Optimized Democracy

Spring 2021, Lecture 8, 2021-02-22

Committee Elections

Dominik Peters, Harvard University

Committee Elections

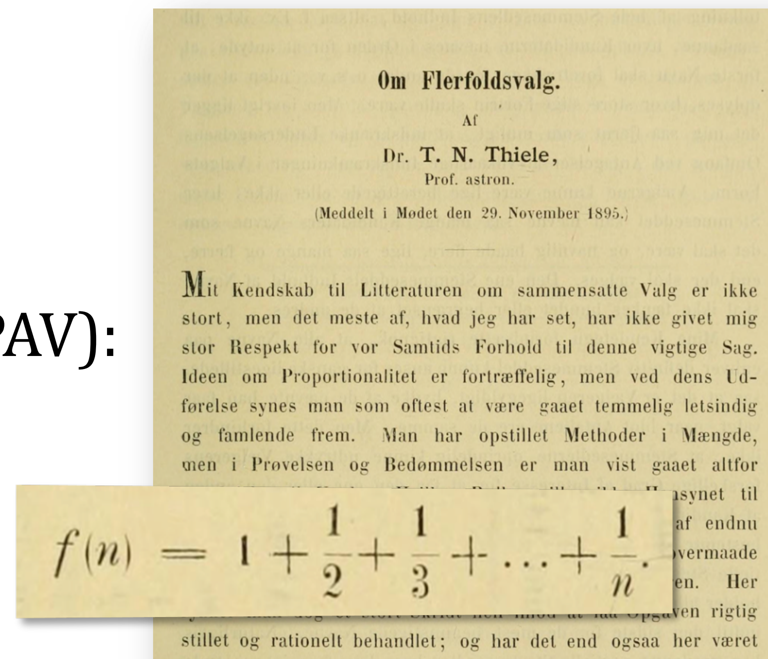
- A set C of candidates, k of which have to be elected
- Outcome: committee $W \subseteq C, |W| = k$.
- A set N of n voters
- Each voter $i \in N$ approves a subset $A_i \subseteq C$.
- We say that i's utility is $u_i(W) = |A_i \cap W|$ (this is a dichotomous preference assumption).

Thiele's methods

- Given a sequence $w_1, w_2, \ldots$, select a committee W that maximizes

$$\sum_{i \in N} w_1 + w_2 + \cdots + w_{u_i(W)}.$$

- Examples:
  - Approval Voting (AV):
    1, 1, 1, …
  - Chamberlin-Courant (CC):
    1, 0, 0, …
  - Proportional Approval Voting (PAV):
    $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$



Om Flerfoldsvalg.

Af

Dr. T. N. Thiele,

Prof. astron.

(Meddelt i Mødet den 29. November 1895.)

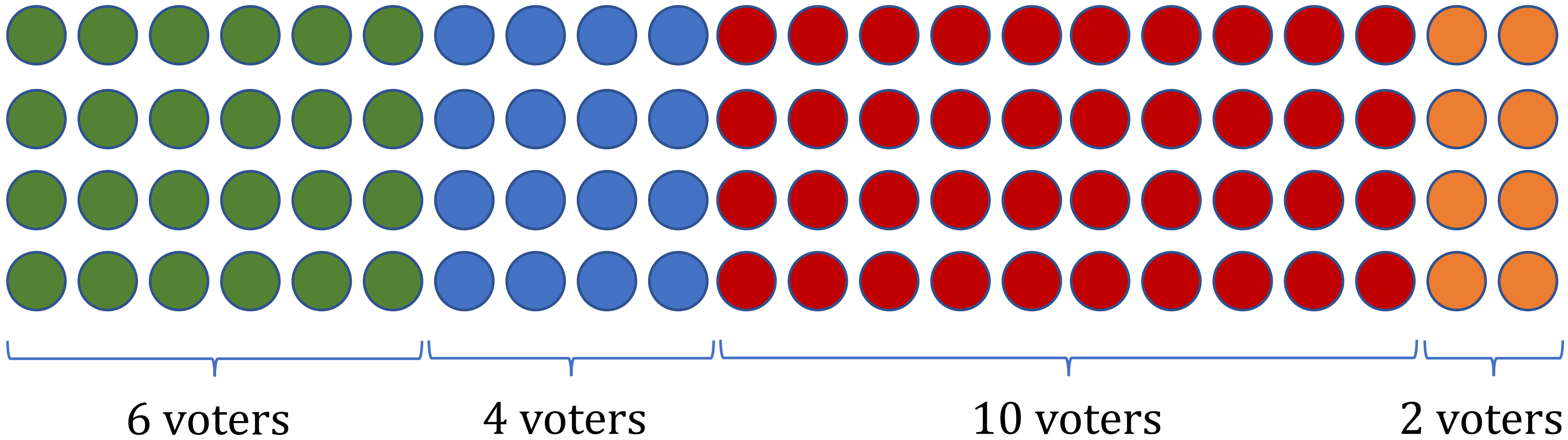
Mit Kendskab til Litteraturen om sammensatte Valg er ikke stort, men det meste af, hvad jeg har set, har ikke givet mig stor Respekt for vor Samtids Forhold til denne vigtige Sag. Ideen om Proportionalitet er fortræffelig, men ved dens Udførelse synes man som oftest at være gaaet temmelig letsindig og famlende frem. Man har opstillet Methoder i Mængde, men i Prøvelsen og Bedømmelsen er man vist gaaet altfor …

$$f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

… stillet og rationelt behandlet; og har det end ogsaa her været

Why harmonic numbers?

