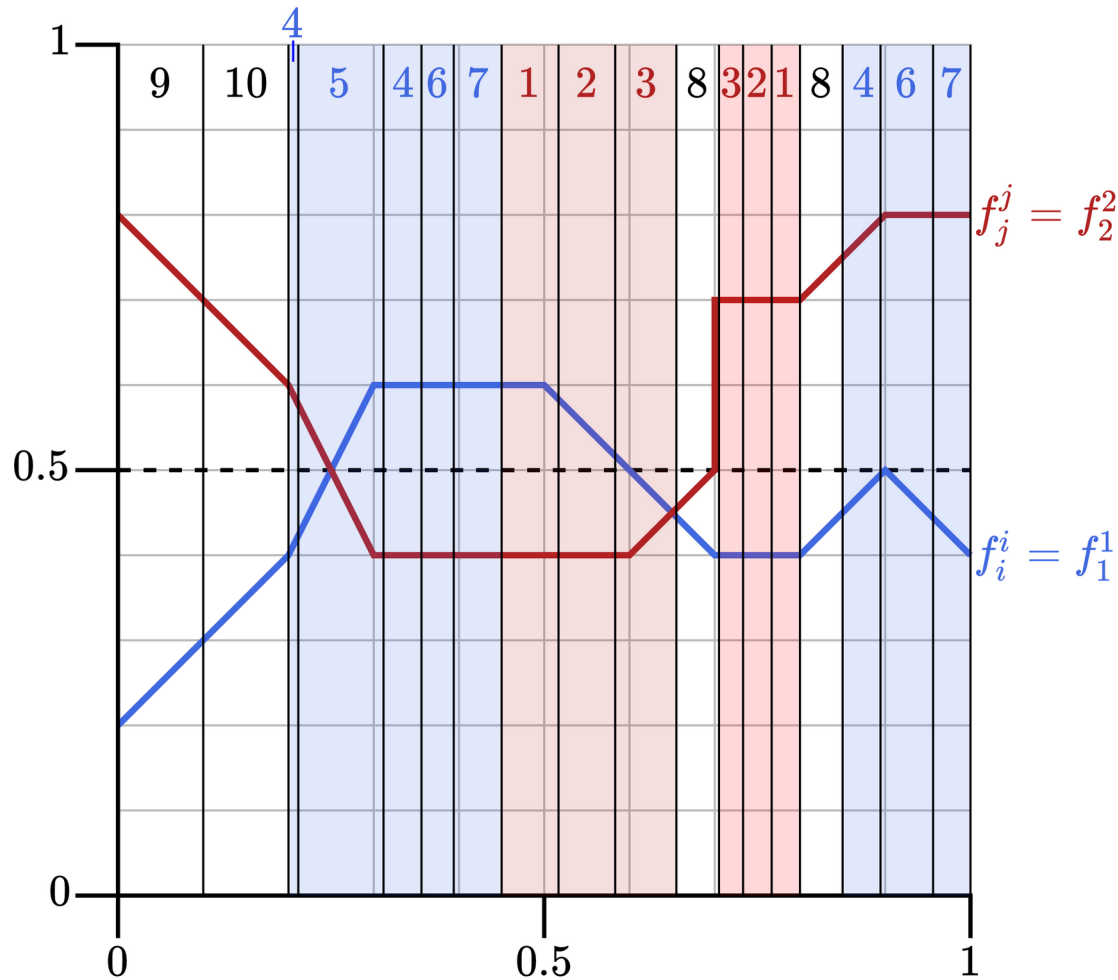
STATE-CUTTING PROTOCOLS

Pros:

- Guarantees geometric target in the state-cutting model

- Works even when parties disagree substantially over how voters are distributed

Cons:

- Protocols are both (somewhat) specific to the state-cutting model
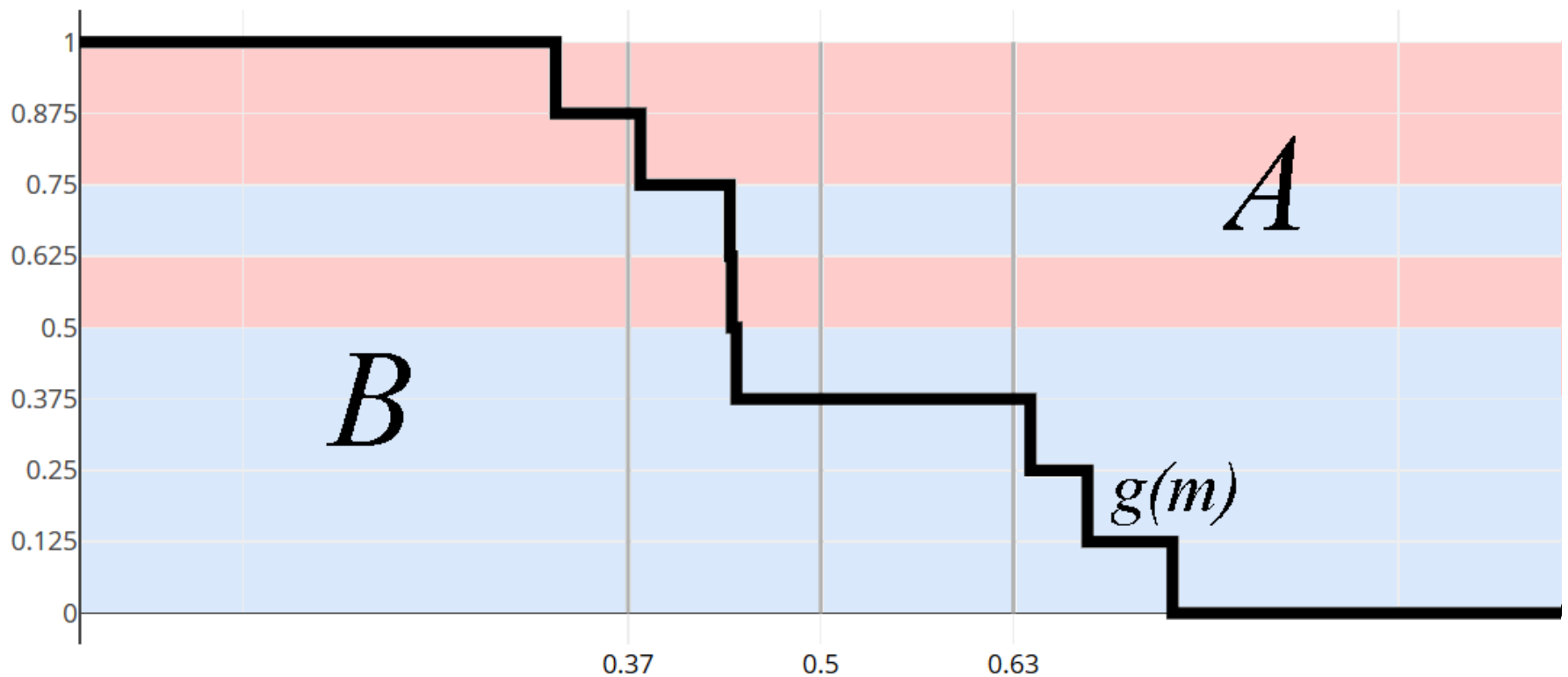
THRESHOLD ELECTION

Simple asymmetric protocol with a non-standard election by T-F.

1. Party 1 constructs the partition.

2. Party 2 observes it and chooses $m \in [0.5, 1)$.

3. An "threshold election" is held in each district. If either party gets a majority strictly greater than m, they win. Otherwise, award the district randomly, with equal probabilities for each party.

THRESHOLD ELECTION

Example from 2012 Wisconsin election, where each row represents a district.

$A =$ Republican party, $B =$ Democratic party.

THRESHOLD ELECTION

Pros:

- Simple rules, simple strategies
- Guarantees expected prop. in geometry-free model
- Randomness not required at equilibrium

Cons:

- Not realistically implementable
- Changes election system
- Need to modify substantially to get proportionality in other models

DISCUSSION

- What are the biggest weaknesses of the fair division approach to gerrymandering?

