Optimized Democracy

Spring 2021 | Lecture 13

Random Assignment

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ASSIGNMENT PROBLEMS

School choice

Assign students to schools

Housing allocation

Assign applicants to public housing

Common thread: Each player requires exactly one good

THE MODEL

- Set of players $N = \{1, \dots, n\}$
- Set G of n goods (we'll talk later about the case of $|G| \neq n$)
- Each player has a ranking $\sigma_i \in \mathcal{L}$ over G
- An assignment is a perfect matching π between players and goods, where $\pi(i)$ is the good assigned to i
- We are interested in rules f that take $\boldsymbol{\sigma} \in \mathcal{L}^n$ and output π

SERIAL DICTATORSHIP

- Players select their favorite goods according to a predetermined order τ
- Example for the order $1 \succ_\tau 2 \succ_\tau 3 \succ_\tau 4$:

1	2	3	4
a	a	d	a
b	b	d	d
c	c	b	c
d	d	a	b

SERIAL DICTATORSHIP: PROPERTIES

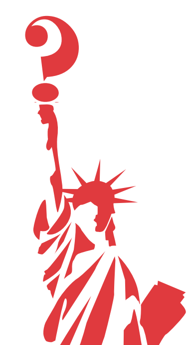
- An assignment π is Pareto efficient if there is no assignment π' such that $\pi'(i) \succcurlyeq_{\sigma_i} \pi(i)$ for all $i \in N$ and $\pi'(j) \succ_{\sigma_j} \pi(j)$ for some $j \in N$
- A rule f is strategyproof (SP) if for all $\boldsymbol{\sigma} \in \mathcal{L}^n$, for all $i \in N$ and for all $\sigma_i' \in \mathcal{L}$,

$$f(\boldsymbol{\sigma})(i) \succcurlyeq_{\sigma_i} f(\sigma_i', \boldsymbol{\sigma}_{-i})(i)$$

Poll 1

Which of the following properties is satisfied by serial dictatorship?

- Pareto efficiency
- Strategyproofness
- Both
- Neither



RANDOM SERIAL DICTATORSHIP

(Serial dictatorship with the order τ chosen uniformly at random.)

1	2	3
a	a	b
b	***b***	***c***
c	c	a

$1 \succ_\tau 2 \succ_\tau 3$

1	3	2
a	***b***	a
b	c	b
c	a	***c***

$1 \succ_\tau 3 \succ_\tau 2$

2	1	3
a	a	b
b	***b***	***c***
c	c	a

$2 \succ_\tau 1 \succ_\tau 3$

2	3	1
a	***b***	a
b	c	b
c	a	***c***

$2 \succ_\tau 3 \succ_\tau 1$

3	1	2
b	***a***	a
c	b	b
a	c	***c***

$3 \succ_\tau 1 \succ_\tau 2$

3	2	1
b	***a***	a
c	b	b
a	c	***c***

$3 \succ_\tau 2 \succ_\tau 1$

A distribution over assignments is called a lottery

LOTTERY TO RANDOM ASSIGNMENT

1	2	3
a	a	b
b	b	c
c	c	a

$1 \succ_\tau 2 \succ_\tau 3$

1	3	2
a	b	a
b	c	b
c	a	c

$1 \succ_\tau 3 \succ_\tau 2$

2	1	3
a	a	b
b	b	c
c	c	a

$2 \succ_\tau 1 \succ_\tau 3$

2	3	1
a	b	a
b	c	b
c	a	c

$2 \succ_\tau 3 \succ_\tau 1$

3	1	2
b	a	a
c	b	b
a	c	c

$3 \succ_\tau 1 \succ_\tau 2$

3	2	1
b	a	a
c	b	b
a	c	c

$3 \succ_\tau 2 \succ_\tau 1$

A random assignment is a bi-stochastic matrix $P = [p_{ix}]$ where p_{ix} is the probability player i is assigned to x

	a	b	c
1	1/2	1/6	1/3
2	1/2	1/6	1/3
3	0	2/3	1/3

RSD: PROPERTIES

- RSD is ex post strategyproof: Players cannot gain from lying regardless of the random coin flips
- In contrast to SD, RSD satisfies equal treatment of equals: For $i, j \in N$ such that $\sigma_i = \sigma_j$ it holds that $p_{ix} = p_{jx}$ for all $x \in G$
- RSD is ex post Pareto efficient: every assignment in its support is Pareto efficient
- Is this a satisfying notion of efficiency for lotteries?

