Optimized Democracy

Spring 2021 | Lecture 16

Apportionment in the 20th Century

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REMINDER: THE MODEL

- Set of states $N = \{1, \dots, n\}$
- K seats to be allocated
- Each state has population p_i, and the total population is $P = \sum_{i=1}^{n} p_i$
- The standard quota of state i is $q_i = \frac{p_i}{P} \cdot K$
- The upper quota of i is $\lceil q_i \rceil$, and the lower quota is $\lfloor q_i \rfloor$
- Let k_i be the number of seats allocated to i

THE CENSUS OF 1910

- The 1910 census counted 91 million people, 20% more than 1900, and showed migration from rural states to urban centers
- At the urging of Prof. Walter F. Willcox from Cornell, Congress adopted the Webster Method in 1912, but increased the number of seats from 386 to 433 such that no state would lose seats (but the power of rural states still eroded due to seat inflation)
- Two additional seats were reserved for Arizona and New Mexico, which had not yet joined the union, for a total of 435 — the number still used today

HUNTINGTON-HILL METHOD

- Define the rounding function

$$f(x) = \begin{cases} \lfloor x \rfloor \text{ if } x < \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil} \\ \lceil x \rceil \text{ if } x \geq \sqrt{\lfloor x \rfloor \cdot \lceil x \rceil} \end{cases}$$

- The Huntington-Hill Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} f(\hat{q}_i) = K$, where $\hat{q}_i = p_i/D$ is the modified quota
  - Each state is allocated $k_i = f(\hat{q}_i)$

THE HUNTINGTON-HILL METHOD

- By changing the rounding function f one can obtain a family of apportionment methods called divisor methods
- f is assumed to satisfy two conditions: $f(x) = x$ if x is an integer and $f(x) \geq f(y)$ if $x \geq y$
- Theorem: A divisor method is the Huntington-Hill Method if and only if for all $i, j \in N$ such that $p_i/k_i \leq p_j/k_j$,

$$\frac{p_i/k_i}{p_j/k_j} > \frac{p_j/(k_j+1)}{p_i/(k_i-1)}$$

		Ratio 0.848		Ratio 0.831	
State	p_i	k_i	p_i/k_i	k_i	p_i/k_i
1	3,300,000	16	206,250	17	194,117
2	700,000	4	175,000	3	233,333
Total	4,000,000	20	...	20	...

PROOF OF THEOREM

- We'll prove the "only if" direction
- The modified quota $\hat{q}_i = p_i/D$ is rounded down to k_i when $k_i \leq p_i/D < \sqrt{k_i(k_i+1)}$ and rounded up to k_i when $k_i \geq p_i/D \geq \sqrt{k_i(k_i-1)}$
- It follows that
$$\sqrt{k_i(k_i-1)} \leq p_i/D < \sqrt{k_i(k_i+1)}$$
- Equivalently,
$$\frac{k_i(k_i-1)}{p_i^2} \leq \frac{1}{D^2} < \frac{k_i(k_i+1)}{p_i^2}$$
- This holds for all $i, j \in N$, therefore
$$\frac{k_i(k_i-1)}{p_i^2} < \frac{k_j(k_j+1)}{p_j^2}$$
- This is equivalent to the desired property ■

A FINAL HISTORICAL DETOUR

- Joseph A. Hill, a statistician at the Census Bureau, initially suggested the method based on the idea of minimizing “relative differences” in citizens per seat
- Edward V. Huntington, a Harvard math professor, formalized the idea and showed that it’s equivalent to rounding at the geometric mean
- This shows that the method slightly favors small states: A fractional seat of 0.41 is needed to be rounded from 1 to 2, whereas 0.49 is needed to be rounded from 31 to 32

A FINAL HISTORICAL DETOUR

- In 1921 Congress considered bills based on Webster and Huntington-Hill, but both were rejected; ultimately there was no reapportionment that decade (!)
- In 1929 Congress turned to the National Academy of Sciences
- The committee that was formed favored Huntington-Hill because it minimizes relative differences and because it "occupies mathematically a neutral position with respect to the emphasis on larger and smaller states"

A FINAL HISTORICAL DETOUR

- Fortunately, based on the census of 1930 there was no disagreement between Webster and Huntington-Hill, and the consensus apportionment was enacted
- Under the census of 1940, Huntington-Hill gave Arkansas an extra seat and Webster gave Michigan an extra seat
- Since Arkansas was Democratic and Michigan was Republican, this became a partisan issue
- In 1941, President Roosevelt (a Democrat) signed into law an act designating Huntington-Hill as the permanent apportionment method

