Optimized Democracy

Spring 2021 | Lecture 1

Voting Rules

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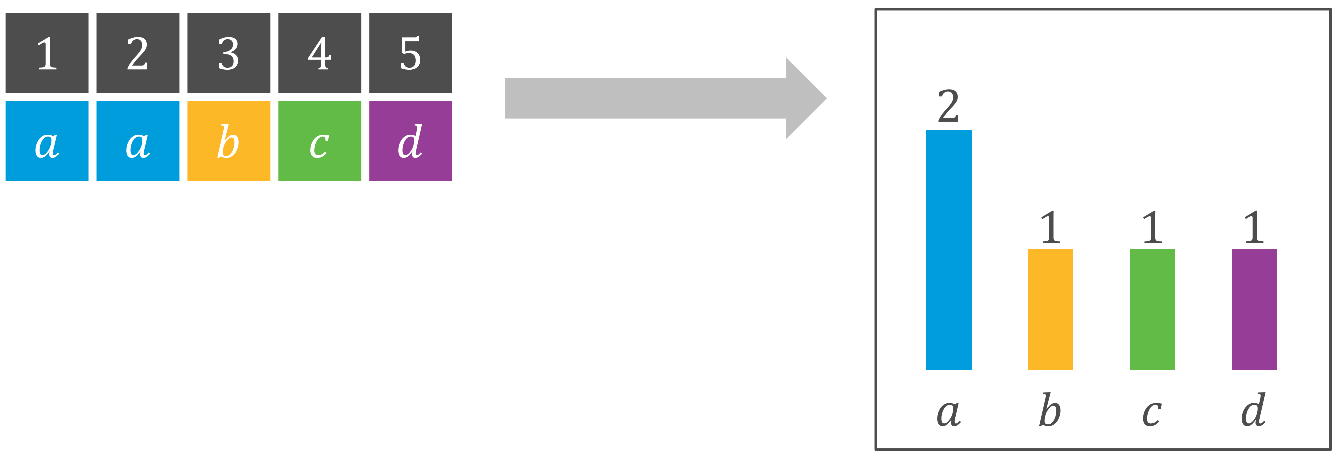
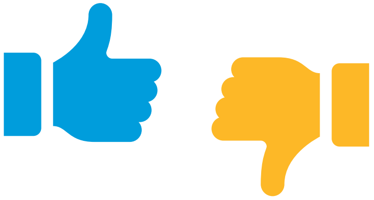
PLURALITY

- Each person votes for a single alternative, and the alternative with most points wins
- A highly problematic voting rule!



SOME BALLOT TYPES

Rankings



Approvals

Scores/stars

We will focus on rankings!

Jean-Charles de Borda

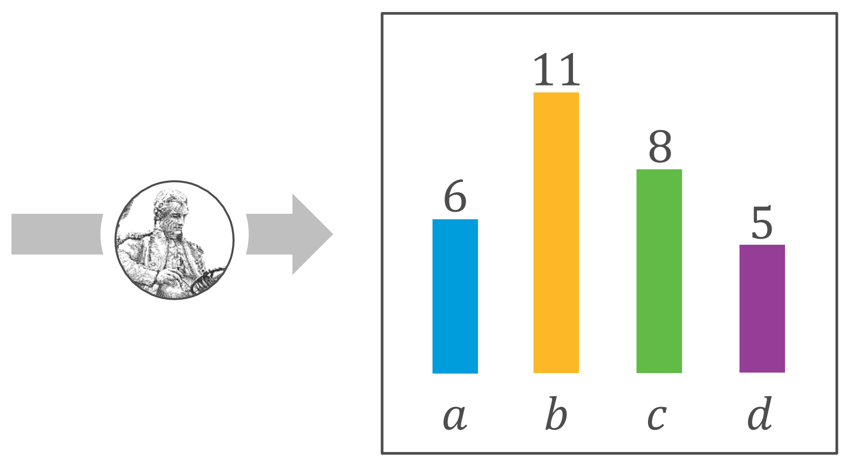
1733–1799

Mathematician, engineer, and naval officer. Also remembered as an instigator of the metric system.

