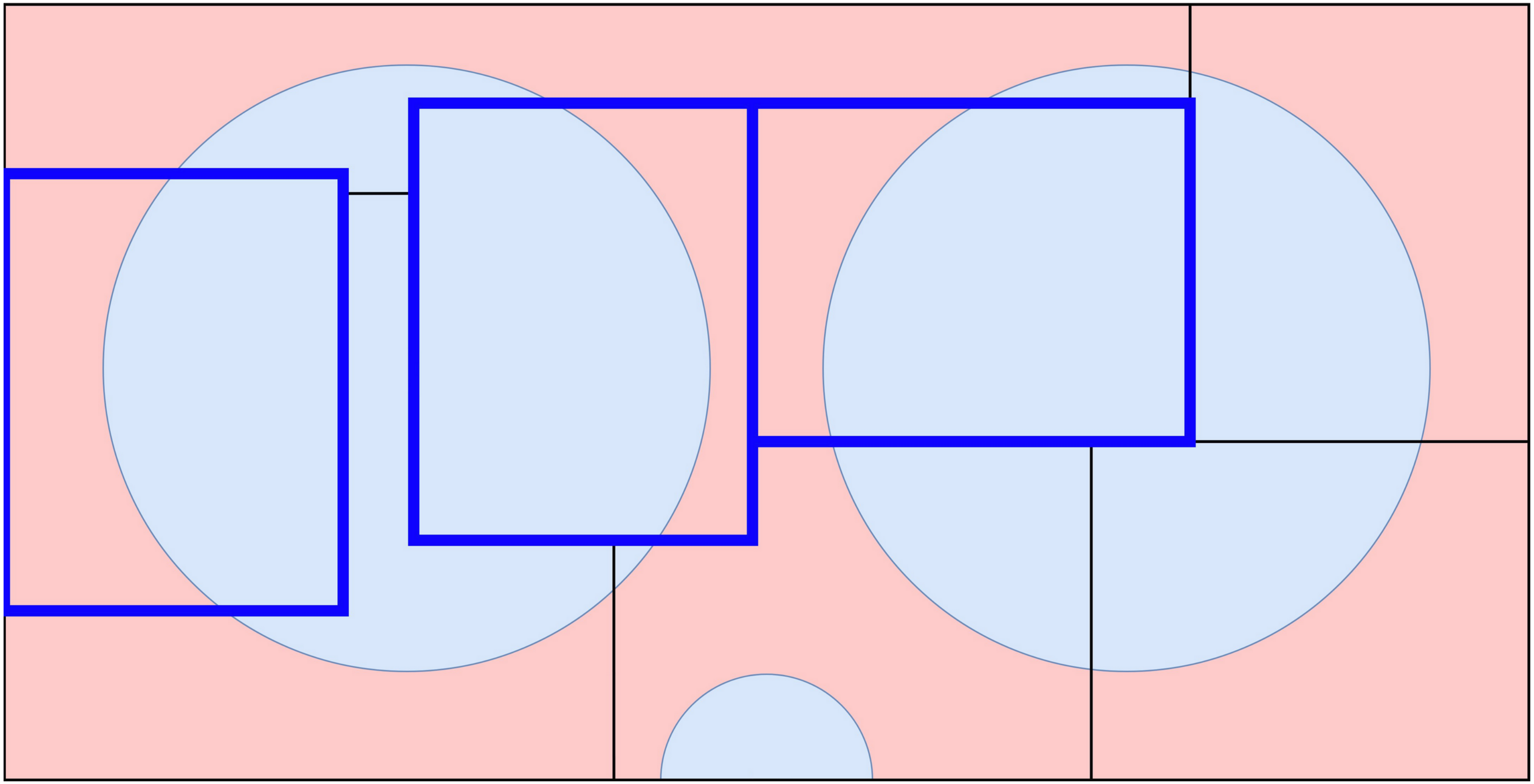
Optimized Democracy

Spring 2021 | Lecture 17

Redistricting As Cake-Cutting

Jamie Tucker-Foltz | Harvard University

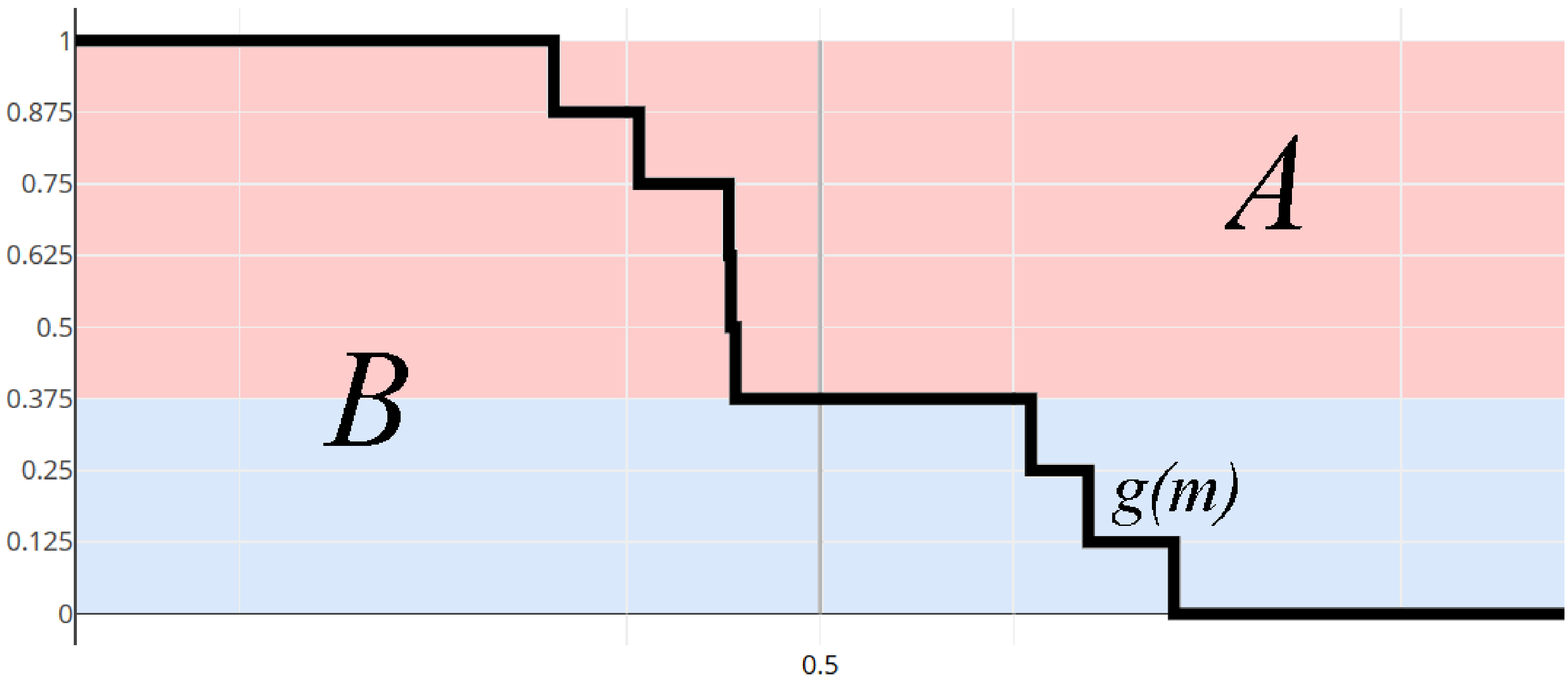
REDISTRICTING



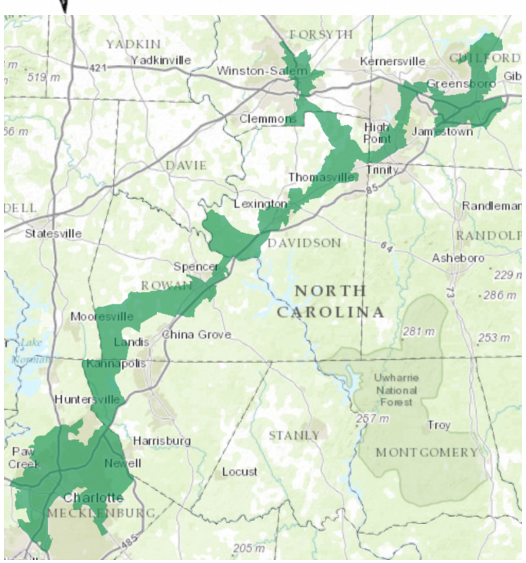
CRACKING AND PACKING

Example from 2012 Wisconsin election, where each row represents a district.

$A =$ Republican party, $B =$ Democratic party.



GERRYMANDERING



Ideas to prevent it:

- Have an independent commission draw fair districts
- Use an interactive protocol with participation from both parties
- Statistically prove a map is gerrymandered

ABSTRACT MODEL

- State S, with set of feasible districts $\mathcal{D} \subseteq 2^S$
- Set of parties $N := \{1, 2, \ldots, n\}$ (today $n := 2$)
- Population measure $\mu : \mathcal{D} \to \mathbb{R}_{\geq 0}$