$k = 11$



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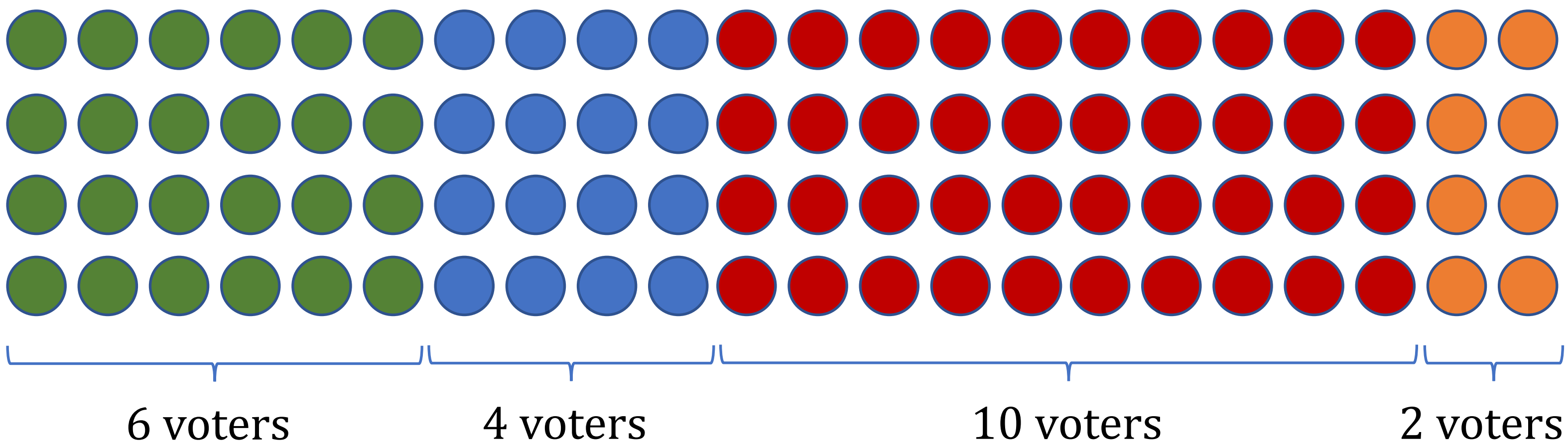

+6	+4	+10	+2
+3	+2	+5	+1
+2	+1.33	+3.33	+0.66
+1.5	+1	+2.5	+0.5
+1.2	+0.8	+2	+0.4
+1	+0.66	+1.66	+0.33

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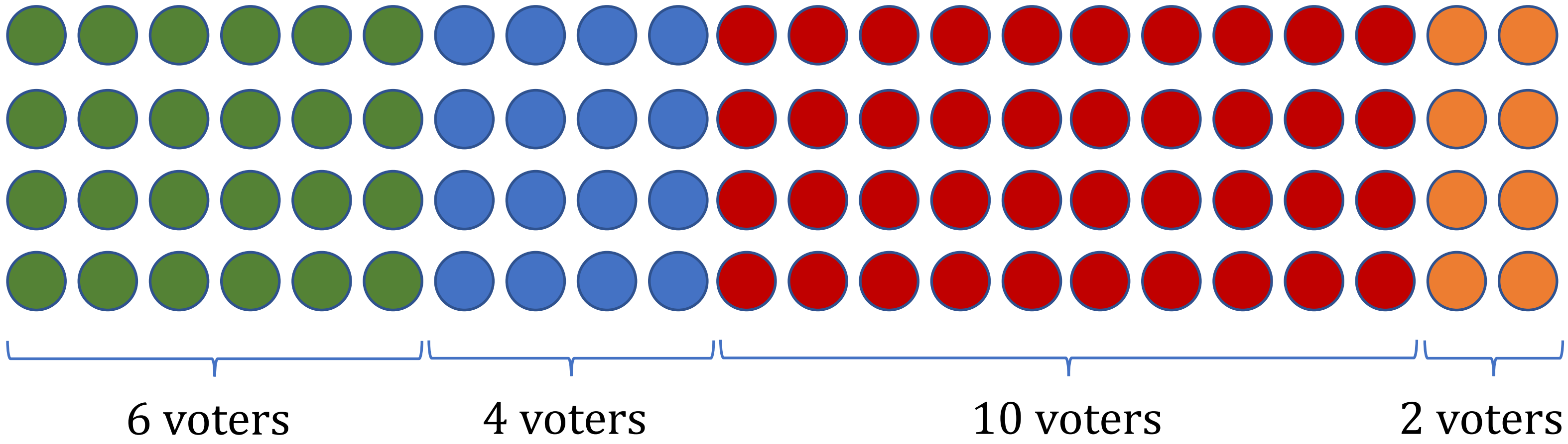