POPULATION MONOTONICITY REDUX

- The population paradox is still relevant while the Alabama Paradox no longer is
- Suppose there are two censuses where the populations in the second are denoted by $p_1', \dots, p_n'$ and the apportionment by $k_1', \dots, k_n'$ (it could be that $K \neq K'$)
- An apportionment method is population monotonic if $k_i < k_i'$ and $k_j > k_j'$ implies that $p_i < p_i'$ or $p_j > p_j'$
- Theorem: All divisor methods are population monotonic

PROOF OF THEOREM

- Suppose $k_i < k_i'$ and $k_j > k_j'$
- It follows that $p_i/D < p_i'/D'$ and $p_j/D > p_j'/D'$
- Rearranging, we get

$$p_i' > \left(\frac{D'}{D}\right)p_i \quad \text{and} \quad p_j' < \left(\frac{D'}{D}\right)p_j$$

- If $D'/D \leq 1$ then $p_j' < p_j$ and if $D'/D \geq 1$ then $p_i' > p_i$ ■

HOUSE MONOTONICITY REDUX

- An apportionment method is house monotonic if $K' > K$, with all other variables unchanged, implies $k_i' \geq k_i$ for all $i \in N$
- Theorem: Any population monotonic apportionment method is house monotonic
- Corollary: All divisor methods are house monotonic

PROOF OF THEOREM

- Let $K' > K$, but $p_i = p_i'$ for all $i \in N$
- Let j such that $k_j' > k_j$ (it must exist)
- For all $i \neq j$, if it was the case that $k_i' < k_i$ then population monotonicity would imply that $p_j' > p_j$ or $p_i' < p_i$, which is false
- We conclude that $k_i' \geq k_i$ for all $i \in N$ ■

THE QUOTA CRITERION

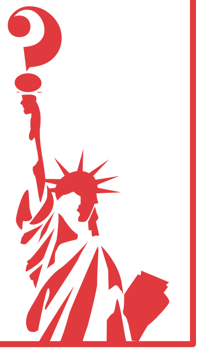
- An apportionment method satisfies the quota criterion if for all $i \in N$,

$$\lfloor q_i \rfloor \leq k_i \leq \lceil q_i \rceil$$

Poll

Of the five methods we discussed (Hamilton, Jefferson, Adams, Webster, Huntington-Hill), how many satisfy the quota criterion?

- 0
- 1
- 2
- 3
- 4
- 5



AN IMPOSSIBILITY

- An apportionment method is neutral if permuting the states permutes the seat allocation
- Theorem: There is no apportionment method that is neutral, population monotonic and satisfies the quota criterion
- Corollary: No divisor method satisfies the quota criterion

PROOF OF THEOREM

- Assume that that the method satisfies all three properties
- We claim that the method satisfies the order-preserving property: if $p_j > p_i$ then $k_j \geq k_i$
- Define an instance with $p_i' = p_j, p_j' = p_i$, and $p_t = p_t'$ for all $t \neq i, j$
- By population monotonicity, either $k_i' \geq k_i$ or $k_j' \leq k_j$
- By neutrality, $k_i' = k_j$ and $k_j' = k_i$
- It follows that $k_j \geq k_i$

PROOF OF THEOREM

State	p_i	q_i
1	69,900	6.99
2	5,200	0.52
3	5,000	0.50
4	19,900	1.99
Total	100,000	10

By the quota criterion, $k_1 \leq 7$ and $k_4 \leq 2$. Therefore, $k_2 \geq 1$ or $k_3 \geq 1$. By the order-preserving property, $k_2 \geq 1$.

State	p_i'	q_i'
1	68,000	8.02
2	5,500	0.65
3	5,600	0.66
4	5,700	0.67
Total	84,800	10

By the quota criterion, $k_1' \geq 8$. Therefore, $k_2' = 0$ or $k_3' = 0$ or $k_4' = 0$. By the order-preserving property, $k_2' = 0$.