- For each $j \in N$, distribution function $v^j : \mathcal{D} \to \mathbb{R}_{\geq 0}$
- Target number of districts $m \in \mathbb{Z}_{>0}$

A *partition* is a set P of m disjoint* districts covering S, each of equal measure. The utility of party j is

$$u^j(P) := \left|\left\{D \in P \mid \forall i \neq j,\ v^j(D) >^* v^i(D)\right\}\right|.$$

DISCRETE GRAPH MODEL

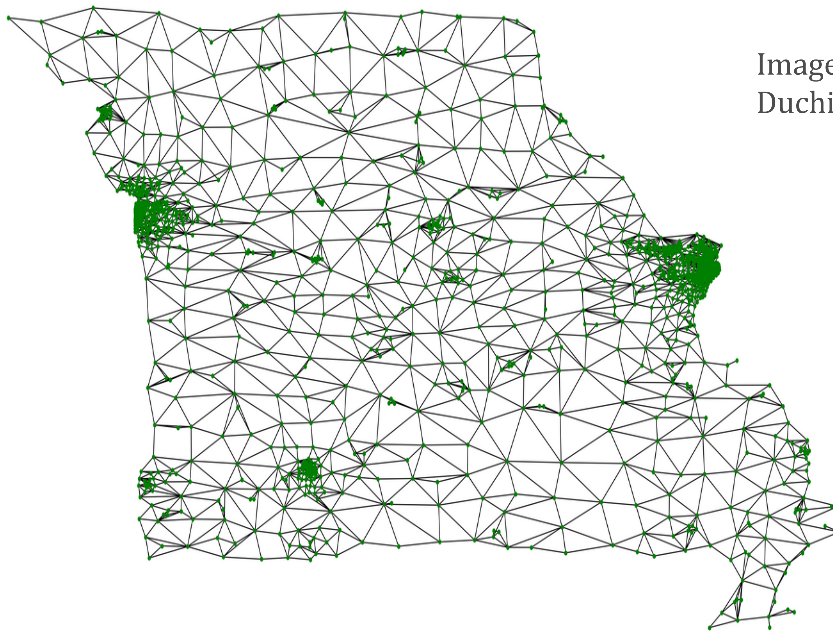
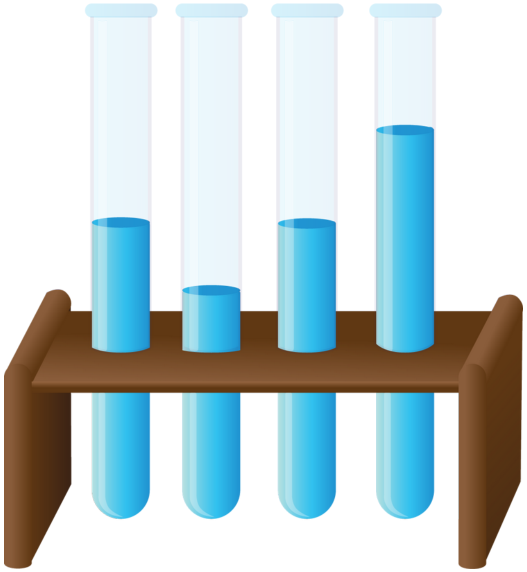


Image credit: DeFord, Duchin, Solomon, 2020

- Graph G of indivisible census blocks
- $S := V(G)$
- $\mathcal{D} := \{D \subseteq S \mid \text{induced subgraph of } D \text{ is connected}\}$
- $\mu(D) := \sum_{b \in D}(\text{population of } b)$
- $v^j(D) := \sum_{b \in D}(\text{number of } j\text{-voters in } b)$

GEOMETRY-FREE MODEL

Continuous model of "placing voters in buckets" with no constraints.