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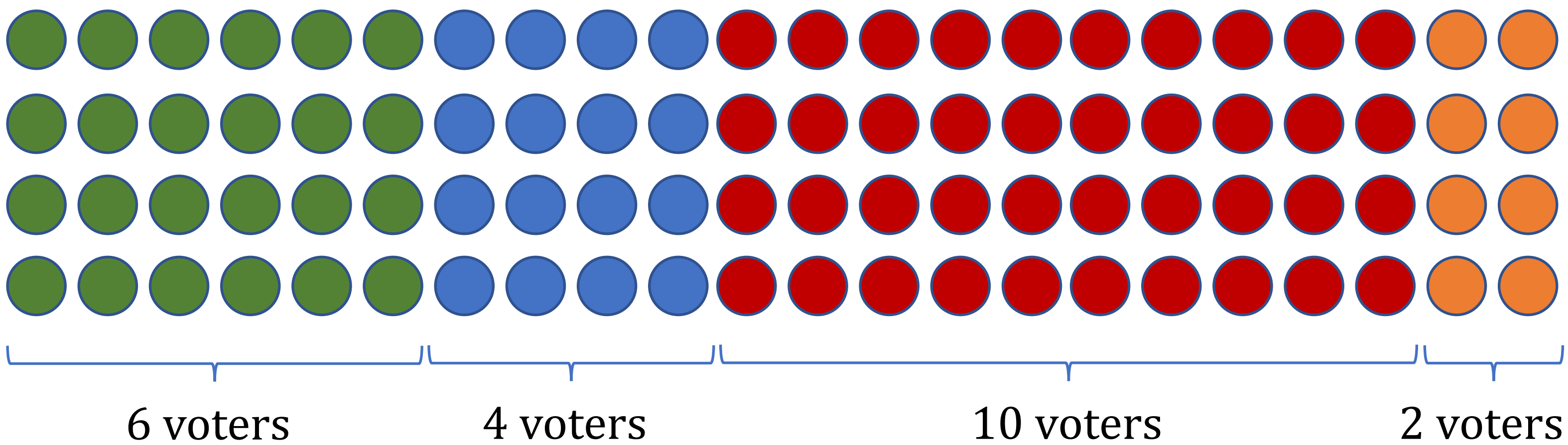

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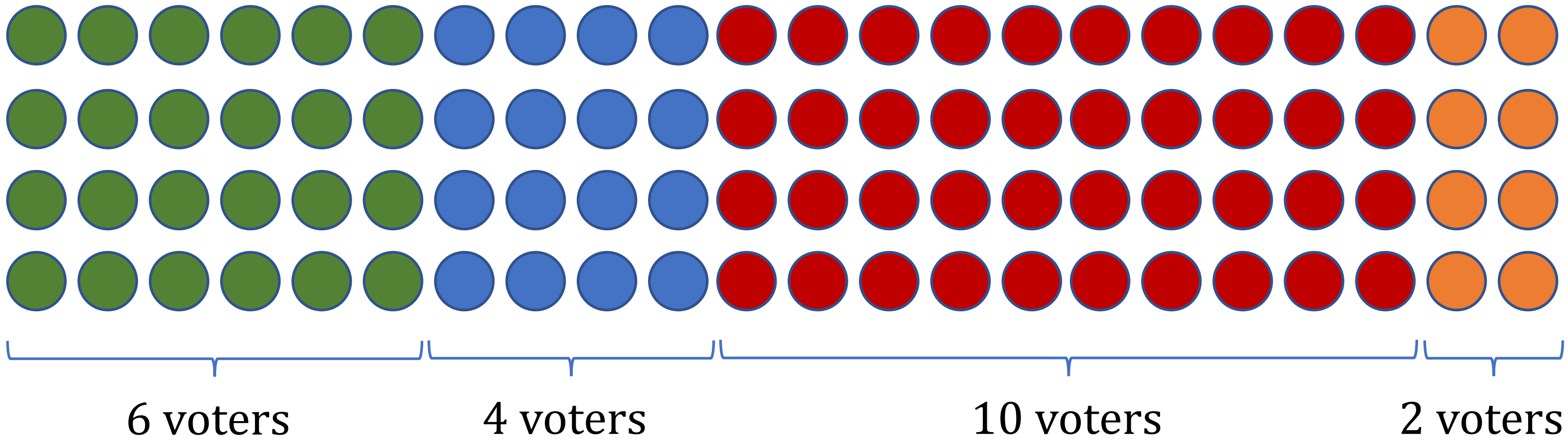

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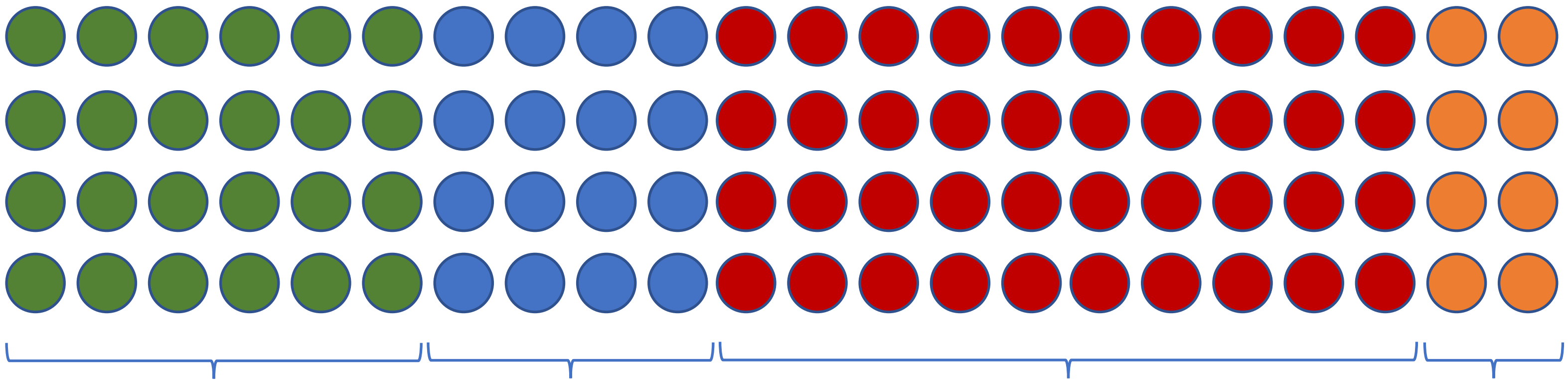

