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Section 4: Convex Optimization, Integer Programming

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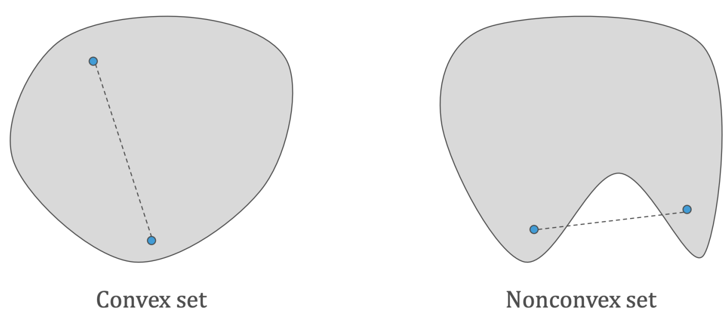
# 1 Convex Optimization

**Definition 1 (Convex Optimization Problem)** *A convex optimization problem is a specialization of a general optimization problem* $\min_{\mathbf{x}} f(\mathbf{x})$ *such that* $\mathbf{x} \in \mathcal{F}$ *where the objective function* $f : \mathbb{R}^n \to \mathbb{R}$ *is a convex function, and the feasible region* $\mathcal{F}$ *is a convex set.*

## 1.1 Convex Sets

**Definition 2 (Convex Sets)** *A set* $\mathcal{F} \subseteq \mathbb{R}^n$ *is convex if for all* $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ *and* $\theta \in [0,1]$, $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in \mathcal{F}$. *In other words, a set is convex if and only if all convex combinations of any two elements results in another element contained within the set.*




Intuitively the definition of convex sets is: for any two points in the set, if I draw the line between the two points, every point in the line must be in the set. Now we'll provide some examples of convex sets:

**Example 1** *Prove that*

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n : \forall i = 1, \dots, n, a \leq x_i \leq b\}$$

*is a convex set.*

**Proof:** Let $\mathbf{x}, \mathbf{y} \in \mathcal{F}$, and $\theta \in [0,1]$. For all $i = 1, \ldots, n$, $a \leq x_i$ and $a \leq y_i$, so $\theta x_i + (1-\theta) y_i \geq \theta a + (1-\theta) a = a$. Similarly, $\theta x_i + (1-\theta) y_i \leq \theta(b) + (1-\theta) b = b$. Therefore, $\theta \mathbf{x} + (1-\theta)\mathbf{y} \in \mathcal{F}$. ■

Moreover, *intersections of convex sets are convex*. An argument works likes this: for any two points within the intersection of several convex sets, the line between them is (because each of the sets is convex) also withing each of the convex sets, so the line must be in the intersection, which proves the intersection is a convex set.

Note that it is *not* true that the union of convex sets is convex.

**Problem 1** *Give an example where the union of convex sets is not convex.*

**Problem 2** *Show that a set is convex if and only if its intersection with any line is convex.*

**Problem 3** *Are the following sets convex?*

1. *A set of the form* $\{x \in \mathbb{R}^n | \alpha \leq w^T x \leq \beta\}$.
2. *The set of points closer to one point than another (by Euclidean distance), i.e.,*

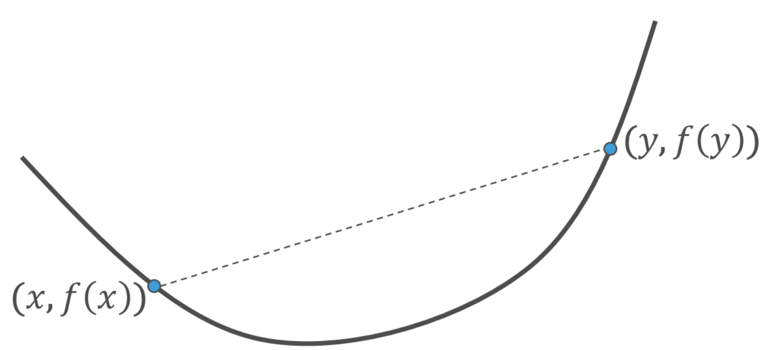
$$\{x | \mathbf{dist}(x, S) \leq \mathbf{dist}(x, Z)\},$$

*where* $S, Z \in \mathbb{R}^n$.

