

# *Optimized Democracy*

Spring 2022 | Lecture 10

**Cake Cutting**

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# CAKE CUTTING



How to **fairly** divide a heterogeneous divisible good between players with different preferences?

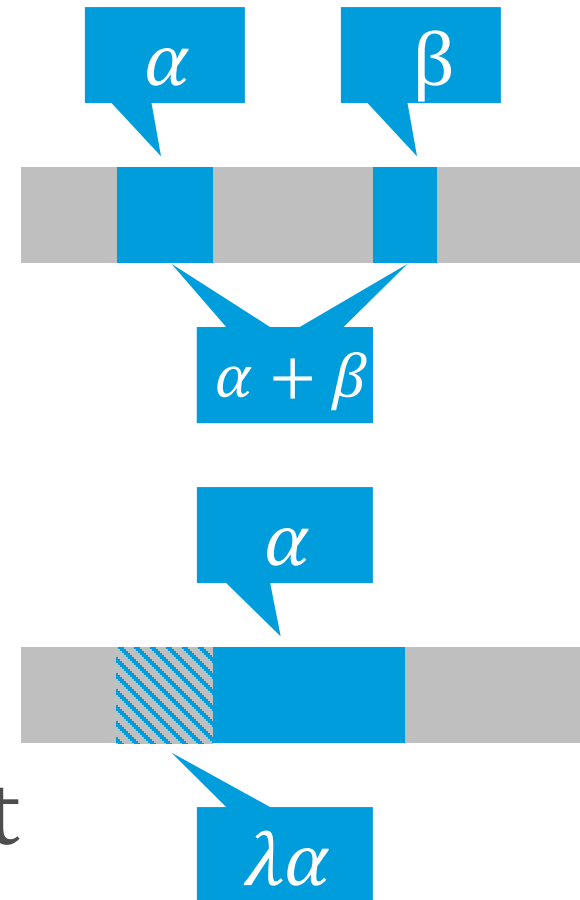
# THE PROBLEM

- Cake is interval  $[0,1]$
- Set of **players**  $N = \{1, \dots, n\}$
- Piece of cake  $X \subseteq [0,1]$ : finite union of subintervals of  $[0,1]$



# THE PROBLEM

- Each player  $i \in N$  has a non-negative valuation  $V_i$  over pieces of cake
- **Additive:** for  $X \cap Y = \emptyset$ ,  
$$V_i(X) + V_i(Y) = V_i(X \cup Y)$$
- **Normalized:** For all  $i \in N$ ,  
$$V_i([0,1]) = 1$$
- **Divisible:**  $\forall \lambda \in [0,1]$  can cut  
$$I' \subseteq I \text{ s.t. } V_i(I') = \lambda V_i(I)$$



# FAIRNESS PROPERTIES

- Our goal is to find an **allocation**  $A_1, \dots, A_n$

- **Proportionality:**

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- **Envy-Freeness (EF):**

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$

## Poll 1

For  $n = 2$ , which is stronger?

- |                   |                |
|-------------------|----------------|
| • Proportionality | • Equivalent   |
| • Envy-Freeness   | • Incomparable |



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## Poll 2

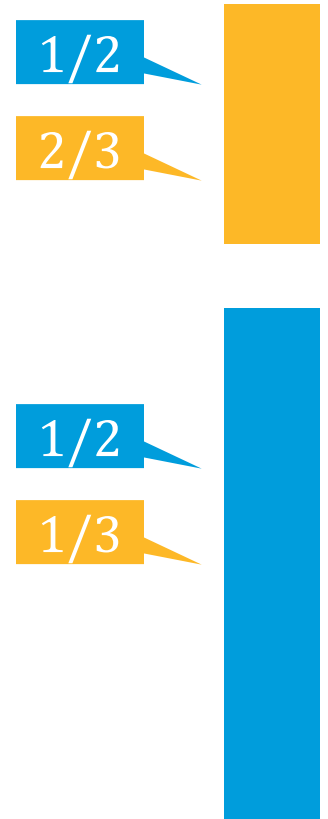
For  $n \geq 3$ , which is stronger?

