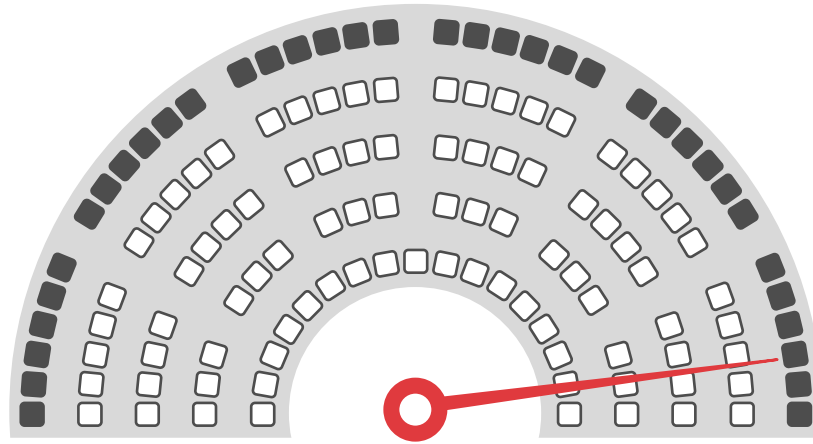




# *Optimized Democracy*

Spring 2022 | Lecture 16

Apportionment in the 20<sup>th</sup> Century

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# REMINDER: THE MODEL

- Set of states  $N = \{1, \dots, n\}$
- $K$  seats to be allocated
- Each state has population  $p_i$ , and the total population is  $P = \sum_{i=1}^n p_i$
- The **standard quota** of state  $i$  is  $q_i = \frac{p_i}{P} \cdot K$
- The **upper quota** of  $i$  is  $\lceil q_i \rceil$ , and the **lower quota** is  $\lfloor q_i \rfloor$
- Let  $k_i$  be the number of seats allocated to  $i$

# THE CENSUS OF 1910

- The 1910 census counted 91 million people, 20% more than 1900, and showed migration from rural states to urban centers
- At the urging of Prof. Walter F. Willcox from Cornell, Congress adopted the Webster Method in 1912, but increased the number of seats from 386 to 433 such that no state would lose seats (but the power of rural states still eroded due to seat inflation)
- Two additional seats were reserved for Arizona and New Mexico, which had not yet joined the union, for a total of 435 — the number still used today

# HUNTINGTON-HILL METHOD

- Define the rounding function

$$f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x < \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil} \\ \lceil x \rceil & \text{if } x \geq \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil} \end{cases}$$

- **The Huntington-Hill Method:**
  - Takes a desired number of seats  $K$
  - Finds a divisor  $D$  such that  $\sum_{i=1}^n f(\hat{q}_i) = K$ , where  $\hat{q}_i = p_i/D$  is the modified quota
  - Each state is allocated  $k_i = f(\hat{q}_i)$

# DIVISOR METHODS

- By changing the rounding function  $f$  one can obtain a family of apportionment methods called **divisor methods**
- $f$  is assumed to satisfy two conditions:  $f(x) = x$  if  $x$  is an integer and  $f(x) \geq f(y)$  if  $x \geq y$
- **Theorem:** Fixing  $f$ , if  $D$  and  $D'$  are two different divisors yielding apportionments  $k_1, \dots, k_n$  and  $k'_1, \dots, k'_n$  then  $k_i = k'_i$  for all  $i \in N$
- **Proof:** (Essentially copied from Jefferson.)
  - Assume w.l.o.g. that  $D \leq D'$ , then  $p_i/D \geq p_i/D'$  for all  $i \in N$
  - We conclude that  $k_i = f(p_i/D) \geq f(p_i/D') = k'_i$  for all  $i \in N$
  - It also holds that  $\sum_{i \in N} k_i = K = \sum_{i \in N} k'_i$
  - It can't be the case that  $k_i > k'_i$  for some  $i \in N$  ■

# DIVISOR METHODS

- **Theorem:** A divisor method is the Huntington-Hill Method if and only if for all  $i, j \in N$  such that  $p_i/k_i \leq p_j/k_j$ ,

$$\frac{p_i/k_i}{p_j/k_j} > \frac{p_j/(k_j + 1)}{p_i/(k_i - 1)}$$

|       |           | Ratio 0.848 |           | Ratio 0.831 |           |
|-------|-----------|-------------|-----------|-------------|-----------|
| State | $p_i$     | $k_i$       | $p_i/k_i$ | $k_i$       | $p_i/k_i$ |
| 1     | 3,300,000 | 16          | 206,250   | 17          | 194,117   |
| 2     | 700,000   | 4           | 175,000   | 3           | 233,333   |
| Total | 4,000,000 | 20          | ...       | 20          | ...       |