## 1.2 Convex Functions

**Definition 3 (Convex Function)** *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is convex if and only if for any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\theta \in [0,1]$,

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$$



*For functions that $f : \mathbb{R} \to \mathbb{R}$ that are twice differentiable, this definition is equivalent to saying that $f''(x) \geq 0$ for all $x \in \mathbb{R}$ (this may have been the definition you have seen before in calculus courses!)*

*Finally, $f$ is concave if and only if $(-1)f$ is convex.*

Now let's give some examples for convex functions:

**Example 2**

- *Exponential.* $f(x) = e^{ax}$. *We can show this is convex via the second derivative:*

$$f''(x) = a^2 e^{ax} \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

- *Euclidean Norm.* $f(\mathbf{x}) = ||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^{n}(x_i)^2}$. *We can show this is convex via the definition (and via the triangle inequality):*

$$\begin{aligned} ||\theta\mathbf{x} + (1-\theta)\mathbf{y}||_2 &\leq ||\theta\mathbf{x}||_2 + ||(1-\theta)\mathbf{y}||_2 \\ &= \theta||\mathbf{x}||_2 + (1-\theta)||\mathbf{y}||_2 \end{aligned}$$

We also have that weighted sums of convex functions are convex:

**Example 3** *Let $f(x) = \sum_{i=1}^{m} a_i f_i(x)$, where $f_i$ is convex and $a_i \geq 0$ for all $i = 1, \ldots, m$. Then $f$ is convex.*

**Proof:**

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \sum_i a_i f_i(\theta x + (1-\theta)y) \\ &\leq \sum_i a_i(\theta f_i(x) + (1-\theta)f_i(y)) \\ &= \sum_i a_i f_i(x) + (1-\theta)\sum_i a_i f_i(y) \\ &= \theta f(x) + (1-\theta)f(y) \end{aligned}$$

■

**Problem 4** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is increasing and convex on its domain $(a, b)$. Let $g$ denote its inverse, i.e., the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. Suppose that $f$ and $g$ are differentiable. What can you say about the convexity or concavity of $g$?*

### 1.3 Convex Optimization Problems

Let's give a few examples of convex optimization problems.

**Example 4 (Linear Programming)** *The linear programming problem can be formulated as finding*

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \qquad \text{such that } A\mathbf{x} = \mathbf{a} \text{ and } B\mathbf{x} \leq \mathbf{b},$$

*where* $\mathbf{x} \in \mathbb{R}^n$ *is the optimization variable, and* $\mathbf{c} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $\mathbf{a} \in \mathbb{R}^m$, $B \in \mathbb{R}^{k \times n}$, $\mathbf{b} \in \mathbb{R}^k$ *are the problem data.*

*Verbally, we are trying to minimize a (convex) linear objective function subject to linear constraints (which gives us a convex set). Taking the dot product of* $\mathbf{x}$ *with* $\mathbf{c}$ *gives us our objective function evaluated at an input of* $\mathbf{x}$*, the pairing* $\mathbf{A}$ *and* $\mathbf{a}$ *encode a set of equality constraints and the pairing* $\mathbf{B}$ *and* $\mathbf{b}$ *encode a set of inequality constraints.*

**Problem 5** *Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example the feed mix contains two active ingredients and a filler to provide bulk. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:*

| Nutrient | A | B | C | D |
|---|---|---|---|---|
| gram | 90 | 50 | 20 | 2 |

*The ingredients have the following nutrient values and cost:*

| | A | B | C | D | Cost/kg |
|---|---|---|---|---|---|
| Ingredient 1 (gram/kg) | 100 | 80 | 40 | 10 | 40 |
| Ingredient 2 (gram/kg) | 200 | 150 | 20 | 0 | 60 |

*What should be the amounts of active ingredients and filler in one kg of feed mix? Model this as an LP.*

# 2 Integer Programming

## 2.1 Introduction

**Definition 4 (Feasibility Problem)** *Find $x_1, \ldots, x_l$ such that*

$$\forall i \in [k], \sum_{j=1}^{l} a_{ij} x_j \leq b_i$$
$$\forall j \in [l], x_j \in \mathbb{Z}.$$

**Definition 5 (Integer Programming Optimization Problem)**

$$\max \sum_{j=1}^{l} c_j x_j$$

*such that*

$$\forall i \in [k], \sum_{j=1}^{l} a_{ij} x_j \leq b_i$$
$$\forall j \in [l], x_j \in \mathbb{Z}.$$

**Example 5 (Envy-Free)** *Suppoes we have players $N = \{1, \ldots, n\}$ and items $M = \{1, \ldots, M\}$, and player $i$ has value $v_{ij}$ for item $j$. Partition the items into bundles $A_1, \ldots, A_n$. We say that the $A_1, \ldots, A_n$ is envy-free iff*

$$\forall i, i', \sum_{j \in A_i} v_{ij} \geq \sum_{j \in A_{i'}} v_{ij}.s$$

*Formulate this as an IP.*

**Proof:** Let $x_{ij} \in \{0, 1\}$, $x_{ij} = 1$ iff $j \in A_i$. Then our feasibility problem is:

$$\text{Find } x_{11}, \ldots, x_{nm} \text{ such that}$$
$$\forall i \in N, \forall i' \in N, \sum_{j \in M} v_{ij} x_{ij} \geq \sum_{j \in M} v_{ij} x_{i'j}$$
$$\forall j \in M, \sum_{i \in N} x_{ij} = 1$$
$$\forall i \in N, j \in M, x_{ij} \in \{0, 1\}$$

■

**Problem 6** *Recall the 8 queens puzzle: on an $8 \times 8$ grid, place 8 queens such that no two are in the same row, column, or diagonal. Formulate this as an integer program.*