BORDA COUNT

- Each voter awards $m - k$ points to the alternative placed in the k'th position, where m is the number of alternatives

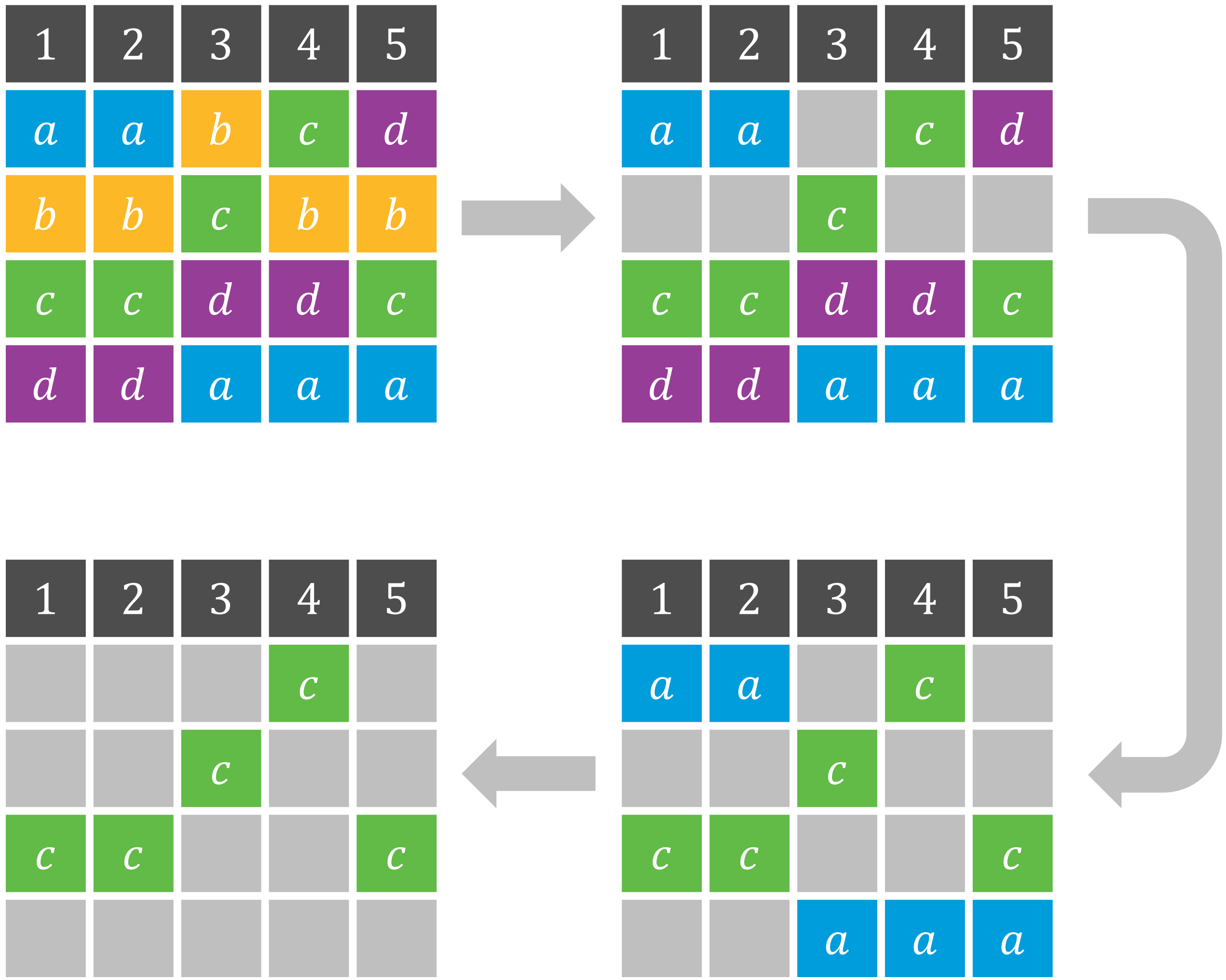
1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a