- $S := \bigcup_{j \in N} V_j$
  $V_j := (j, [0, p_j])$
  $\sum_{j \in N} p_j = 1$
- $\mathcal{D} := \{\bigcup_{j \in N} (j, [a_j, b_j]) \mid \forall j \in N,\ 0 \le a_j \le b_j \le p_j\}$
- $\mu(D) := \sum_{j \in N} (b_j - a_j)$
- $v^j(D) := b_j - a_j$

PROPORTIONALITY

$$\text{For all } j \in N, u^j(P) \geq \left\lfloor m \cdot \frac{v^j(S)}{\sum_{i \in N} v^i(S)} \right\rfloor.$$

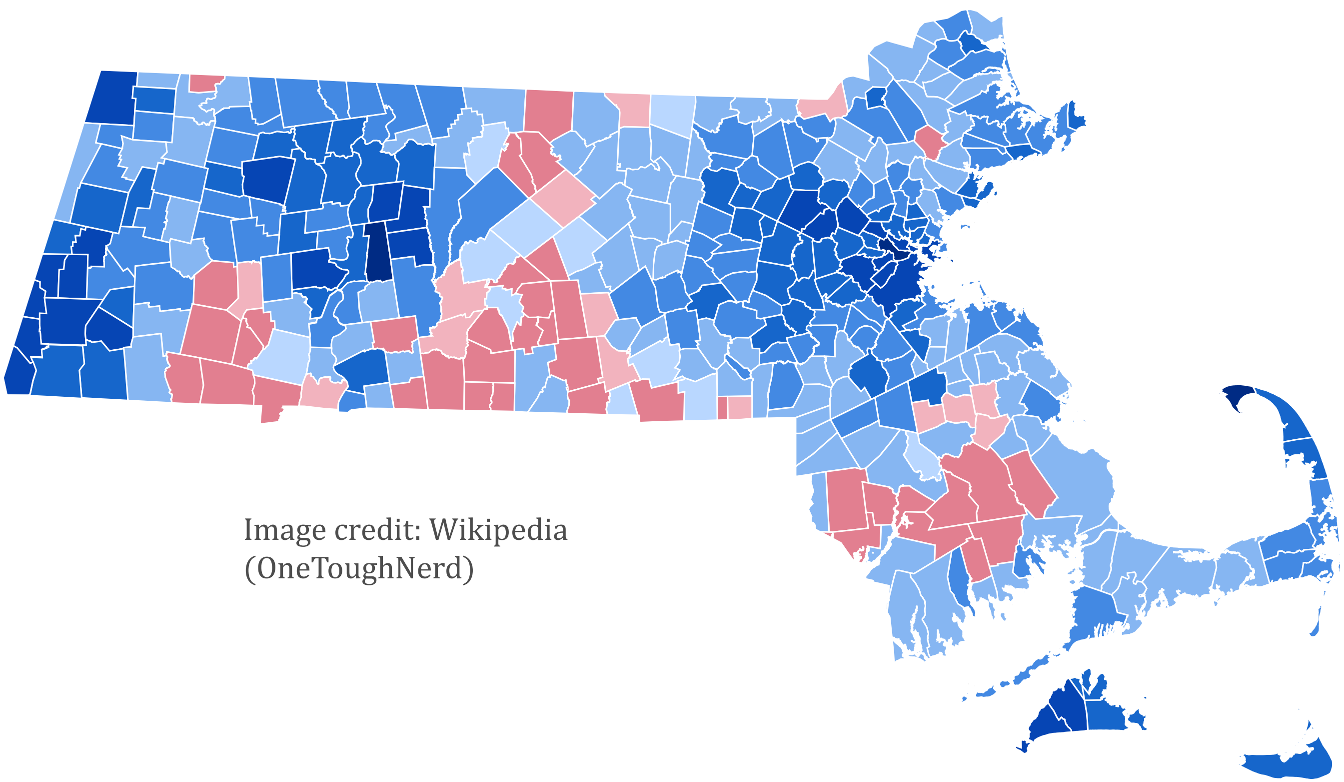


Image credit: Wikipedia
(OneToughNerd)

GEOMETRIC TARGET

For all $j \in N$, let $P^j_{\max}$ be partition maximizing u^j and let $P^j_{\min}$ be a partition minimizing u^j. Then

$$u^j(P) \geq \left\lfloor \frac{u^j(P^j_{\max}) + u^j(P^j_{\min})}{2} \right\rfloor.$$