- |                   |                |
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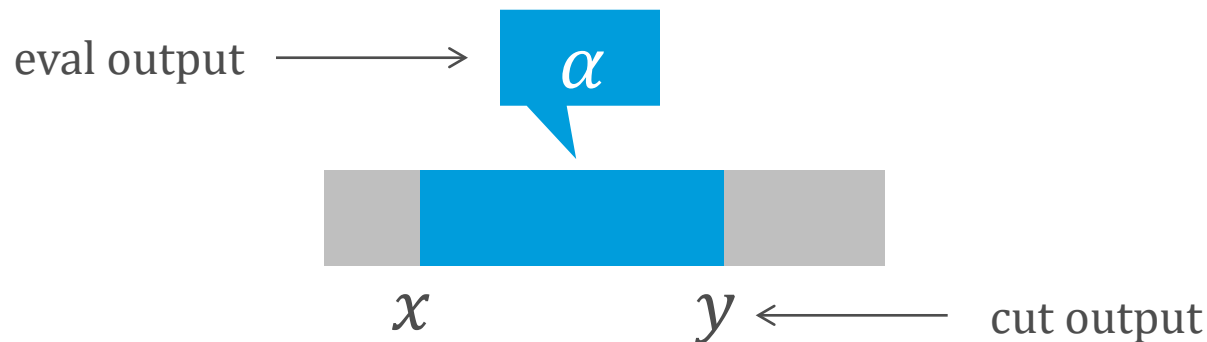
# CUT-AND-CHOOSE

- Algorithm for  $n = 2$  [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces  $X, Y$  s.t.  
 $V_1(X) = 1/2, V_1(Y) = 1/2$
- Player 2 chooses preferred piece
- This is EF (hence proportional)



# THE ROBERTSON-WEBB MODEL

- What is the complexity of Cut-and-Choose?
- Input size is  $n$
- Two types of operations
  - $\text{Eval}_i(x, y)$  returns  $V_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$  returns  $y$  such that  $V_i([x, y]) = \alpha$





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## Poll 3

#Operations needed to find an EF allocation  
when  $n = 2$ ?

- One
- Two
- Three
- Four



# DUBINS-SPANIER

- Referee continuously moves knife
- Repeat: when piece left of knife is worth  $1/n$  to player, player shouts “stop” and gets piece
- That player is removed
- Last player gets remaining piece

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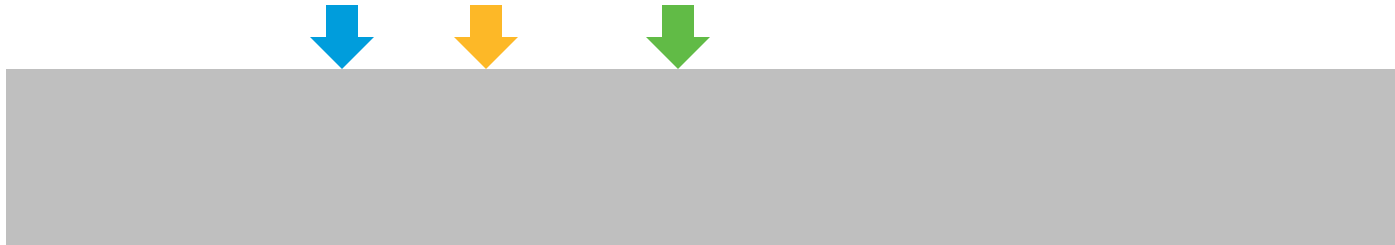
## Poll 4

What is the complexity of DS?