# PROOF OF THEOREM

- We'll prove the “only if” direction
- The modified quota  $\hat{q}_i = p_i/D$  is rounded down to  $k_i$  when  $k_i \leq p_i/D < \sqrt{k_i(k_i + 1)}$  and rounded up to  $k_i$  when  $k_i \geq p_i/D \geq \sqrt{k_i(k_i - 1)}$

- It follows that

$$\sqrt{k_i(k_i - 1)} \leq p_i/D < \sqrt{k_i(k_i + 1)}$$

- Equivalently,

$$\frac{k_i(k_i - 1)}{p_i^2} \leq \frac{1}{D^2} < \frac{k_i(k_i + 1)}{p_i^2}$$

- This holds for all  $i, j \in N$ , therefore

$$\frac{k_i(k_i - 1)}{p_i^2} < \frac{k_j(k_j + 1)}{p_j^2}$$

- This is equivalent to the desired property ■

# A FINAL HISTORICAL DETOUR

- Joseph A. Hill, a statistician at the Census Bureau, initially suggested the method based on the idea of minimizing “relative differences” in citizens per seat
- Edward V. Huntington, a Harvard math professor, formalized the idea and showed that it’s equivalent to rounding at the geometric mean
- This shows that the method slightly favors small states: A fractional seat of 0.41 is needed to be rounded from 1 to 2, whereas 0.49 is needed to be rounded from 31 to 32

# A FINAL HISTORICAL DETOUR

- In 1921 Congress considered bills based on Webster and Huntington-Hill, but both were rejected; ultimately there was no reapportionment that decade (!)
- In 1929 Congress turned to the National Academy of Sciences
- The committee that was formed favored Huntington-Hill because it minimizes relative differences and because it “occupies mathematically a neutral position with respect to the emphasis on larger and smaller states”

# A FINAL HISTORICAL DETOUR

- Fortunately, based on the census of 1930 there was no disagreement between Webster and Huntington-Hill, and the consensus apportionment was enacted
- Under the census of 1940, Huntington-Hill gave Arkansas an extra seat and Webster gave Michigan an extra seat
- Since Arkansas was Democratic and Michigan was Republican, this became a partisan issue
- In 1941, President Roosevelt (a Democrat) signed into law an act designating Huntington-Hill as the permanent apportionment method

# POPULATION MONOTONICITY

- Suppose there are two censuses where the populations in the second are denoted by  $p'_1, \dots, p'_n$  and the apportionment by  $k'_1, \dots, k'_n$  (it could be that  $K \neq K'$ )
- An apportionment method is **population monotonic** if  $k_i < k'_i$  and  $k_j > k'_j$  implies that  $p_i < p'_i$  or  $p_j > p'_j$
- **Theorem:** All divisor methods are population monotonic

# PROOF OF THEOREM

- Suppose  $k_i < k'_i$  and  $k_j > k'_j$
- It follows that  $p_i/D < p'_i/D'$  and  $p_j/D > p'_j/D'$
- Rearranging, we get

$$p'_i > \left(\frac{D'}{D}\right) p_i \quad \text{and} \quad p'_j < \left(\frac{D'}{D}\right) p_j$$

- If  $D'/D \leq 1$  then  $p'_j < p_j$  and if  $D'/D \geq 1$  then  $p'_i > p_i$  ■

# HOUSE MONOTONICITY

- An apportionment method is **house monotonic** if  $K' > K$ , with all other variables unchanged, implies  $k'_i \geq k_i$  for all  $i \in N$
- **Theorem:** Any population monotonic apportionment method is house monotonic
- **Corollary:** All divisor methods are house monotonic

# PROOF OF THEOREM

- Let  $K' > K$ , but  $p_i = p'_i$  for all  $i \in N$
- Let  $j$  such that  $k'_j > k_j$  (it must exist)
- For all  $i \neq j$ , if it was the case that  $k'_i < k_i$  then population monotonicity would imply that  $p'_j > p_j$  or  $p'_i < p_i$ , which is false
- We conclude that  $k'_i \geq k_i$  for all  $i \in N$  ■

# THE QUOTA CRITERION

- An apportionment method satisfies the **quota criterion** if for all  $i \in N$ ,

$$\lfloor q_i \rfloor \leq k_i \leq \lceil q_i \rceil$$

## Poll

Of the five methods we discussed (Hamilton, Jefferson, Adams, Webster, Huntington-Hill), how many satisfy the quota criterion?