Poll

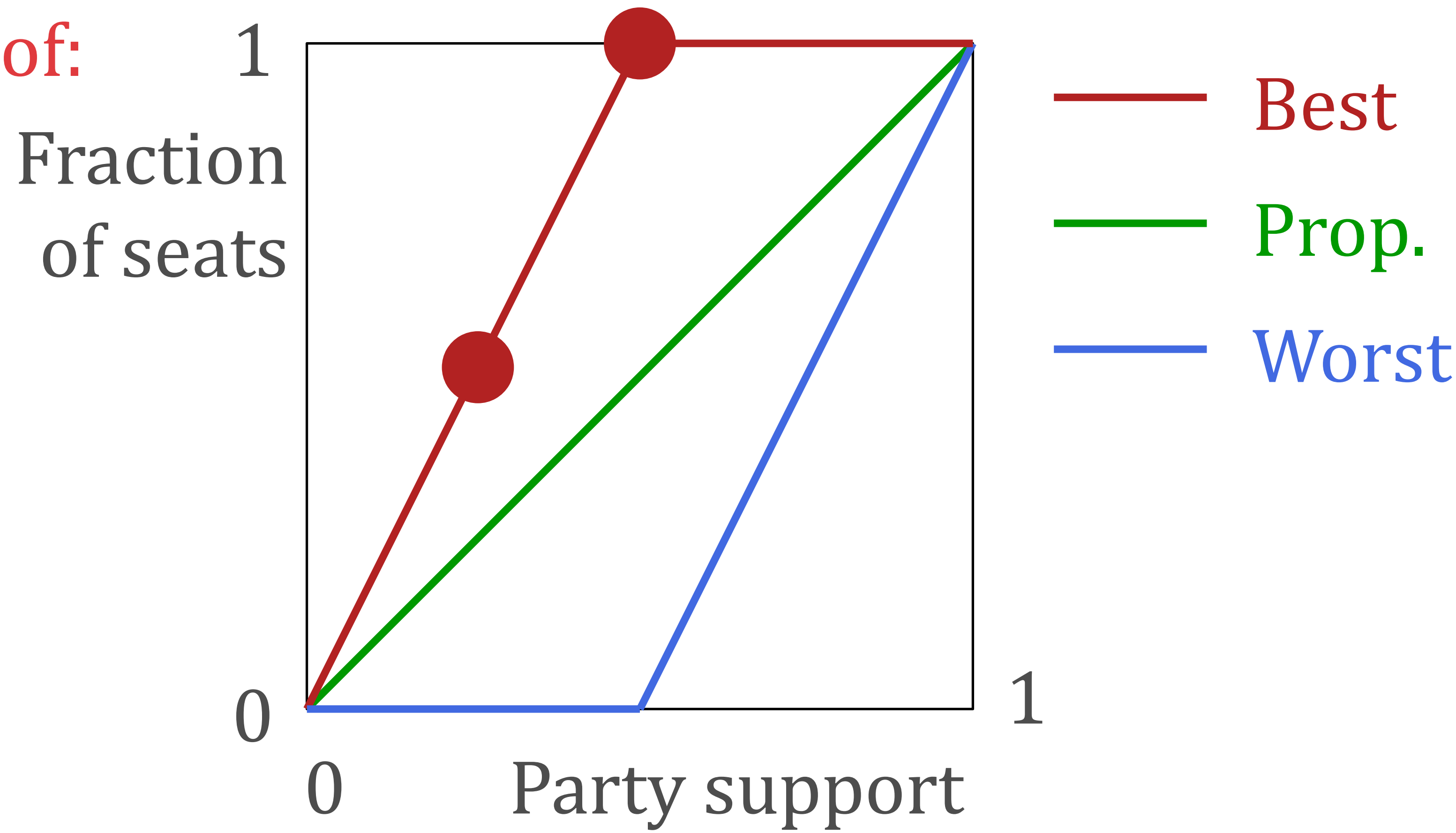
In the geometry-free model, for the minority party, which is easier for a given partition to satisfy?

- Proportionality
- Geometric target
- Equivalent
- Incomparable

GEOMETRIC TARGET

Theorem: In the geometry-free model, a partition satisfies proportionality if and only if it satisfies the geometric target (up to ties).

Proof:



LRY PROTOCOL

Interactive protocol by Landau, Reid, and Yershov that uses a neutral administrator.

1. Administrator presents both parties with a series of bipartitions $(L_1, R_1), (L_2, R_2), \dots, (L_{m-1}, R_{m-1})$ of S, such that each $L_i \subseteq L_{i+1}$.