  - What would make these ideas more palatable to lawmakers?
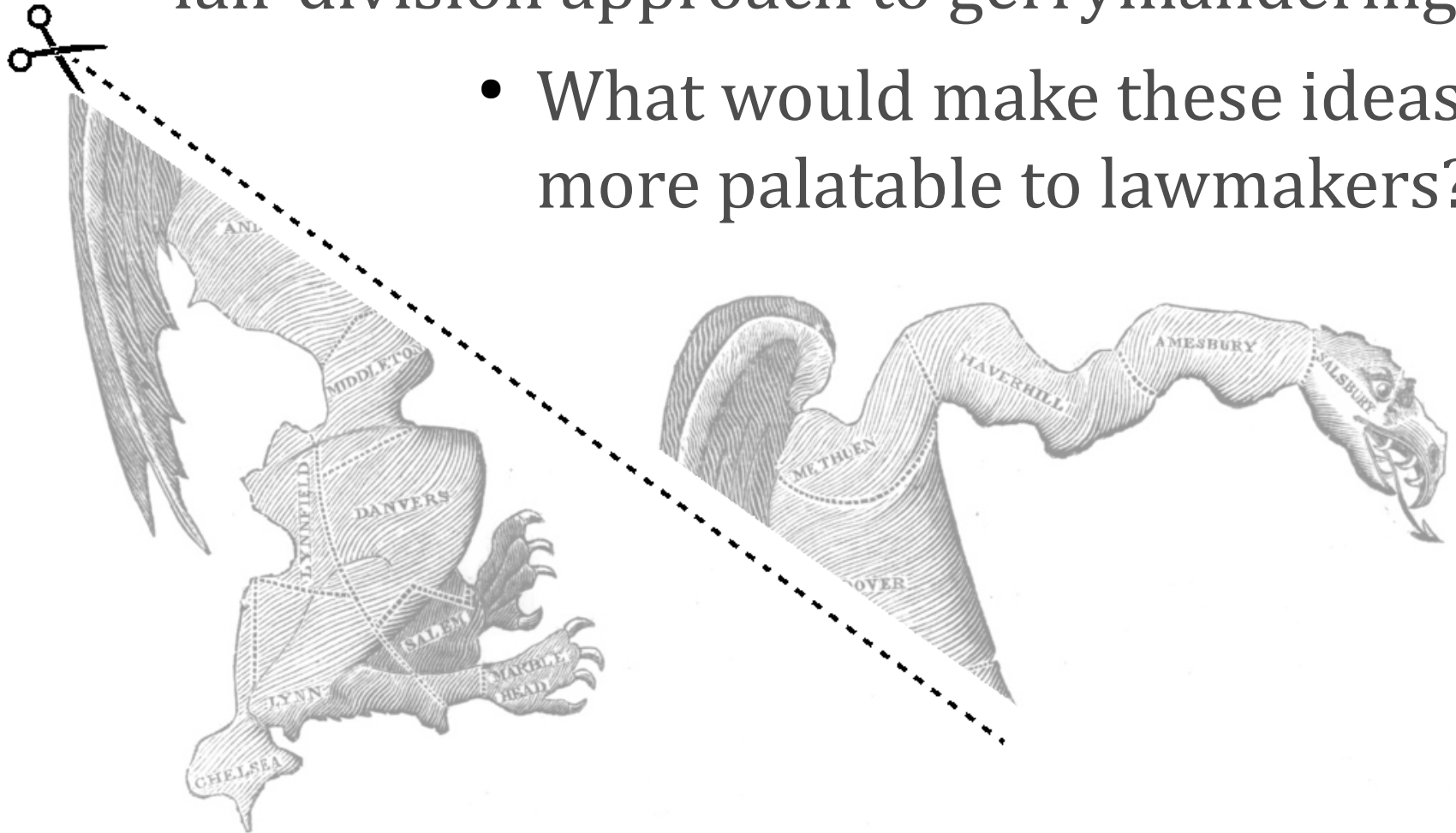
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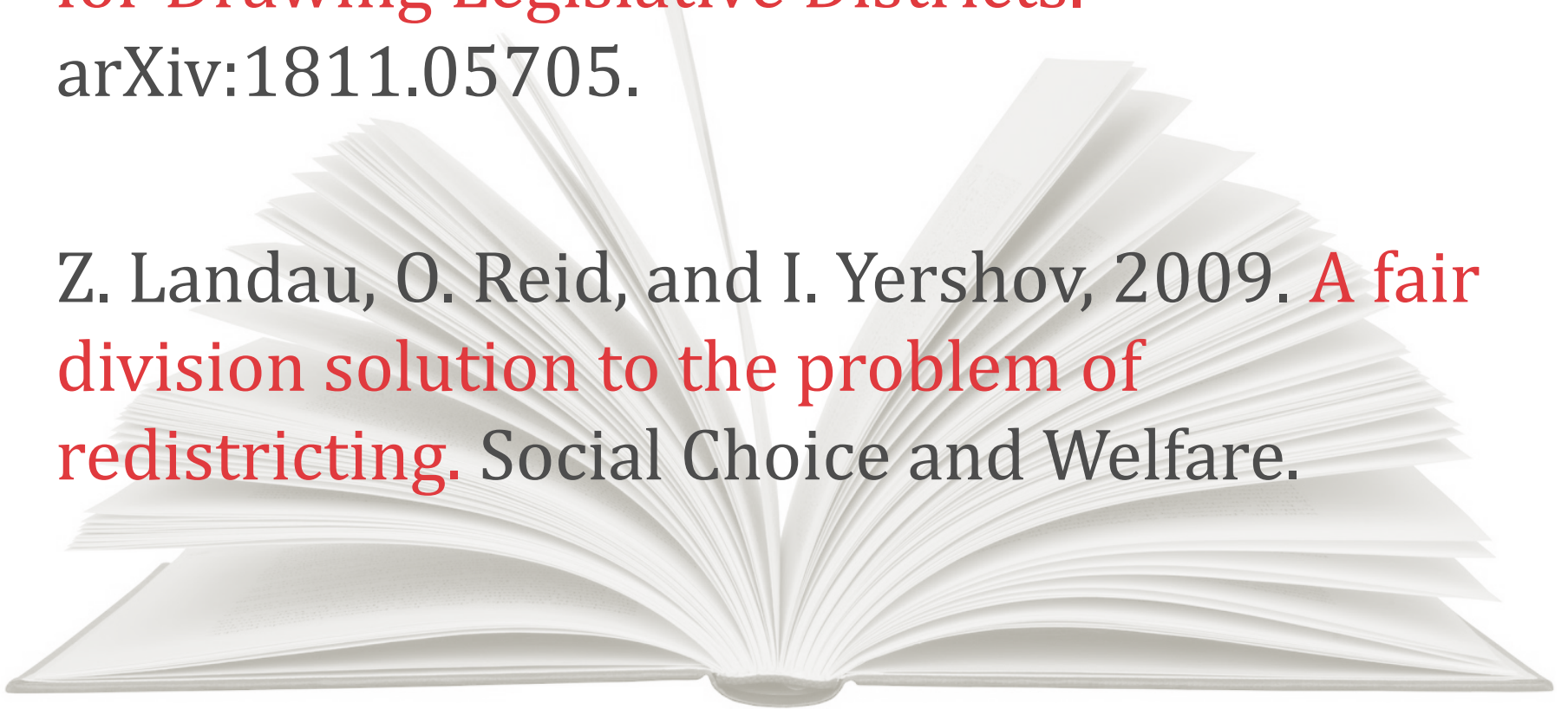