- $\Theta(n)$
- $\Theta(n^2)$
- $\Theta(n \log n)$
- $\Theta(n^2 \log n)$



# DUBINS-SPANIER



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# DUBINS-SPANIER



# EVEN-PAZ

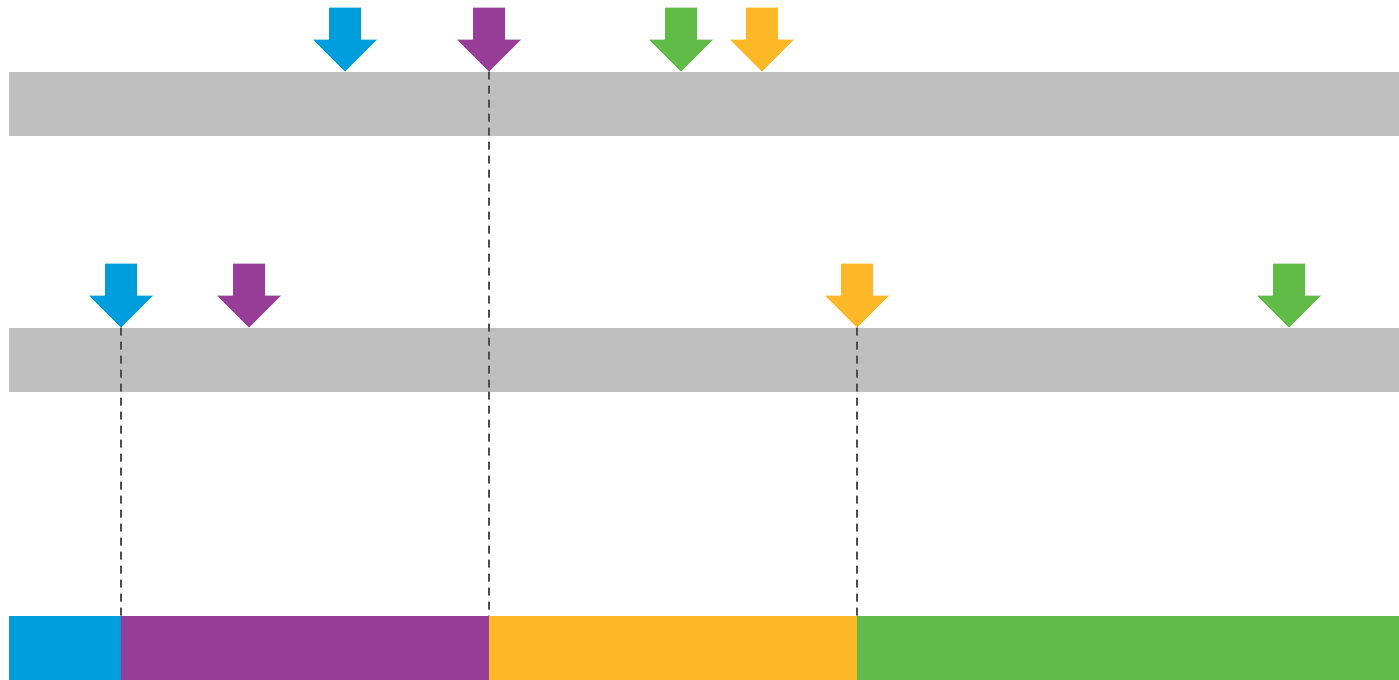
- Given  $[x, y]$ , assume  $n = 2^k$  for ease of exposition
- If  $n = 1$ , give  $[x, y]$  to the single player
- Otherwise, each player  $i$  makes a mark  $z_i$  s.t.

$$V_i([x, z_i]) = \frac{1}{2} V_i([x, y])$$

- Let  $z^*$  be the  $n/2$  mark from the left
- Recurse on  $[x, z^*]$  with the left  $n/2$  players, and on  $[z^*, y]$  with the right  $n/2$  players