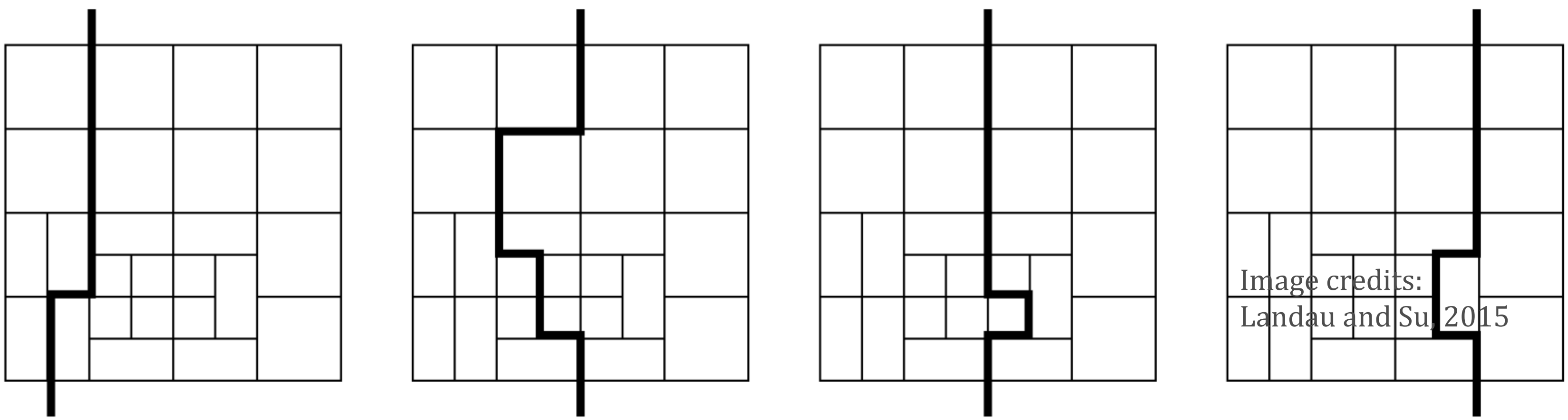
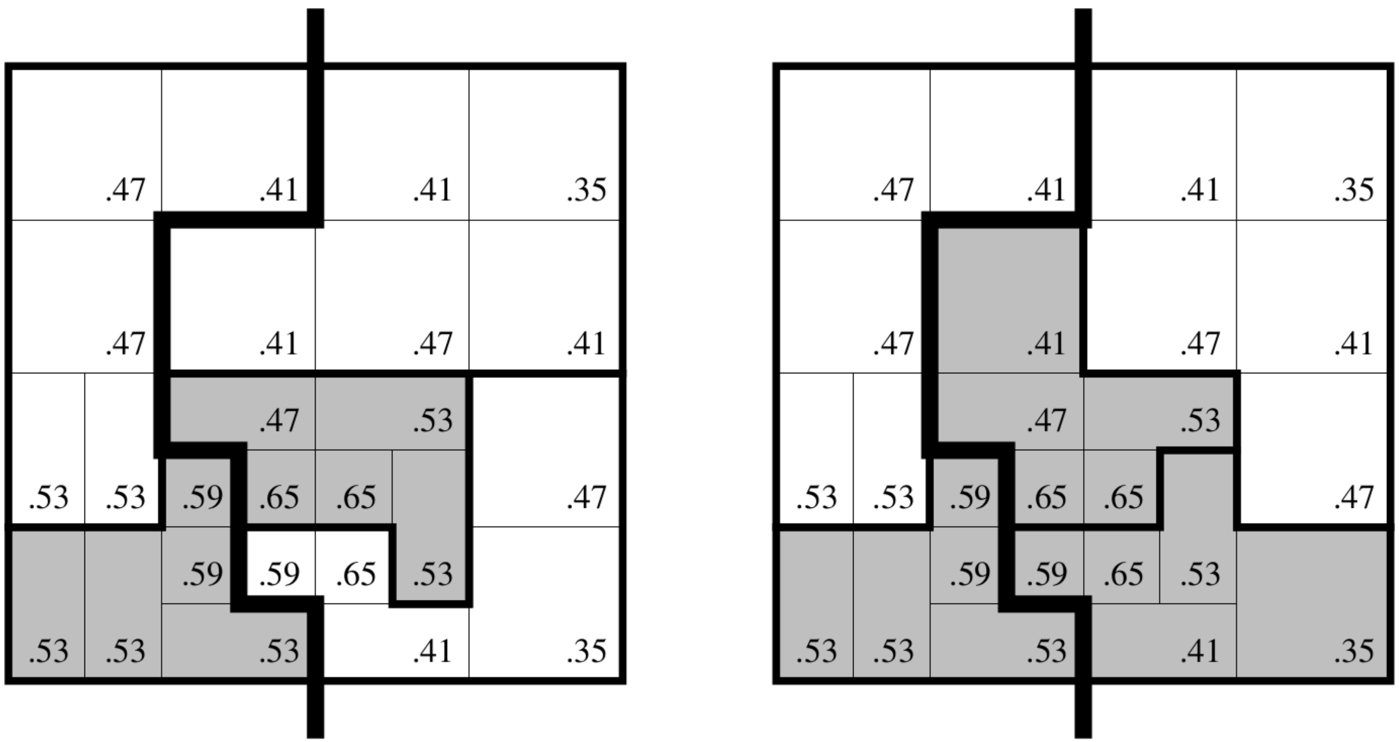


Image credits:
Landau and Su, 2015

LRY PROTOCOL

2. For each $i \in [m]$, each party is asked, "Would you rather redistrict L_i, with the other party redistricting R_i, or vice versa?"

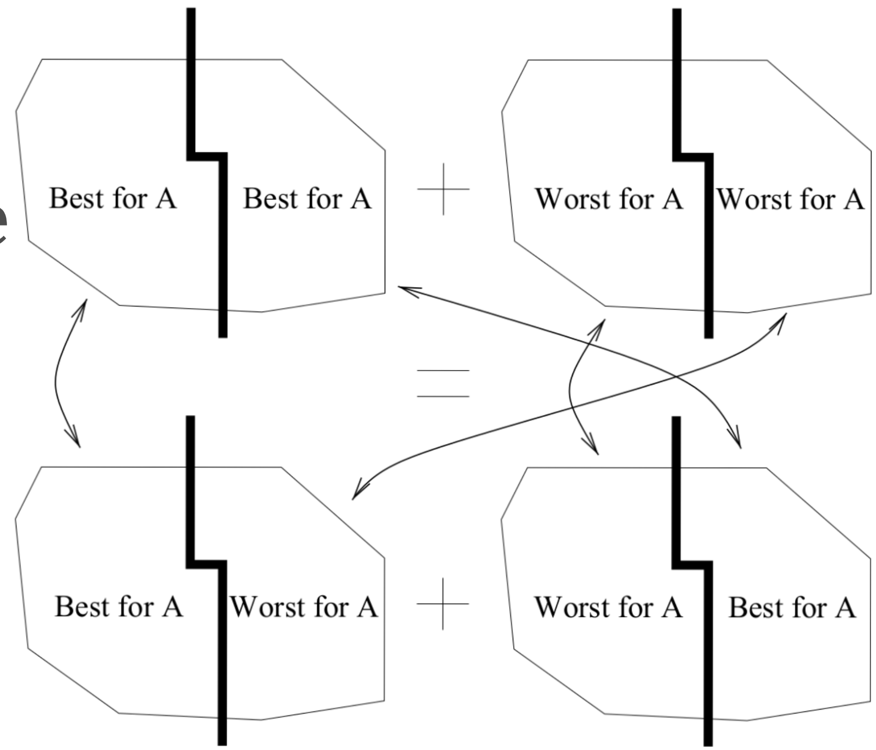


LRY PROTOCOL

3. Try to find an i such that one party prefers redistricting L_i and the other prefers redistricting R_i. If no such i exists, randomly select an outcome at the cross-over point.

Theorem (Good Choice Property): Restricting the feasible set of partitions to respect a given split, a party's preferred choice satisfies its geometric target.