6 voters | 4 voters | 10 voters | 2 voters

+6	**+4**	**+10**	+2
+3	+2	**+5**	+1
+2	+1.33	+3.33	+0.66
+1.5	+1	+2.5	+0.5
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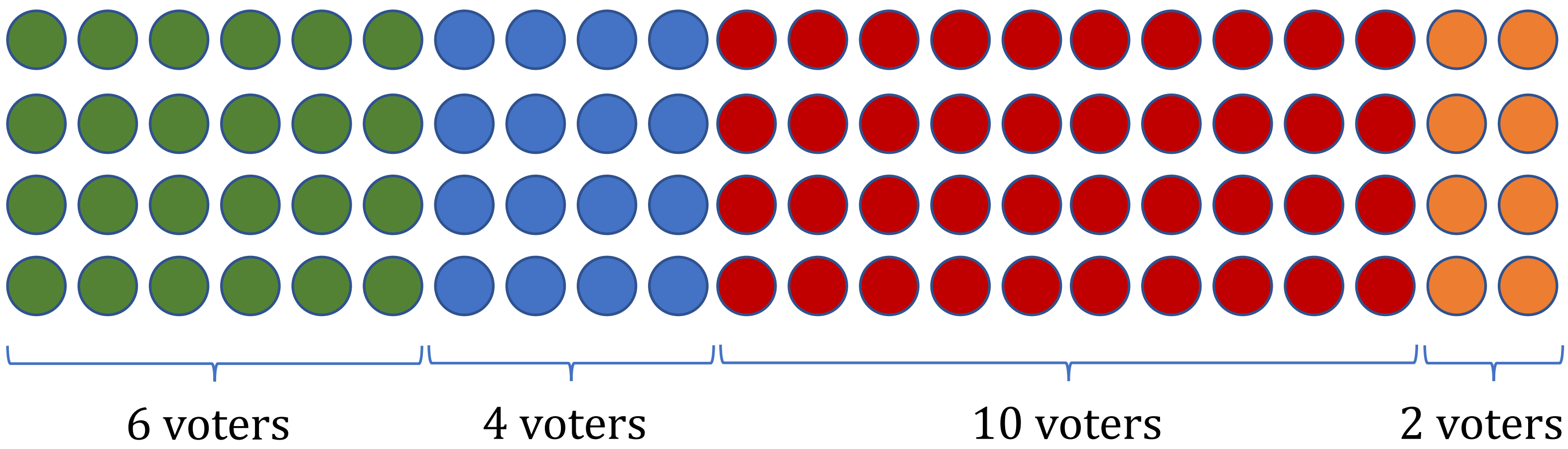

+6	**+4**	**+10**	**+2**
+3	**+2**	**+5**	+1
+2	+1.33	**+3.33**	+0.66
+1.5	+1	**+2.5**	+0.5
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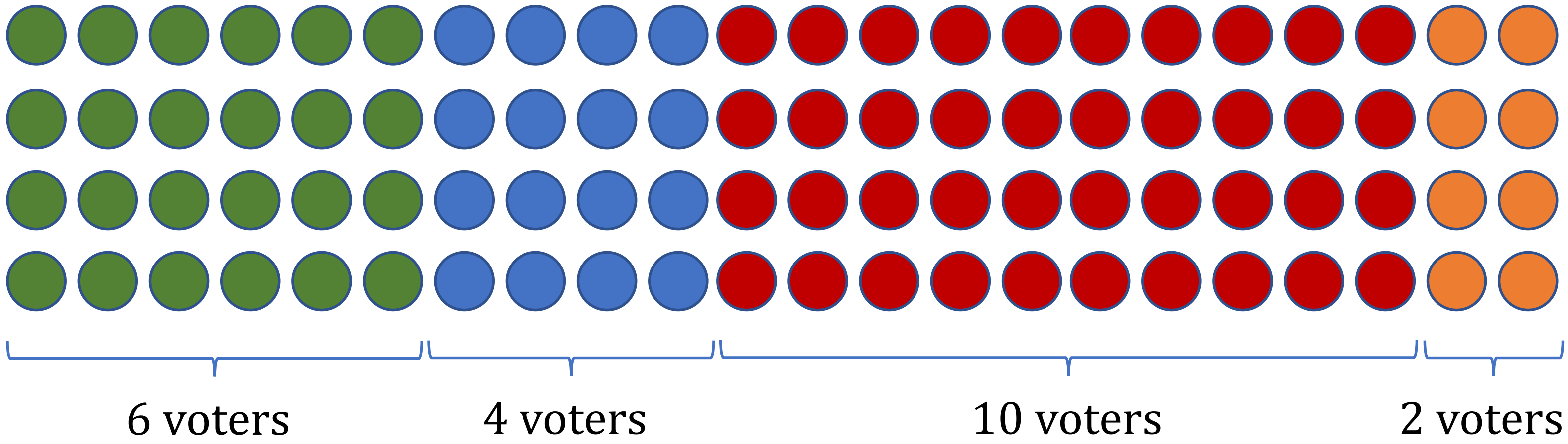


Suppose a party has x supporters, with $x \geqslant \ell \frac{n}{k}$. Then the party deserves at least ℓ seats. Note that

$$\frac{x}{1} > \frac{x}{2} > \frac{x}{3} > \cdots > \frac{x}{\ell} = \frac{n}{k}.$$

It follows that if we elect all seats with marginal increment $\geqslant \frac{n}{k}$, then all parties obtain what they deserve.