SINGLE TRANSFERABLE VOTE

- Also known as “alternative vote,” “instant-runoff voting” and (misleadingly) “ranked-choice voting”
- Votes are tabulated in rounds, where in each round the alternative with the lowest plurality score is eliminated; last alternative left standing is the winner

SINGLE-TRANSFERABLE VOTE



STV AROUND THE WORLD

■ Ireland
Used for all public elections

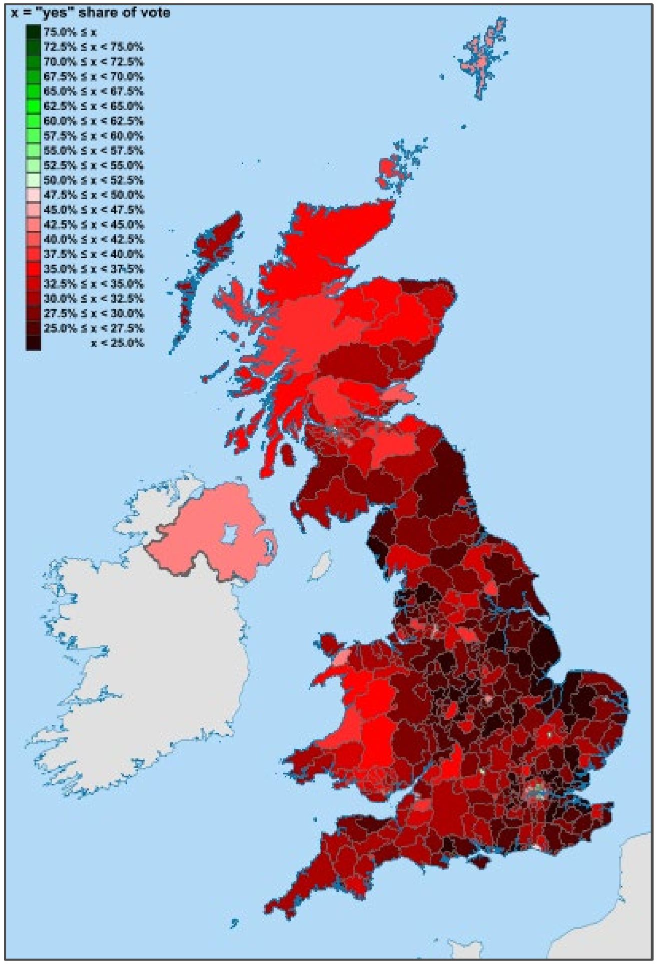
■ Canada
Used in Ontario for municipal elections

■ Australia
Used for parliamentary elections

■ USA
Used for statewide elections in Maine and in cities such as Cambridge, MA

BARRIERS TO ADOPTION

- UK referendum (2011): Choose between plurality and STV as a method for electing MPs
- Academics agreed STV is better
- But STV was seen as beneficial to a particular politician



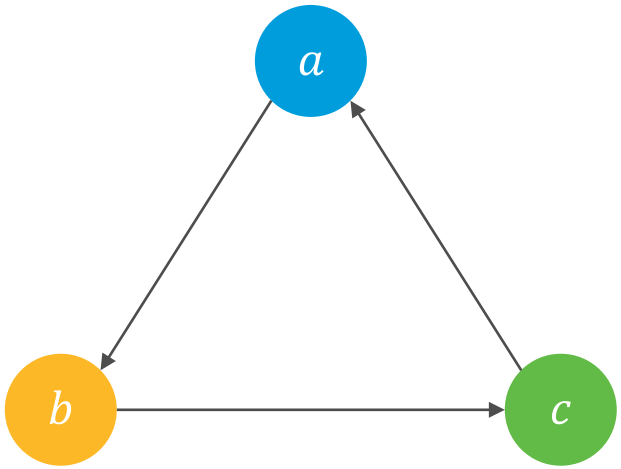
Marquis de Condorcet

1743–1794

Philosopher, mathematician, enlightened nobleman. Also known for dying mysteriously in prison.