Proof:



LRY PROTOCOL

Pros:

- Realistically implementable
- Simple party participation
- Guaranteed to be within 2 of prop. / geometric target in geometry-free model

Cons:

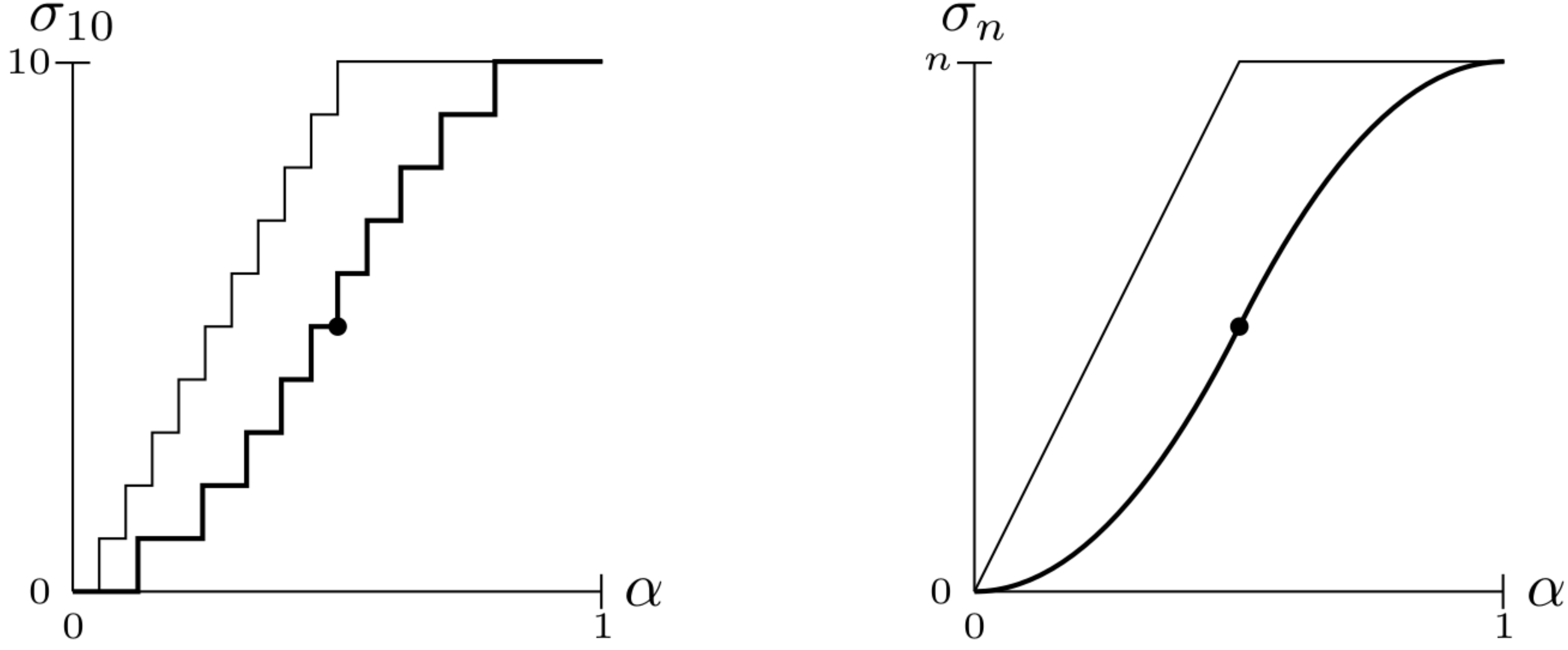
- Relies heavily on neutrality of the administrator
- Can be arbitrarily far from geometric target in grid-based model

CUT AND FREEZE

By Pegden and Procaccia: partition, freeze, and re-partition until all districts are frozen.

CUT AND FREEZE

Theorem: In the geometry-free model, under optimal play, each party can guarantee a number of seats as in the following graphs.



CUT AND FREEZE

Pros:

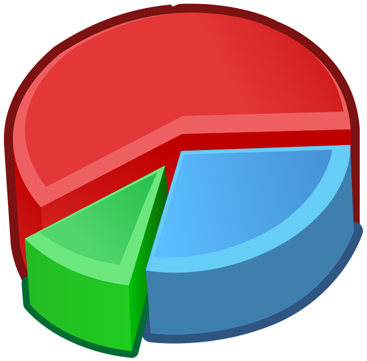
- Realistically implementable
- Approximate proportionality in geometry-free model
- Hard to pack specific groups into one district

Cons:

- Requires complicated strategies
- Requires several rounds of interaction

STATE-CUTTING MODEL 1

Cake-cutting analogue introduced by Benade, Procaccia, and T-F.



- $S := [0, 1]$
- $\mathcal{D} :=$ {finite unions of closed intervals}
- $\mu :=$ Lebesgue measure
- $v^j(D) := \int_D f^j(D)$ where, for all $x \in S$,

$$\sum_{j \in N} f^j(x) = 1$$

STATE-CUTTING PROTOCOL 1

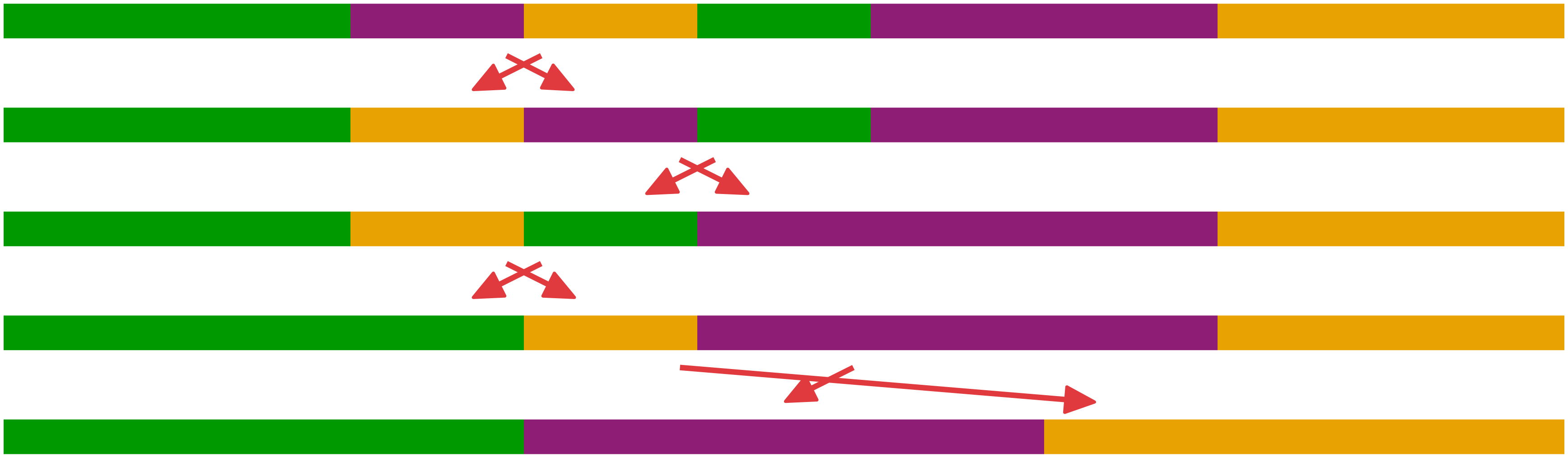
1. Ask each party j to construct an optimal partition P_j.