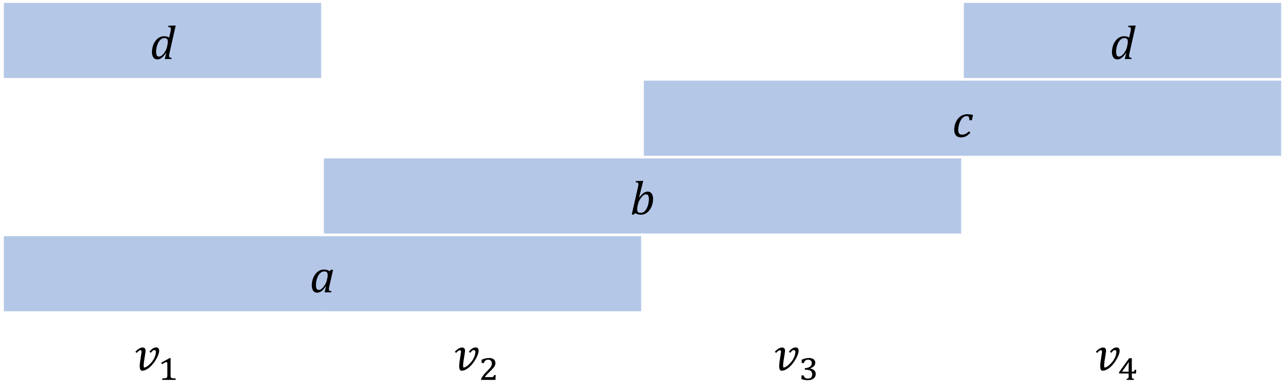
Why harmonic numbers?

- $\boldsymbol{w} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ is the unique sequence such that Thiele's method is proportional in the party list case.
- PAV is the unique approval-based committee rule that satisfies
  - symmetry
  - continuity
  - reinforcement
  - proportionality (D'Hondt) on party list profiles
- *Next*: define proportionality when approval sets can intersect.

A representation axiom that is too strong

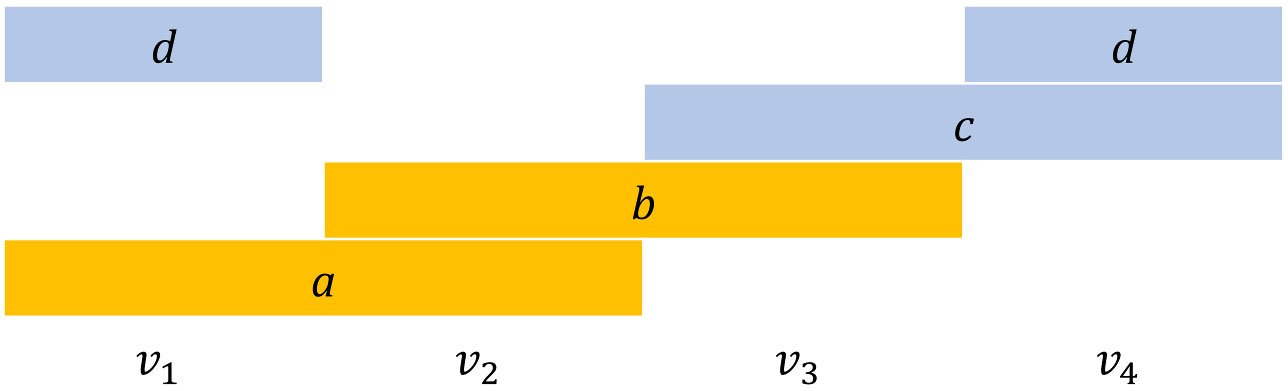
$k = 2$

"if $\frac{n}{k}$ voters have at least 1 candidate in common, then one of their common candidates should be elected"



Justified Representation

If $S \subseteq N$ with $|S| \geq \frac{n}{k}$ have a candidate in common, $|\bigcap_{i \in S} A_i| \geqslant 1$, then it cannot be that $u_i(W) = 0$ for all $i \in S$.



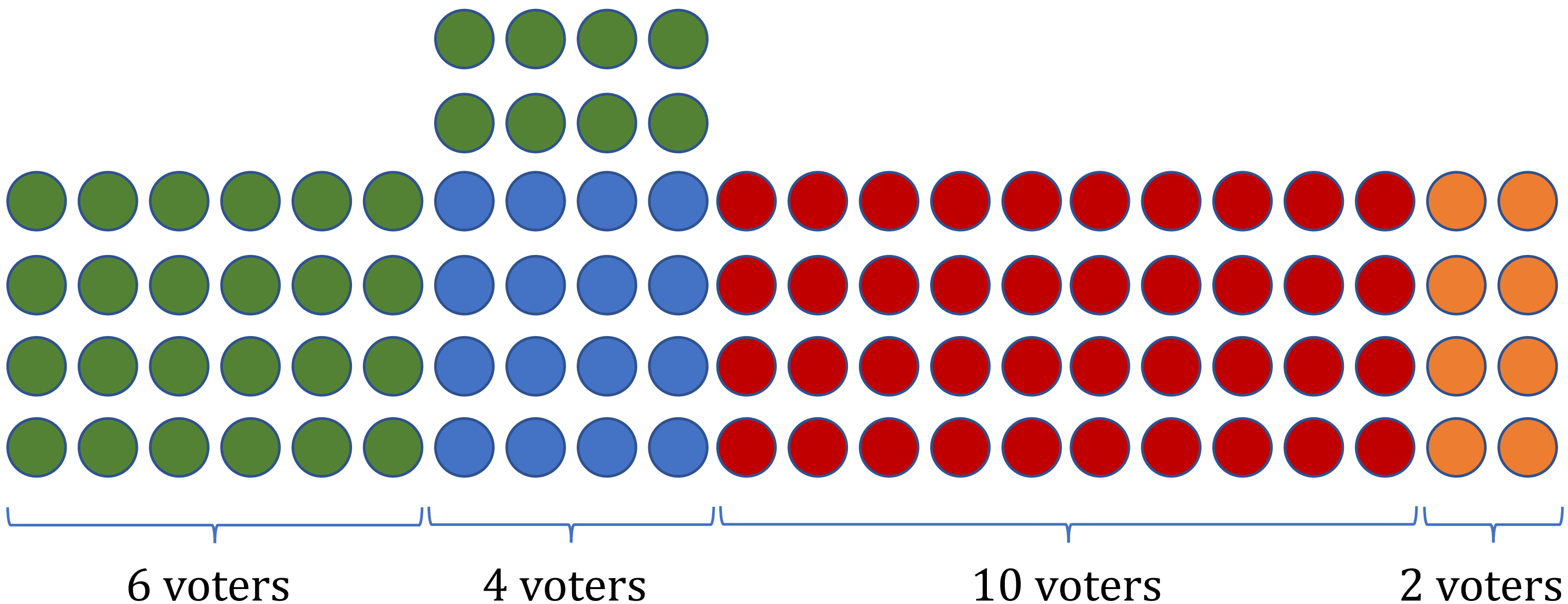
AV fails JR. CC and PAV satisfy JR.