THE CONDORCET PARADOX

1	2	3
a	*c*	*b*
b	*a*	*c*
c	*b*	*a*



The preferences of the majority may be cyclical!

CONDORCET CONSISTENT RULES

- A Condorcet winner is an alternative that defeats every other alternative in a head-to-head comparison
- A rule is Condorcet consistent if it always selects a Condorcet winner whenever it is presented with a profile that contains one

Poll

Which rule is Condorcet consistent?

- Plurality
- Borda Count
- Both rules
- Neither one

Ramon Llull

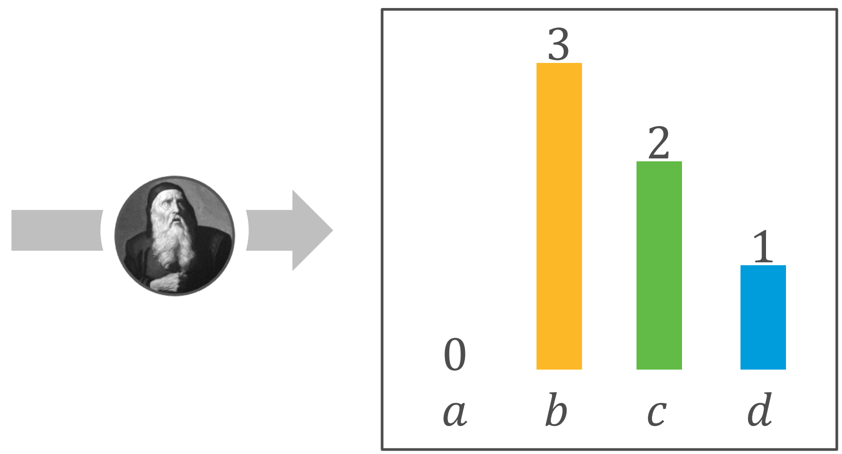
c. 1232–1315

Monk, missionary, and philosopher; one of the most influential intellectuals of his time. Also remembered for publishing a medieval parenting guide.

LLULL'S RULE

- Each alternative receives one point for each head-to-head comparison it wins (as well as for tied comparisons)

1	2	3	4	5
a	*a*	*b*	*c*	*d*
b	*b*	*c*	*b*	*b*
c	*c*	*d*	*d*	*c*
d	*d*	*a*	*a*	*a*



- Llull's rule is Condorcet consistent

Charles Lutwidge Dodgson

1832–1898

Professor of mathematics at Oxford, pioneer photographer, and beloved author. Also known for not plagiarizing Condorcet's work.

DODGSON'S RULE

- The Dodgson score of an alternative x is the minimum number of swaps between adjacent alternatives needed to make x a Condorcet winner; select an alternative with minimum score
- Dodgson's rule is Condorcet consistent
- Dodgson's rule is NP-hard to compute [Bartholdi et al. 1989]

DODGSON'S RULE

What is the Dodgson score of *b*?

1	2	3	4	5
a	*a*	*d*	*d*	*d*
b	*b*	*c*	*c*	*c*
c	*c*	*a*	*b*	*b*
d	*d*	*b*	*a*	*a*

1	2	3	4	5
a	*a*	*d*	*d*	*d*
b	*b*	*c*	*b*	*c*
c	*c*	*a*	*c*	*b*
d	*d*	*b*	*a*	*a*

1	2	3	4	5
a	*a*	*d*	*b*	*d*
b	*b*	*c*	*d*	*c*
c	*c*	*a*	*c*	*b*
d	*d*	*b*	*a*	*a*