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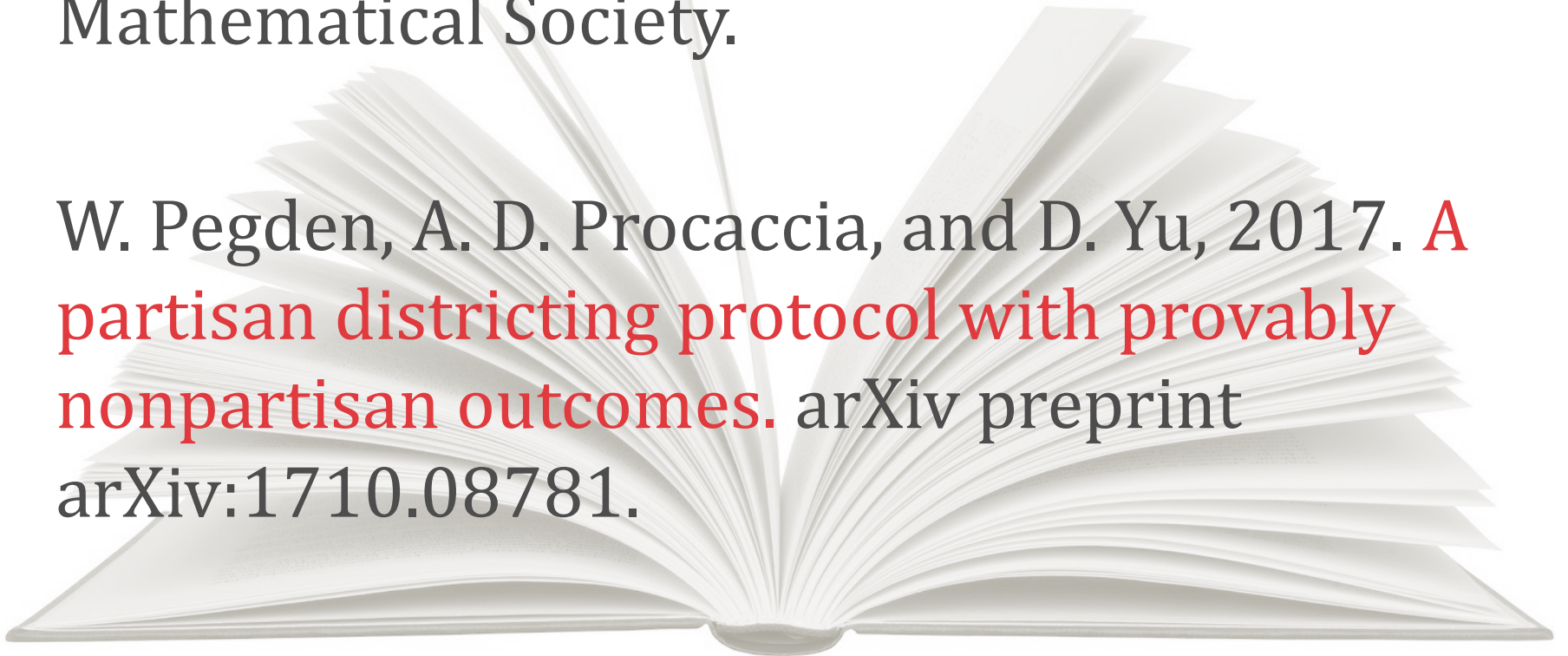
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