ORDINAL EFFICIENCY

- Random assignment P stochastically dominates P' if for all $i \in N$ and $x \in G$, $\sum_{y \succcurlyeq_{\sigma_i} x} p'_{iy} \geq \sum_{y \succcurlyeq_{\sigma_i} x} p_{iy}$, with at least one strict inequality
- A random assignment is ordinally efficient if it isn't stochastically dominated by any other assignment

Poll 2

What is the relation between ex post efficiency and ordinal efficiency?

- Ex post $\Rightarrow$ ordinal
- Ordinal $\Rightarrow$ ex post
- Ex post $\Leftrightarrow$ ordinal
- Incomparable

RSD IS NOT ORDINALLY EFFICIENT

1	2	3	4
a	*a*	*b*	*b*
b	*b*	*a*	*a*
c	*c*	*c*	*c*
d	*d*	*d*	*d*

	a	*b*	*c*	*d*
1	5/12	1/12	1/4	1/4
2	5/12	1/12	1/4	1/4
3	1/12	5/12	1/4	1/4
4	1/12	5/12	1/4	1/4

Random serial dictatorship

	a	*b*	*c*	*d*
1	1/2	0	1/4	1/4
2	1/2	0	1/4	1/4
3	0	1/2	1/4	1/4
4	0	1/2	1/4	1/4

Stochastically dominating assignment

PROBABILISTIC SERIAL RULE

- The probabilistic serial rule is directly defined by a random assignment (more on this later)
- Each good is a "divisible" good consisting of "probability shares"
- At every point in time, all players "eat" their favorite remaining goods at the same rate
- When all goods are eaten, each player has probability shares adding up to 1

PROBABILISTIC SERIAL RULE

1	2	3	4
a	*b*	*b*	*b*
b	*c*	*c*	*d*
c	*d*	*d*	*c*
d	*a*	*a*	*a*


Good *a*


Good *c*

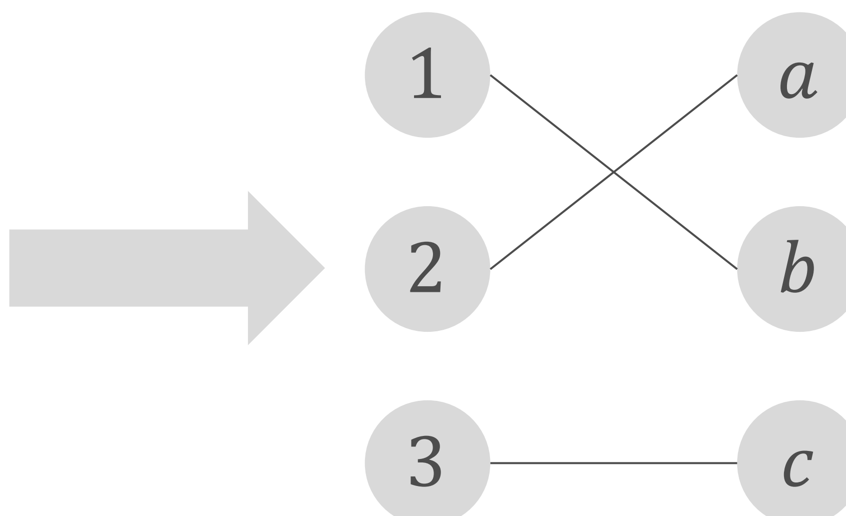

Good *b*


Good *d*

RANDOM ASSIGNMENT TO LOTTERY

- We saw that every lottery induces a random assignment, is the converse also true?
- A permutation matrix is a bistochastic matrix consisting only of zeros and ones
- A permutation matrix represents an assignment