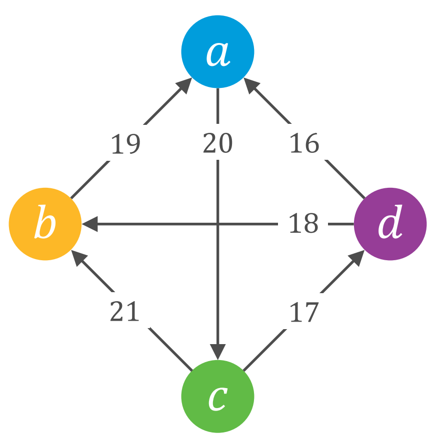
1	2	3	4	5
b	*a*	*d*	*b*	*d*
a	*b*	*c*	*d*	*c*
c	*c*	*a*	*c*	*b*
d	*d*	*b*	*a*	*a*

SCHULZE'S RULE

- Let $P(x, y)$ denote the number of voters who prefer x to y
- A path from x to y of strength p is a sequence of alternatives $x = a_1, \dots, a_k = y$ such that for all $i = 1, \dots, k-1$, $P(a_i, a_{i+1}) > P(a_{i+1}, a_i)$ and $P(a_i, a_{i+1}) \geq p$
- Let $S(x, y)$ be the strength of the strongest path from x to y (it's 0 if there's no path)
- Exercise: if $S(x, y) > S(y, x)$ and $S(y, z) > S(z, y)$ then $S(x, z) > S(z, x)$
- Therefore there exists a winning alternative $x^\star$ such that $S(x^\star, y) \geq S(y, x^\star)$ for all y
- Schulze's rule is Condorcet consistent

SCHULZE'S RULE

5 voters	2 voters	3 voters	4 voters	3 voters	3 voters	1 voter	5 voters	4 voters
a	*a*	*a*	*b*	*c*	*c*	*d*	*d*	*d*
c	*c*	*d*	*a*	*b*	*d*	*a*	*b*	*c*
b	*d*	*c*	*c*	*d*	*b*	*c*	*a*	*b*
d	*b*	*b*	*d*	*a*	*a*	*b*	*c*	*a*



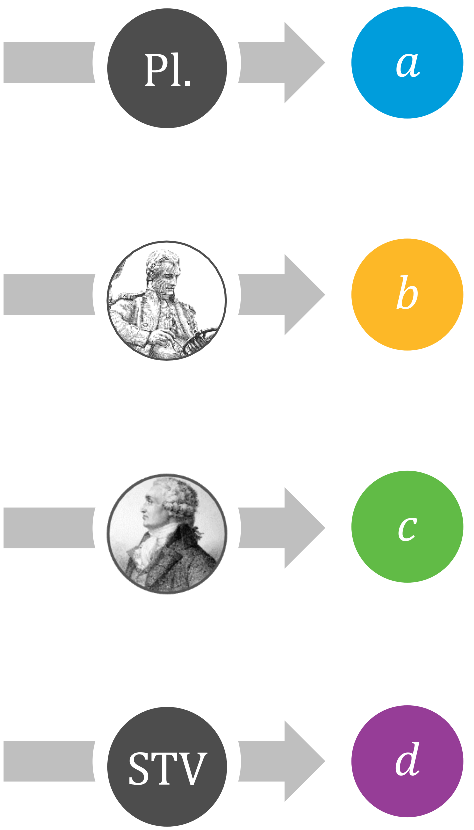
Pairwise comparisons

	a	*b*	*c*	*d*
a	–	20	20	17
b	19	–	19	17
c	19	21	–	17
d	18	18	18	–

Strength of paths $S(x, y)$

AWESOME EXAMPLE

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	*b*	*c*	*c*	*d*	*e*
b	*d*	*d*	*e*	*e*	*c*
c	*c*	*b*	*b*	*c*	*b*
d	*e*	*a*	*d*	*b*	*d*
e	*a*	*e*	*a*	*a*	*a*



One rule
to rule
them all?