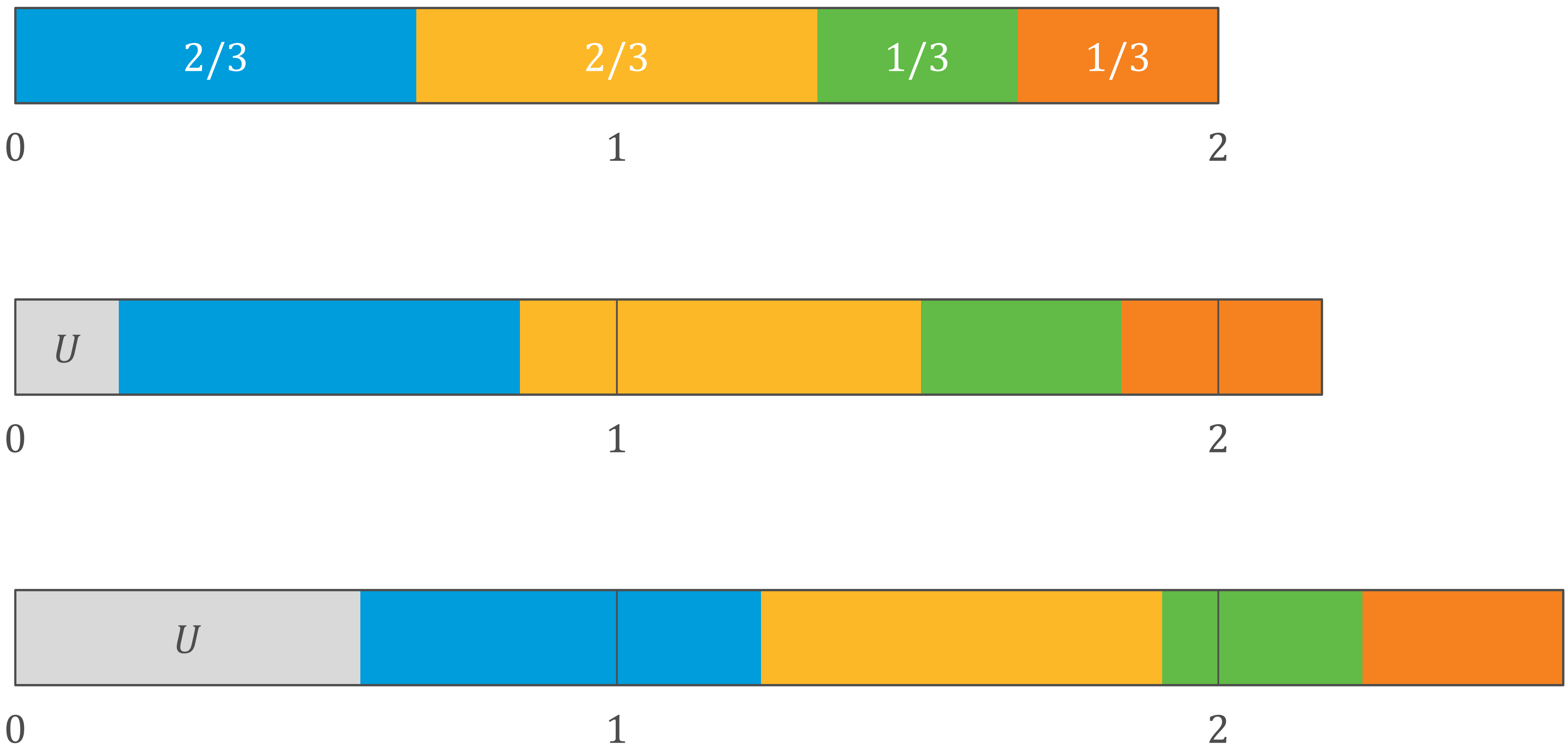
We have constructed an example where $k_1' > k_1$ and $k_2' < k_2$ yet $p_1' < p_1$ and $p_2' > p_2$ ■

RANDOMIZED APPORTIONMENT

Consider the following algorithm:

1. Take a random permutation of the label of the states (w.l.o.g. it's identity)
2. Provisionally allocate $\lfloor q_i \rfloor$ seats to each state $i \in N$, and let $r_i = q_i - \lfloor q_i \rfloor$
3. Draw $U \sim \mathcal{U}([0,1])$
4. Let $Q_i = U + \sum_{j=1}^{i} r_i$
5. For each $i \in N$, allocate an extra seat to state i if $[Q_{i-1}, Q_i)$ contains an integer

RANDOMIZED APPORTIONMENT



RANDOMIZED APPORTIONMENT

- The randomized apportionment algorithm gives each state its standard quota in expectation
- It follows that it's population monotonic in expectation
- It satisfies the quota criterion ex post

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