CC satisfies JR

- Let W be the CC committee, violating JR.
- Some number $n' < n$ of voters is covered by W.
- On average, each member of W covers $< \frac{n}{k}$ voters.
- Thus, some member $c^\dagger \in W$ covers $< \frac{n}{k}$ voters.
- Remove $c^\dagger$, and add the candidate approved by the JR group. This gives higher CC score.

Extended Justified Representation

If $S \subseteq N$ with $|S| \geq \ell \frac{n}{k}$ have ℓ candidate in common, $|\bigcap_{i \in S} A_i| \geqslant \ell$, then it cannot be that $u_i(W) < \ell$ for all $i \in S$.



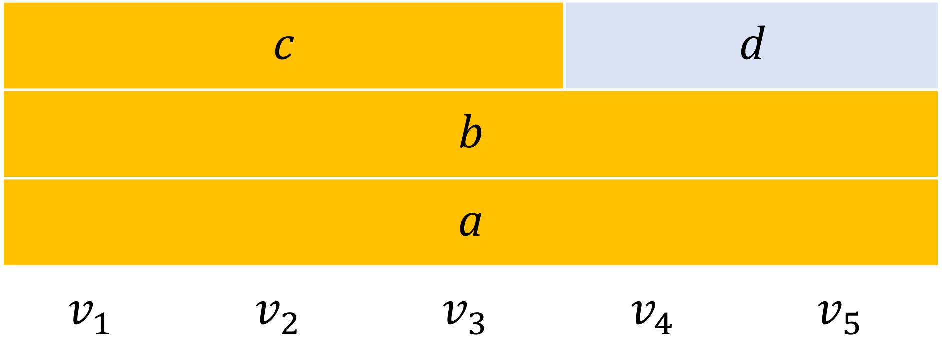
AV and CC fail EJR. PAV satisfies EJR.

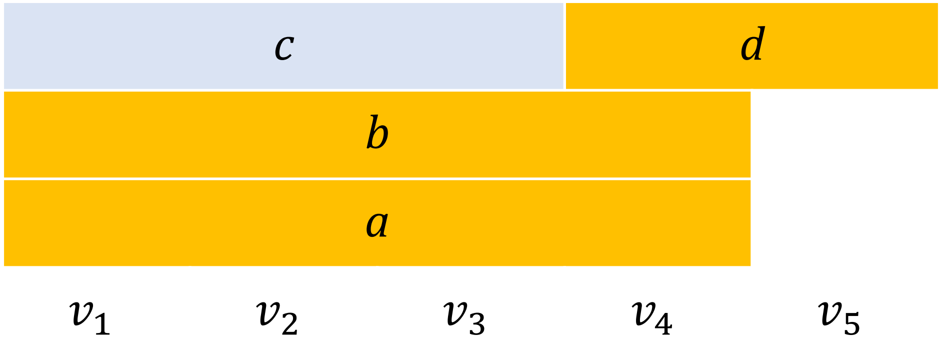
PAV satisfies EJR

- Let W be the PAV committee. Suppose $S \subseteq N$ has size $\geqslant \ell \frac{n}{k}$, and $u_i(W) < \ell$ for all $i \in S$, but there is $c^* \in \bigcap_{i \in S} A_i \setminus$W.
- Let $\widetilde{W} = W \cup \{c^*\}$.
- Note PAV-score$\left(\widetilde{W}\right) \geqslant$ PAV-score$(W) + |S|\frac{1}{\ell} \geqslant$ PAV-score$(W) + \frac{n}{k}$.
- Claim: Can remove a member from $\widetilde{W}$ and lower PAV-score by $< \frac{n}{k}$.
- What is the average loss of PAV score from removal?
- $\frac{1}{k+1}\sum_{c \in \widetilde{W}} \sum_{i : c \in A_i} \frac{1}{u_i(\widetilde{W})} = \frac{1}{k+1}\sum_{i \in N} \sum_{c \in A_i \cap \widetilde{W}} \frac{1}{u_i(\widetilde{W})} \leq \frac{1}{k+1}\sum_{i \in N} 1 < \frac{n}{k}$.
- Hence there is some $c^\dagger \in \widetilde{W}$ with PAV-score$(\widetilde{W} \setminus \{c^\dagger\}) >$ PAV-score(W), contradiction.

PAV is not strategyproof

$k = 3$

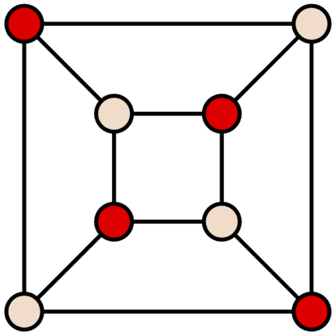




Theorem. No committee rule is strategyproof and satisfies EJR.