	a	b	c
1	0	1	0
2	1	0	0
3	0	0	1



RANDOM ASSIGNMENT TO LOTTERY

	a	*b*	*c*
1	1/2	1/6	1/3
2	1/2	1/2	0
3	0	1/3	2/3

Theorem [Birkhoff-von Neumann]: Any bistochastic matrix can be obtained as a convex combination of permutation matrices

	a	*b*	*c*
1	0	1	0
2	1	0	0
3	0	0	1

× 1/6

	a	*b*	*c*
1	1	0	0
2	0	1	0
3	0	0	1

× 1/2

	a	*b*	*c*
1	0	0	1
2	1	0	0
3	0	1	0

× 1/3

PS: PROPERTIES

- Probabilistic serial obviously satisfies equal treatment of equals
- Theorem: Probabilistic serial is ordinally efficient
- Given a random assignment P and a profile $\boldsymbol{\sigma}$, define a graph $\Gamma_{P,\boldsymbol{\sigma}} = (G, E)$ where $(x, y) \in E$ iff $\exists i \in N$ such that $x \succ_{\sigma_i} y$ and $p_{iy} > 0$
- Lemma: If $\Gamma_{P,\boldsymbol{\sigma}}$ is acyclic then P is ordinally efficient

PROOF OF THEOREM

- If P is the output of PS, we claim that $\Gamma_{P,\boldsymbol{\sigma}}$ is acyclic, and conclude by the lemma
- Suppose for contradiction that $\Gamma_{P,\boldsymbol{\sigma}}$ has a cycle
- Let x be the first good in the cycle to be fully eaten at time t
- There is an edge (y, x) in $\Gamma_{P,\boldsymbol{\sigma}}$ so there is $i \in N$ such that $y \succ_{\sigma_i} x$ and $p_{ix} > 0$
- But at any point up to t, player i should have been eating y or a more preferred good, which contradicts the fact that $p_{ix} > 0$ ■

ENVY-FREENESS

- A random assignment P is envy free if for all $i, j \in N$ and $x \in G$, $\sum_{y \succcurlyeq_{\sigma_i} x} p_{iy} \geq \sum_{y \succcurlyeq_{\sigma_i} x} p_{jy}$
- Random serial dictatorship isn't envy free
- Theorem: Probabilistic serial is envy free
- This is a direct consequence of the fact that at each moment in time, all players are eating their favorite goods at the same rate