2. Construct a sequence of partitions from P_1 to P_2, each differing from the previous one on at most two districts.

3. Select an intermediate partition that satisfies the geometric targets of both parties.

STATE-CUTTING PROTOCOL 1

How to achieve step 2? Bubble sort!



Can transition from P_1 to P_2 via the simplest possible partition $\{[\frac{k-1}{m}, \frac{k}{m}] \mid k \in [m]\}$ (the bottom one). Each swap modifies only two districts.

STATE-CUTTING PROTOCOL 1

Theorem: If two partitions differ on at most two districts, the balance of power can differ by at most one.

Proof: Suppose P and P' differ on districts $D_1, D_2 \in P$ and $D'_1, D'_2 \in P'$. Suppose party 1 has a majority in D_1 and D_2, but a minority in D'_1 and D'_2. Then:

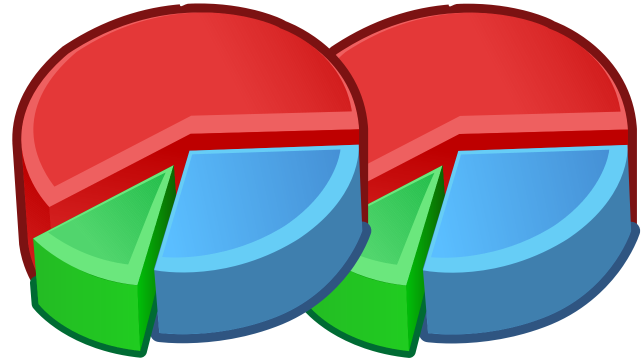
$$\frac{1}{m} < v^1(D_1) + v^1(D_2) = v^1(D_1 \cup D_2)$$

$$= v^1(D'_1 \cup D'_2) = v^1(D'_1) + v^1(D'_2) < \frac{1}{m}$$

Contradiction.

STATE-CUTTING MODEL 2

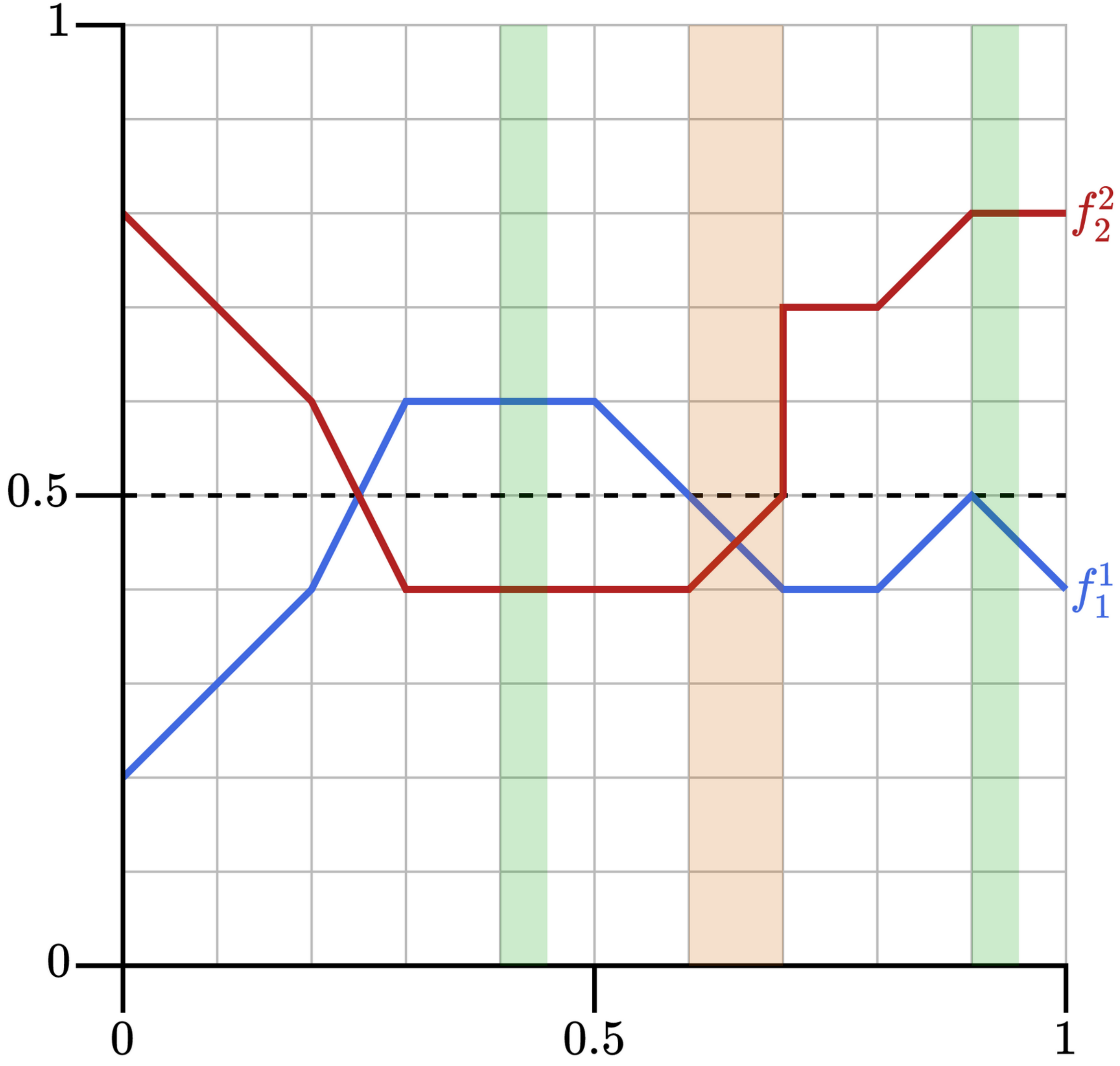
Now parties are allowed to disagree over the distribution of voters!