- 0    • 1    • 2    • 3    • 4    • 5



# AN IMPOSSIBILITY

- An apportionment method is **neutral** if permuting the states permutes the seat allocation
- **Theorem:** There is no apportionment method that is neutral, population monotonic and satisfies the quota criterion
- **Corollary:** No divisor method satisfies the quota criterion

# PROOF OF THEOREM

- Assume that the method satisfies all three properties
- We claim that the method satisfies the **order-preserving property**: if  $p_j > p_i$  then  $k_j \geq k_i$
- Define an instance with  $p'_i = p_j$ ,  $p'_j = p_i$ , and  $p_t = p_t$  for all  $t \neq i, j$
- By population monotonicity, either  $k'_i \geq k_i$  or  $k'_j \leq k_j$
- By neutrality,  $k'_i = k_j$  and  $k'_j = k_i$
- It follows that  $k_j \geq k_i$

# PROOF OF THEOREM

| State | $p_i$   | $q_i$ |
|-------|---------|-------|
| 1     | 69,900  | 6.99  |
| 2     | 5,200   | 0.52  |
| 3     | 5,000   | 0.50  |
| 4     | 19,900  | 1.99  |
| Total | 100,000 | 10    |

By the quota criterion,  
 $k_1 \leq 7$  and  $k_4 \leq 2$ .

Therefore,  $k_2 \geq 1$  or  $k_3 \geq 1$ . By the order-preserving property,  $k_2 \geq 1$ .

| State | $p'_i$ | $q'_i$ |
|-------|--------|--------|
| 1     | 68,000 | 8.02   |
| 2     | 5,500  | 0.65   |
| 3     | 5,600  | 0.66   |
| 4     | 5,700  | 0.67   |
| Total | 84,800 | 10     |

By the quota criterion,  
 $k'_1 \geq 8$ . Therefore,  $k'_2 = 0$   
or  $k'_3 = 0$  or  $k'_4 = 0$ . By the  
order-preserving property,  
 $k'_2 = 0$ .

We have constructed an example where  $k'_1 > k_1$  and  
 $k'_2 < k_2$  yet  $p'_1 < p_1$  and  $p'_2 > p_2$  ■

# Let's Raffle Off Congressional Seats

New York lost its 27th Congress member after the latest U.S. census. A dose of randomness could have given it a fairer chance to hold on.



Give or take a few ... *Photographer: Andrew Harrer/Bloomberg*

By Ariel Procaccia

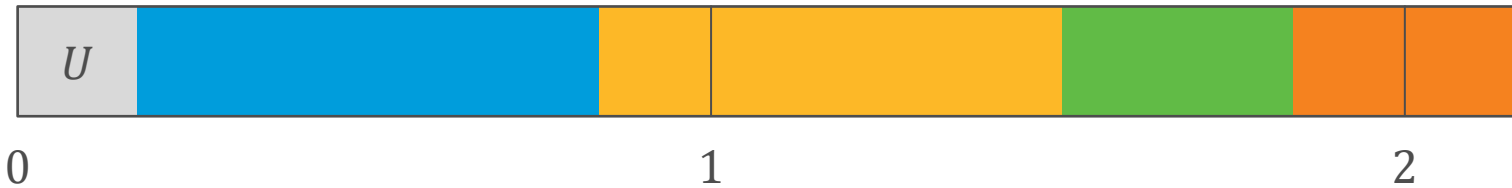
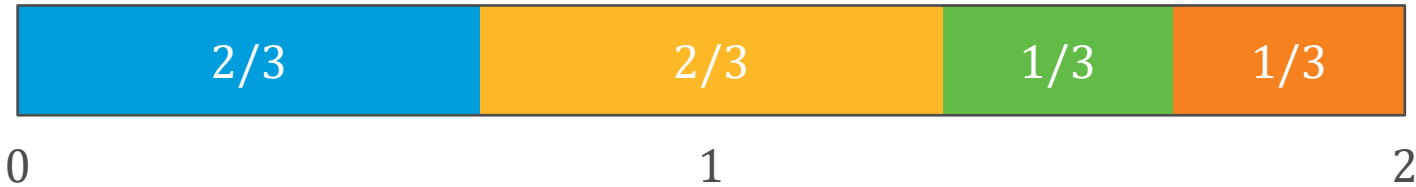
May 5, 2021, 8:30 AM EDT

# RANDOMIZED APPORTIONMENT

Consider the following algorithm:

1. Take a random permutation of the label of the states (w.l.o.g. it's identity)
2. Provisionally allocate  $\lfloor q_i \rfloor$  seats to each state  $i \in N$ , and let  $r_i = q_i - \lfloor q_i \rfloor$
3. Draw  $U \sim \mathcal{U}([0,1])$
4. Let  $Q_i = U + \sum_{j=1}^i r_j$
5. For each  $i \in N$ , allocate an extra seat to state  $i$  if  $[Q_{i-1}, Q_i)$  contains an integer

# RANDOMIZED APPORTIONMENT



# RANDOMIZED APPORTIONMENT

- The randomized apportionment algorithm gives each state its standard quota in expectation
- It follows that it's population monotonic in expectation
- It satisfies the quota criterion ex post

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