PAV is NP-complete

- *Instance*: Profile P, size k, number $B \geqslant 0$.
- *Question*: Is there a committee W with $|W| = k$ such that PAV-score$(W) \geqslant B$?



- Clearly in NP. We'll show this is NP-hard by reducing from CUBIC INDEPENDENT SET:

- *Instance*: Graph $G = (V, E)$ with $d(v) = 3$ for all $v \in V$, size k.
- *Question*: Is there $V' \subseteq V$ with $|V'| = k$ such that for each $e = \{u, v\} \in E$, either $u \notin V'$ or $v \notin V'$?

PAV is NP-complete

- Let $G = (V, E)$ be a cubic graph and let $1 \leqslant k \leqslant |V|$.
- Introduce candidates $C = V$, and voters $N = E$. Each voter approves its endpoints. Set $B = 3k$.
- We prove: There is a k-committee with PAV-score B if and only if G has an independent set of size k.
- $\Leftarrow$: Let V' be an independent set of size k. Then no voter approves 2 candidates in V'. Each candidate in V' is approved by the 3 incident edges. So the PAV-score of V' is $3k$.
- $\Rightarrow$: Suppose W has PAV-score $3k$. Each candidate is approved by 3 voters, so can contribute at most 3 to the PAV score. Since the total score is $3k$, each member of W contributes 3. This can only happen if no voter approves more than 1 candidate in W, so it's an independent set.

PAV can be computed by ILP

- In practice, using modern solvers like Gurobi, we can compute PAV as an integer linear program:

- Maximize $\sum_{i \in N} \sum_{\ell=1}^{k} \frac{1}{l} x_{i,\ell}$

  subject to $\sum_{\ell=1}^{k} x_{i,\ell} = \sum_{c \in A_i} y_c$ for all $i \in N$

  $\sum_{c \in C} y_c = k$

  $y_c \in \{0,1\}, \;\; x_{i,\ell} \in \{0,1\}$ for all i, ℓ, c.

- Fun fact: If profile is single-peaked (i.e. candidates ordered left-to-right, everyone approves an interval), the ILP can be solved in polynomial time.

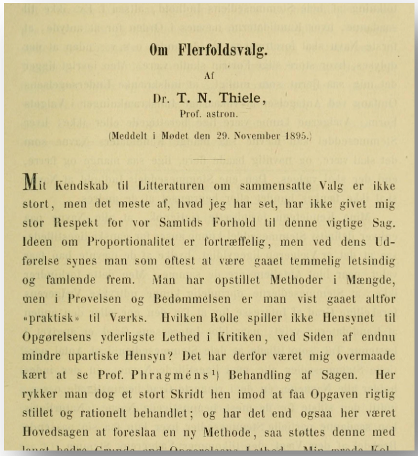
Sequential PAV

- Greedy procedure for calculating PAV:
- $W \leftarrow \emptyset$
- **while** $|W| < k$ **do**
  - Find $c \in C$ that maximizes PAV-score$(W \cup \{c\})$
  - $W \leftarrow W \cup \{c\}$
- **return** W
- *Theorem:* Let W be the optimum PAV committee, and let W' be the committee identified by seqPAV. Then PAV-score$(W') \geqslant \left(1 - \frac{1}{e}\right)$ PAV-score(W).

63%

- Proof: PAV-score is submodular, and approximation is true in general for the greedy algorithm for maximizing a submodular function.

$f(W \cup \{c\}) - f(W) \geqslant f(W' \cup \{c\}) - f(W')$
if $W \subseteq W'$.



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1	× 1	1		a	b	c	d	e	
1	× 1	1		a	b	c	d		f
9	× 1	9		a	b		d	e	
8	× 1	8		a	b		d		f
8	× 1	8		a		c		e	
10	× 1	10		a		c			f
1	× 1	1		a			d		f
4	× 1	4			b	c	d		
5	× 1	5			b	c			f
7	× 1	7			b			e	
2	× 1	2			b				f
4	× 1	4				c	d		
3	× 1	3				c		e	
1	× 1	1				c			f
9	× 1	9					d		
8	× 1	8						e	
9	× 1	9							f
18	× 1	18	z						
			18	38	37	37	37	36	37

1	× 1/2	1/2		a	b	c	d	e	
1	× 1/2	1/2		a	b	c	d		f
9	× 1/2	9/2		a	b		d	e	
8	× 1/2	4		a	b		d		f
8	× 1/2	4		a		c		e	
10	× 1/2	5		a		c			f
1	× 1/2	1/2		a			d		f
4	× 1	4			b	c	d		
5	× 1	5			b	c			f
7	× 1	7			b			e	
2	× 1	2			b				f
4	× 1	4				c	d		
3	× 1	3				c		e	
1	× 1	1				c			f
9	× 1	9					d		
8	× 1	8						e	
9	× 1	9							f
18	× 1	18	z						
			18	✓	55/2	27	27	27	27

1	× 1/3	1/3		a	b	c	d	e	
1	× 1/3	1/3		a	b	c	d		f
9	× 1/3	3		a	b		d	e	
8	× 1/3	8/3		a	b		d		f
8	× 1/2	4		a		c		e	
10	× 1/2	5		a		c			f
1	× 1/2	1/2		a			d		f
4	× 1/2	2			b	c	d		
5	× 1/2	5/2			b	c			f
7	× 1/2	7/2			b			e	
2	× 1/2	1			b				f
4	× 1	4				c	d		
3	× 1	3				c		e	
1	× 1	1				c			f
9	× 1	9					d		
8	× 1	8						e	
9	× 1	9							f
18	× 1	18	z						
			18	✓	✓	133/6	131/6	131/6	22