- $S := [0, 1]$
- $\mathcal{D} :=$ {finite unions of closed intervals}
- $\mu :=$ Lebesgue measure
- $v_i^j(D) := \int_D f_i^j(D)$ where, for all $x \in S$ and $i \in N$,

$$\sum_{j \in N} f_i^j(x) = 1$$

STATE-CUTTING MODEL 2



STATE-CUTTING PROTOCOL 2

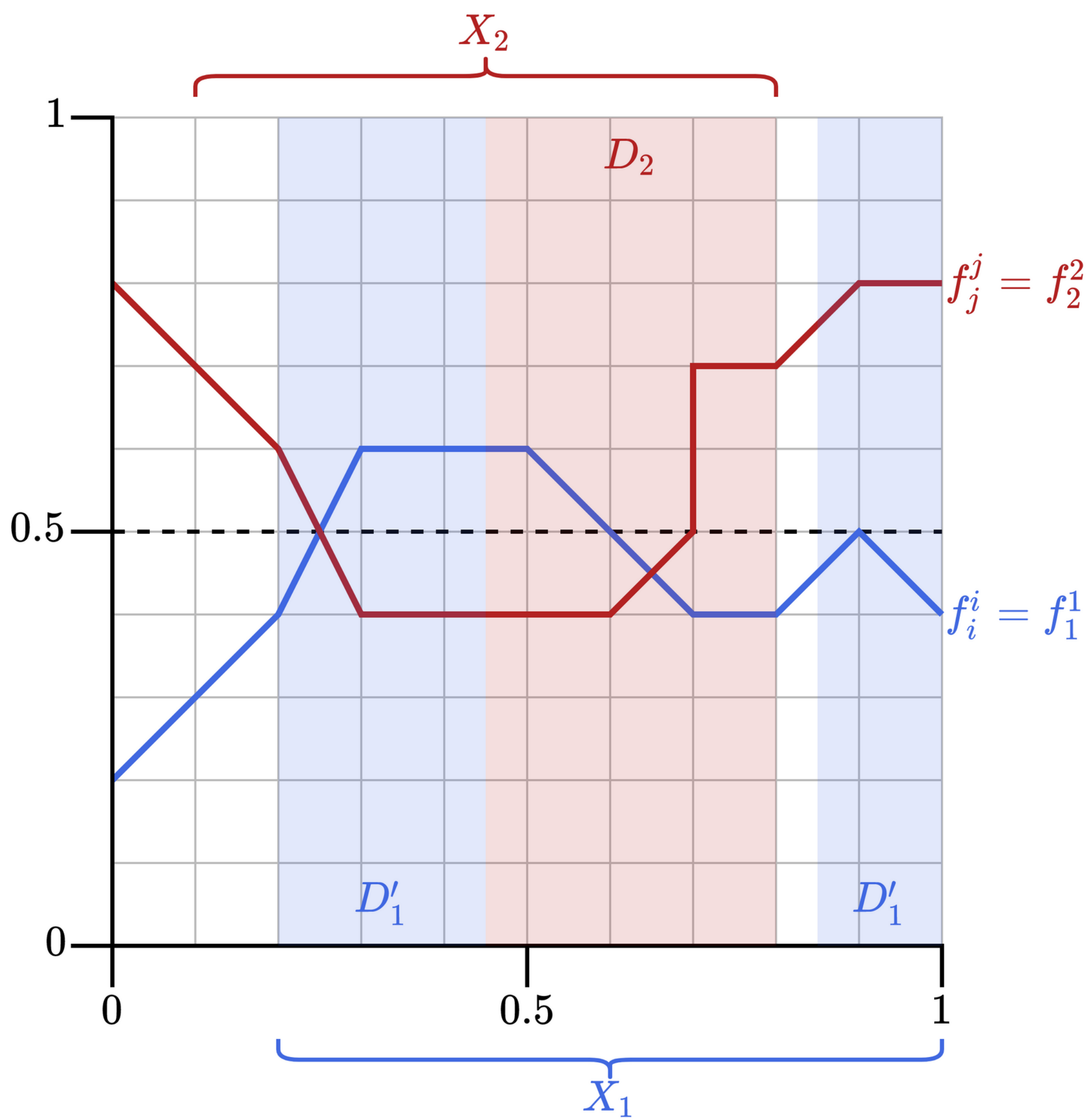
Theorem: Even when parties disagree, there always exists a partition satisfying the geometric targets of both parties:

$$u_i^i(P) \geq \left\lfloor \frac{\min_{P'} u_i^i(P') + \max_{P'} u_i^i(P')}{2} \right\rfloor$$

STATE-CUTTING PROTOCOL 2

1. Each party i computes a maximal set $X_i \subseteq S$ such that $m\mu(X_i) \in \mathbb{Z}$ and $v_i^i(X_i) = \frac{\mu(X_i)}{2}$.
2. Let i be the party with the larger X_i set, and let j be the other party.
3. Party j divides X_j into two pieces of equal size and equal party support according to j.
4. Party i chooses a piece for j to redistrict.
5. Party i redistricts the rest of S.

STATE-CUTTING PROTOCOL 2



Best partition: Divide [0, 1] into 10 equal districts, winning all.

Worst partition: Divide X_2 into 7 equal districts, barely losing all. GT = $\lfloor 7/2 \rfloor + 3 = 6$.

STATE-CUTTING PROTOCOLS

Pros:

- Guarantees geometric target in the state-cutting model
- Works even when parties disagree substantially over how voters are distributed

Cons:

- Protocols are both (somewhat) specific to the state-cutting model

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