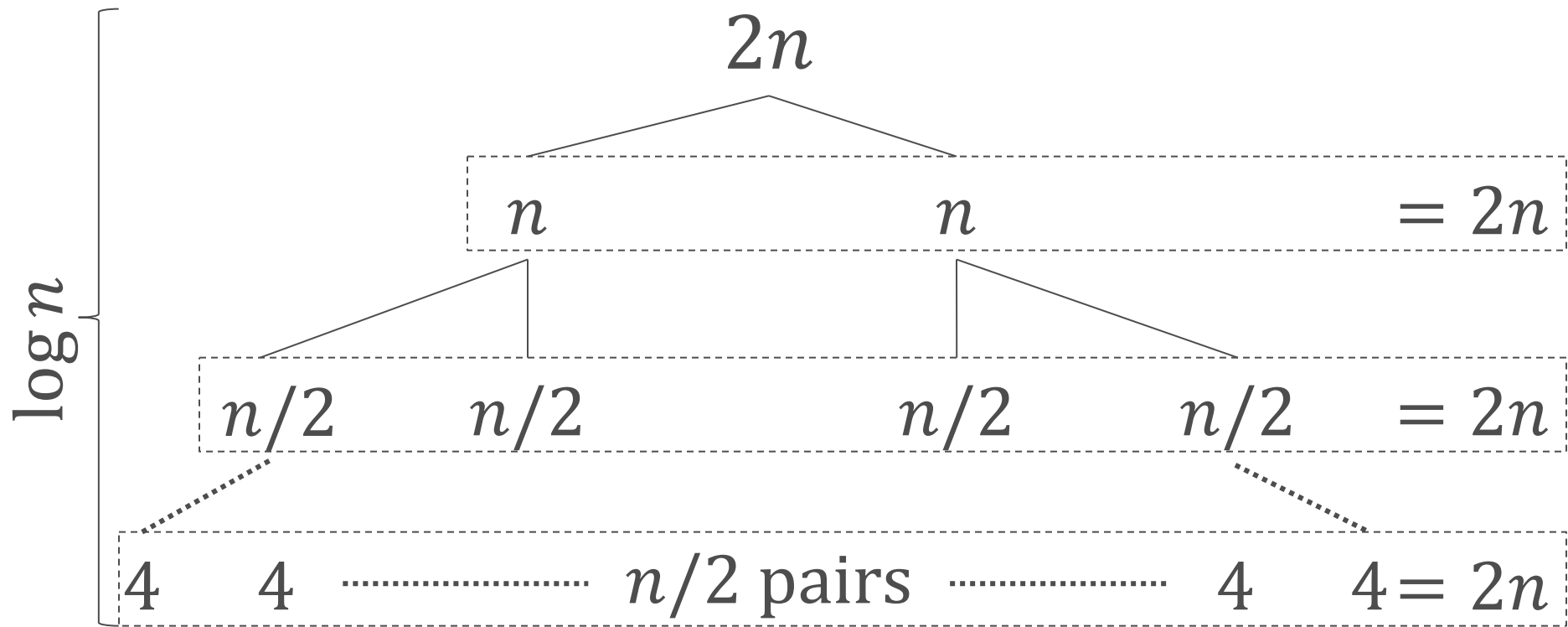
# EVEN-PAZ



# EVEN-PAZ

- **Theorem:** The Even-Paz protocol produces a proportional allocation
- **Proof:**
  - At stage 0, each of the  $n$  players values the whole cake at 1
  - At each stage the players who share a piece of cake value it at least at  $V_i([x, y])/2$
  - Hence, if at stage  $k$  each player has value at least  $1/2^k$  for the piece they're sharing, then at stage  $k + 1$  each player has value at least  $\frac{1}{2^{k+1}}$
  - The number of stages is  $\log n$  ■

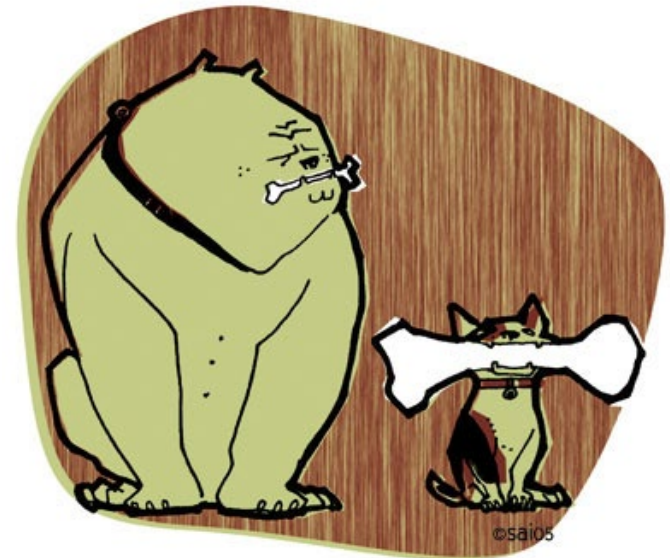
$$T(1) = 0, T(n) = 2n + 2T\left(\frac{n}{2}\right)$$



Overall:  $2n \log n$

# COMPLEXITY OF PROPORTIONALITY

- **Theorem:** Any proportional protocol needs  $\Omega(n \log n)$  operations in the RW model
- The Even-Paz protocol is provably optimal!
- What about envy?



# SELFRIDGE-CONWAY

- **Stage 0**
  - Player 1 divides the cake into three equal pieces according to  $V_1$
  - Player 2 trims the largest piece s.t. there is a tie between the two largest pieces according to  $V_2$
  - Cake 1 = cake w/o trimmings, Cake 2 = trimmings
- **Stage 1 (division of Cake 1)**
  - Player 3 chooses one of the three pieces of Cake 1
  - If player 3 did not choose the trimmed piece, player 2 is allocated the trimmed piece
  - Otherwise, player 2 chooses one of the two remaining pieces
  - Player 1 gets the remaining piece
  - Denote the player  $i \in \{2, 3\}$  that received the trimmed piece by  $T$ , and the other by  $T'$
- **Stage 2 (division of Cake 2)**
  - $T'$  divides Cake 2 into three equal pieces according to  $V_{T'}$
  - Players  $T$ , 1, and  $T'$  choose the pieces of Cake 2, in that order

# THE COMPLEXITY OF EF

- **Theorem [Brams and Taylor 1995]:** There is an EF cake cutting algorithm in the RW model
- But it is **unbounded**
- **Theorem [Aziz and Mackenzie 2016]:** There is a bounded EF algorithm for any  $n$ , whose complexity is

$$O\left(n^{n^{n^{n^n}}}\right)$$

- **Theorem [Procaccia 2009]:** Any EF algorithm requires  $\Omega(n^2)$  queries in the RW model

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