Optimized Democracy (Spring 2021)
Assignment #3

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Due: 3/18/2021 11:59pm ET

Instructions:

- It is fine to look up a complicated sum or inequality, but please do not look up an entire solution. In particular, the solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You may discuss the problems with classmates but please write down solutions completely on your own.
- Please *type up* your solution and send a PDF to Jamie by email (`jtuckerfoltz@gmail.com`), including "CS238HW3" somewhere in the subject. Attach your solutions as a PDF of the form "FirstLast.pdf", where "First" is replaced by your preferred first name and "Last" is replaced by your last name.

Problems:

1. Consider approval-based committee elections with a set N of n voters and a target committee size k, where each voter $i \in N$ approves a set of candidates $A_i \subseteq C$. Let $q := \frac{n}{k}$. We say that a set of $S \subseteq N$ of voters is *ℓ-cohesive* if $|S| \geq \ell \cdot q$ and $|\bigcap_{i \in S} A_i| \geq \ell$. Recall that we write $u_i(W) = |W \cap A_i|$.

   (a) **[10 points]** Assume that q is an integer. Suppose that a committee $W \subseteq C$, $|W| = k$, satisfies Extended Justified Representation (EJR), so for each $1 \leq \ell \leq k$ and every ℓ-cohesive group S, there exists $i \in S$ with $u_i(W) \geq \ell$. Now let S be an ℓ-cohesive group with $|S| = \ell \cdot q$. Prove that S obtains high average welfare, namely that

   $$\sum_{i \in S} \frac{1}{|S|} u_i(W) \geq \frac{\ell - 1}{2}.$$

   (b) **[20 points]** Suppose that every candidate $c \in C$ has many "copies". In particular, suppose we can write $C = C_1 \cup \cdots \cup C_p$ where the C_j's are pairwise disjoint and $|C_j| \geq 2k$ for each $j = 1, \ldots, p$, such that for every voter $i \in N$ and every C_j we have either $C_j \subseteq A_i$ or $C_j \cap A_i = \emptyset$. We call the sets C_j *parties*: thus, each voter is allowed to approve an arbitrary number of parties, and each party has enough candidates to fill the entire committee if needed.

Prove that PAV applied to such a profile selects a committee W that is in the core; in other words, there does not exist a group $S \subseteq N$ with $|S| \geq \ell \cdot q$ and a set $T \subseteq C$ with $|T| = \ell$ such that $u_i(T) > u_i(W)$ for all $i \in S$.

Hint: Obtain a lower bound for the average marginal increase in PAV score of adding a candidate from T to W and then emulate the proof that PAV satisfies EJR.

2. Consider the cake cutting problem with n players and valuation functions $V_1, \ldots, V_n$ satisfying additivity, normalization, and divisibility. Denote the *social welfare* of an allocation $\boldsymbol{A}$ by $\mathrm{sw}(\boldsymbol{A}) = \sum_{i=1}^{n} V_i(A_i)$.

   (a) **[25 points]** Show that, for all valuation functions $V_1, \ldots, V_n$,

$$\frac{\sup\{\mathrm{sw}(\boldsymbol{A}) \,:\, \boldsymbol{A} \text{ is an allocation of the cake}\}}{\sup\{\mathrm{sw}(\boldsymbol{A}) \,:\, \boldsymbol{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = O(\sqrt{n}).$$

   Hint: For an allocation $A^\star$ with maximum social welfare, let $L = \{i \in N : V_i(A_i^\star) \geq 1/\sqrt{n}\}$. Analyze two cases: $|L| \geq \sqrt{n}$ and $|L| < \sqrt{n}$.

   (b) **[10 points]** Give a family of examples of $V_1, \ldots, V_n$ (one example for each value of n) such that

$$\frac{\sup\{\mathrm{sw}(\boldsymbol{A}) \,:\, \boldsymbol{A} \text{ is an allocation of the cake}\}}{\sup\{\mathrm{sw}(\boldsymbol{A}) \,:\, \boldsymbol{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = \Omega(\sqrt{n}).$$

3. Consider a setting with a set M of m divisible goods and a set N of n players. Define an allocation $x \in \mathbb{R}^{n \times m}$ as an $n \times m$ matrix in which x_{ij} denotes the fraction of good j allocated to player i. Let $\mathcal{F} = \{x \mid x_{ij} \geq 0 \text{ and } \sum_i x_{ij} \leq 1\}$ denote the set of feasible allocations. Lastly, assume that each player i has a homogeneous valuation function $v_i : \mathbb{R}^{n \times m} \to \mathbb{R}$; i.e., each player i's valuation for the allocation $x' = c \cdot x$ satisfies $v_i(x') = c \cdot v_i(x)$ for any $c \geq 0$.

   We define *Nash fairness (NF)* as follows. An allocation $x^\star$ is Nash fair if, for any other allocation x', the total proportional change in valuations is not positive; i.e.,

$$\sum_{i \in N} \frac{v_i(x') - v_i(x^\star)}{v_i(x^\star)} \leq 0.$$

   It is known that an NF allocation exists, and, in fact, it is the unique allocation that maximizes the Nash product $\prod_{i \in N} v_i(x)$; you may rely on this fact in your solution.

   The Partial Nash (PN) algorithm first computes the NF allocation $x^\star$, and then assigns each player i a fraction of $x_i^\star$ that depends on the extent to which the presence of i inconveniences the other players (i.e., decreases the value of other players).

Partial Nash

- Compute the NF allocation $x^\star$ based on the reported bids.
- For each player i, remove her and compute the NF allocation $x^\star_{-i}$ that would occur in her absence.
- Allocate to each player i a fraction f_i of everything she receives according to $x^\star$, where
$$f_i = \frac{\prod_{i' \neq i} v_{i'}(x^\star)}{\prod_{i' \neq i} v_{i'}(x^\star_{-i})}.$$

(a) **[10 points]** Show that the allocation produced by the PN algorithm is feasible.

(b) **[10 points]** Prove that the PN algorithm is strategyproof; that is, no player can benefit by reporting untruthfully.

(c) **[15 points]** Prove that the PN algorithm always yields an allocation such that, for every player i, $v_i(x) \geq \frac{1}{e} \cdot v_i(x^\star)$; i.e., it provides a $1/e$ approximation of the optimal allocation.
Hint: Given a sequence of n real numbers $d_1, \ldots, d_n \geq -1$ such that $\sum_{i=1}^n d_i \leq 1$, $\prod_{i=1}^n (1 + d_i) \leq (1 + 1/n)^n$.