1	× 1/4	1/4		*a*	*b*	*c*	*d*	*e*	
1	× 1/4	1/4		*a*	*b*	*c*	*d*		*f*
9	× 1/3	3		*a*	*b*		*d*	*e*	
8	× 1/3	8/3		*a*	*b*		*d*		*f*
8	× 1/3	8/3		*a*		*c*		*e*	
10	× 1/3	10/3		*a*		*c*			*f*
1	× 1/2	1/2		*a*			*d*		*f*
4	× 1/3	4/3			*b*	*c*	*d*		
5	× 1/3	5/3			*b*	*c*			*f*
7	× 1/2	7/2			*b*			*e*	
2	× 1/2	1			*b*				*f*
4	× 1/2	2				*c*	*d*		
3	× 1/2	3/2				*c*		*e*	
1	× 1/2	1/2				*c*			*f*
9	× 1	9					*d*		
8	× 1	8						*e*	
9	× 1	9							*f*
18	× 1	18	*z*						
			18	✓	✓	✓	227/12	227/12	227/12

1	× 1/5	1/5		a	b	c	d	e	
1	× 1/5	1/5		a	b	c	d		f
9	× 1/4	9/4		a	b		d	e	
8	× 1/4	2		a	b		d		f
8	× 1/3	8/3		a		c		e	
10	× 1/3	10/3		a		c			f
1	× 1/3	1/3		a			d		f
4	× 1/4	1			b	c	d		
5	× 1/3	5/3			b	c			f
7	× 1/2	7/2			b			e	
2	× 1/2	1			b				f
4	× 1/3	4/3				c	d		
3	× 1/2	3/2				c		e	
1	× 1/2	1/2				c			f
9	× 1/2	9/2					d		
8	× 1	8						e	
9	× 1	9							f
18	× 1	18	z						
			18	✓	✓	✓	✓	1087/60	541/30

1	× 1/6	1/6		*a*	*b*	*c*	*d*	*e*	
1	× 1/5	1/5		*a*	*b*	*c*	*d*		*f*
9	× 1/5	9/5		*a*	*b*		*d*	*e*	
8	× 1/4	2		*a*	*b*		*d*		*f*
8	× 1/4	2		*a*		*c*		*e*	
10	× 1/3	10/3		*a*		*c*			*f*
1	× 1/3	1/3		*a*			*d*		*f*
4	× 1/4	1			*b*	*c*	*d*		
5	× 1/3	5/3			*b*	*c*			*f*
7	× 1/3	7/3			*b*			*e*	
2	× 1/2	1			*b*				*f*
4	× 1/3	4/3				*c*	*d*		
3	× 1/3	1				*c*		*e*	
1	× 1/2	1/2				*c*			*f*
9	× 1/2	9/2					*d*		
8	× 1/2	4						*e*	
9	× 1	9							*f*
18	× 1	18	*z*						
			18	✓	✓	✓	✓	✓	541/30

1	× 1/6	1/6		a	b	c	d	e	
1	× 1/6	1/6		a	b	c	d		f
9	× 1/5	9/5		a	b		d	e	
8	× 1/5	8/5		a	b		d		f
8	× 1/4	2		a		c		e	
10	× 1/4	5/2		a		c			f
1	× 1/4	1/4		a			d		f
4	× 1/4	1			b	c	d		
5	× 1/4	5/4			b	c			f
7	× 1/3	7/3			b			e	
2	× 1/3	2/3			b				f
4	× 1/3	4/3				c	d		
3	× 1/3	1				c		e	
1	× 1/3	1/3				c			f
9	× 1/2	9/2					d		
8	× 1/2	4						e	
9	× 1/2	9/2							f
18	× 1	18	z						
			18	✓	✓	✓	✓	✓	✓

$n = 108, k = 6, \frac{n}{k} = 18$

So EJR requires $z \in W$.

Sequential PAV fails EJR

- This example is the smallest counterexample! (Though for $k = 7/8/9, n = 35/24/17$ is enough.)
- How to find such counterexamples? ILP!
- Fix k. In any given counterexample, we can relabel alternatives such that SeqPAV selects them in the order $c_1, c_2, \dots, c_k$, and does not select c_{k+1}. Since unselected candidates have no influence, we can take $C = k + 1$.
- For each $S \subseteq C$, add variable $z_S \in \mathbb{Z}$.
- Add constraints that for $j > i$, PAV-score($\{c_1, \dots, c_i\}$) $>$ PAV-score($\{c_1, \dots, c_{i-1}, c_j\}$)
- Add constraint that $z_{\{c_{k+1}\}} \geqslant \frac{1}{k} \sum_S z_S$.
- Minimize $\sum_S z_S$.

Is PAV always right?

$k = 12$

4	5	6	10	14	18
	3		9	13	17
	2		8	12	16
	1		7	11	15
v_1	v_2	v_3	v_4	v_5	v_6

4	5	6	10	14	18
	3		9	13	17
	2		8	12	16
	1		7	11	15
v_1	v_2	v_3	v_4	v_5	v_6

EJR not strong enough to capture this!

Core

- Let W be a committee.
- A group $S \subseteq N$ with $|S| \geq \ell \frac{n}{k}$ *blocks* W if there is $T \subseteq C$ with $|T| = \ell$ such that $u_i(T) > u_i(W)$ for all $i \in S$.
- W is in the *core* if it is not blocked.
- Core implies EJR: An EJR failure is a blocking coalition where $T \subseteq \bigcap_{i \in S} A_i$.
- *Open Problem*: does there always exist a committee in the core?

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<tr><td>v_1</td><td>v_2</td><td>v_3</td><td>v_4</td><td>v_5</td><td>v_6</td></tr>
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