PS IS NOT STRATEGYPROOF

1	2	3	4
a	a	b	b
b	c	c	c
c	d	d	d
d	b	a	a

PS →

	a	b	c	d
1	1/2	0	1/4	1/4
2	1/2	0	1/4	1/4
3	0	1/2	1/4	1/4
4	0	1/2	1/4	1/4

1	2	3	4
b	a	b	b
a	c	c	c
c	d	d	d
d	b	a	a

PS →

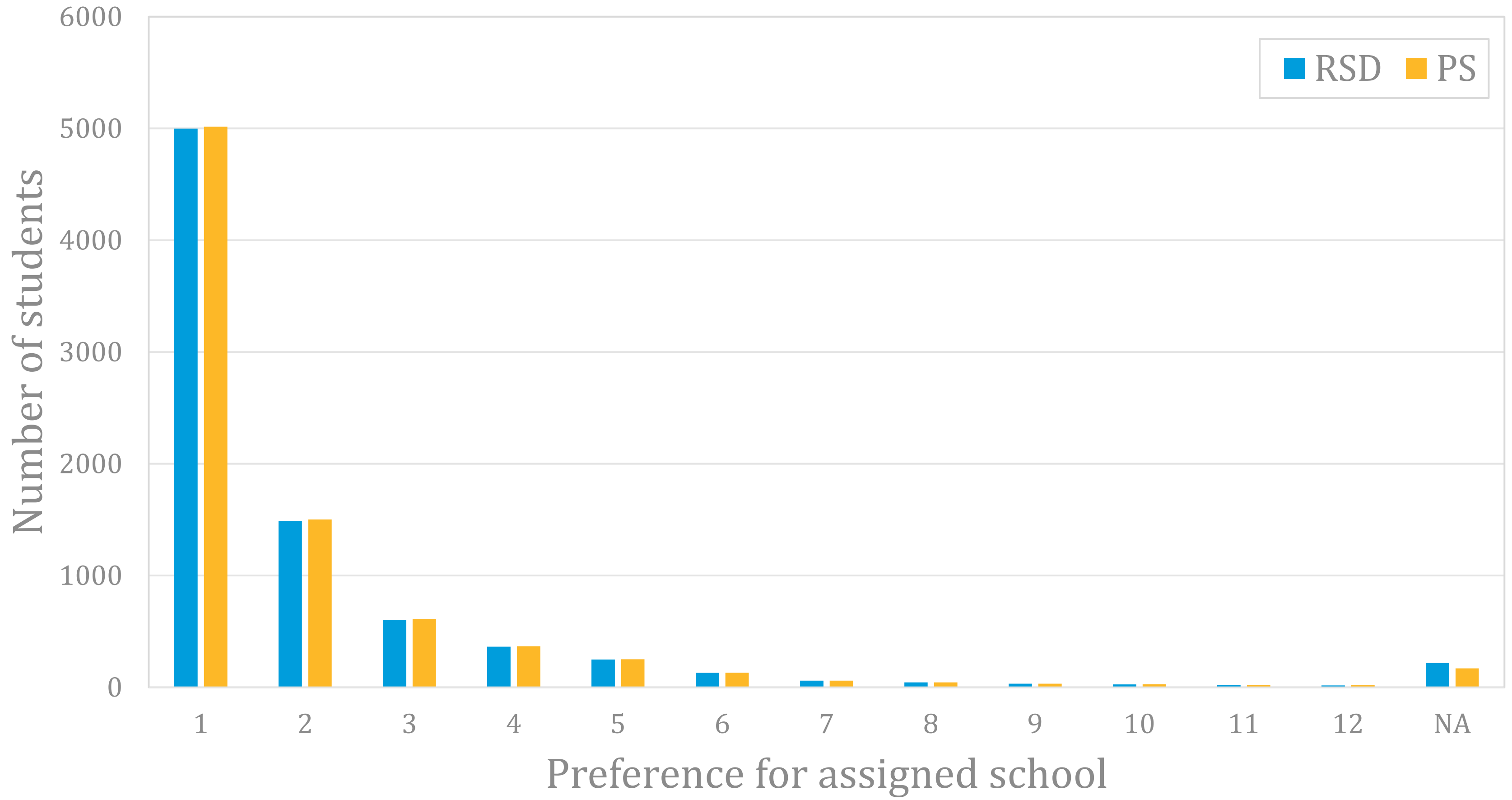
	a	b	c	d
1	1/3	1/3	1/12	1/4
2	2/3	0	1/12	1/4
3	0	1/3	5/12	1/4
4	0	1/3	5/12	1/4

AN IMPOSSIBILITY RESULT

- Theorem: There is no rule that satisfies ordinal efficiency, strategyproofness and equal treatment of equals
- If we accept equal treatment of equals as non-negotiable then the tradeoff between ordinal efficiency and strategyproofness is unavoidable

PS VS. RSD ON NYC DATA

Pathak [2006] ran RSD and PS on ("truthful") data from 8255 students in NYC



FINAL REMARKS

- A result by Che and Kojima [2010] formalizes this “equivalence in the large” between RSD and PS; does this suggest we should prefer one rule over the other?
- We’ve assumed that the number of players is equal to the number of goods, but the general case reduces to this special case

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