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Section 3: CSPs and Backtracking Search

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Constraint Satisfaction Problems (CSPs)

A constraint satisfaction problem, also known as a CSP, is characterized by 3 main components outlined below:

- Decision variables denoted: $\{X_1, \ldots, X_n\}$ i.e. what number of assignments need to be made? How many blanks need to be filled in? Generally the more decision variables, the more challenging the CSP.
- Domains for each variable denoted: $\{D_1, \ldots, D_n\}$ i.e. what values are plausible for each decision variable. E.g. only integer values 1,2, or 3? Any real number? If the decision variables are binary the associated domains are $\{0, 1\}$.
- A set of constraints defined on subsets of variables i.e. equalities and inequalities that describe the way our assignment of values to decision variables is limited e.g. how choosing value A for X_1 might limit what we choose for X_2 if there exists a constraint relating X_1 and X_2.

Types of Constraints

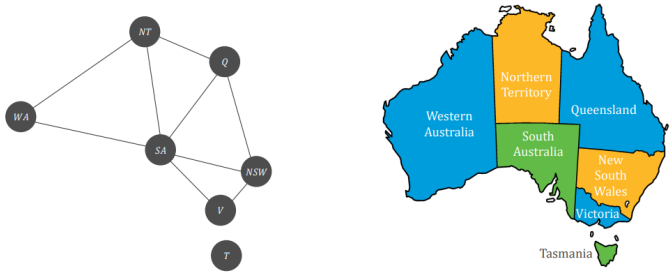
- Unary constraint - Restricts the value of a single variable e.g. $X_1 \geq 5$
- Binary constraint - Relates 2 variables to one another e.g. $X_1 + X_2 \leq 5$
- A CSP with only unary and binary constraints is called a binary CSP and can be represented using a constraint graph (see Example 1 below)
- Other constraints may involve more than 2 decision variables e.g. $X_1 + X_2 - X_3 = 12$

A solution to a CSP is a complete and consistent assignment. This means that every variable has been assigned a value and all of the constraints are satisfied.

Example 1 *Map coloring is a classic example of CSPs in action. Maps are drawn such that no 2 countries which share a border, also share the same color. Or said another way, you can easily see the borders between countries on the map because all borders have different colors on either side. Here the decision variables are the color assignments to each state/country/region of which there are N e.g. in the US there might be* 50 *states plus Washington DC, Puerto Rico etc. depending on the map. The domains would be the*

colors used for the map of which there my be 3 or 4 e.g. [Red, Orange, Green, Blue]. The constraints, as mentioned above, would be a set of binary constraints which stipulate that no to bordering countries can share the same color.

- Variables = {WA,NT,SA,Q,NSW,V,T}
- D_i = {■■■}
- Constraints: adjacent regions have different colors

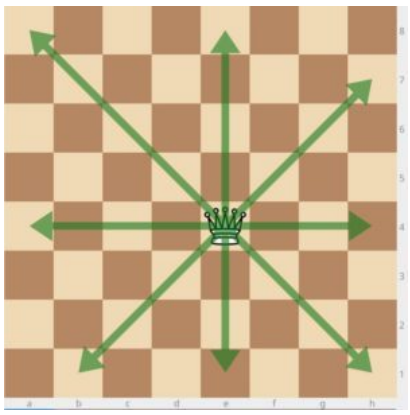


Problem 1 *Consider the N-queens problem, where we wish to place N queens on a N-by-N chess board in such a configuration that no 2 queens threaten one another, i.e. in 1 move, no queen could take another queen. That is, no queen is in the same row, column, or diagonal as another queen.*

Formulate this problem as a CSP. What are the decision variables? What are the variable domains? What are the constraints?

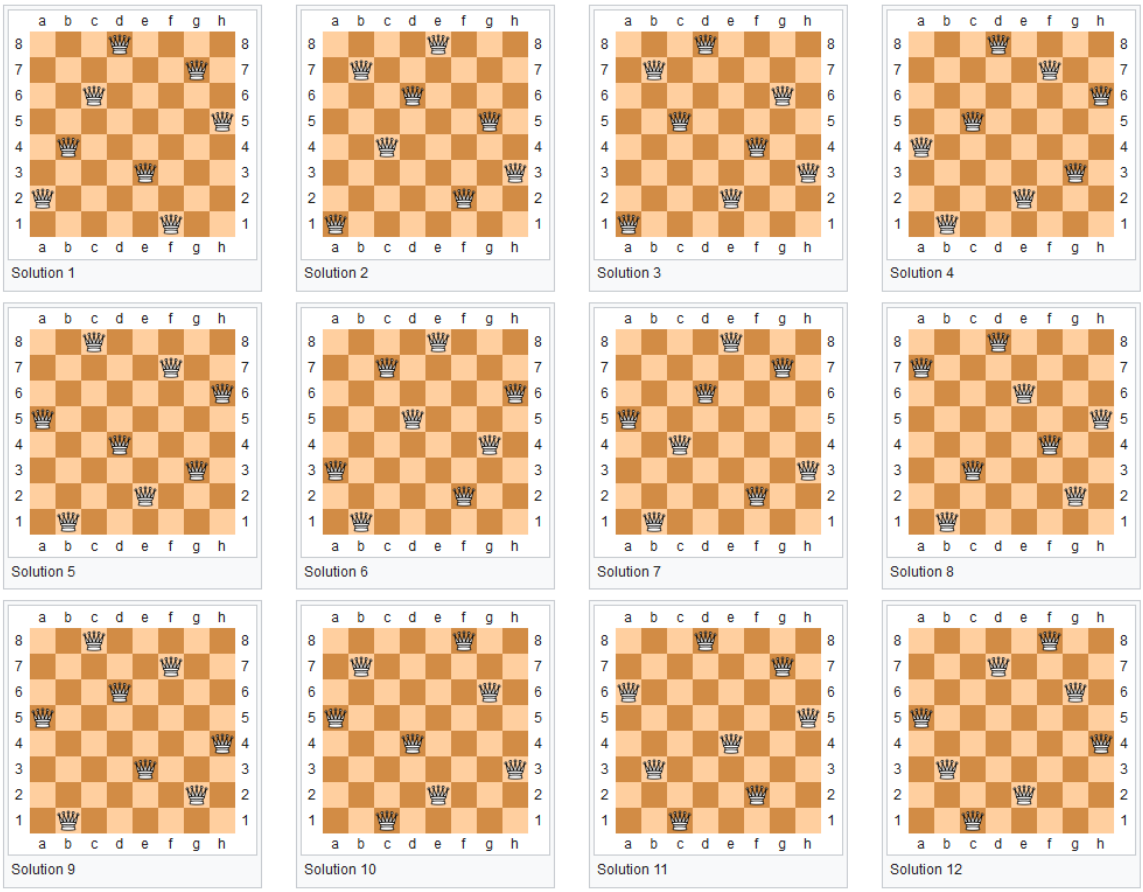
Draw a valid solution (if one exists) for $N = 1, \ldots, 5$.

Hint: Recall that a queen can move any number of tiles up/down, left/right or along the diagonals going through its current square. See below for an illustration of a queen's movement on a chess board.



Source: https://www.ichess.net

Below are 12 fundamental solutions to the $N = 8$ queens problem (a standard chess board). Note, while there are 92 possible solutions, we tend to focus only on the 12 fundamental solutions, which are able to generate all 92 distinct board arrangements satisfying the 8-queens puzzle criteria using the symmetry operations of rotation and reflection.



Source: https://en.wikipedia.org/wiki/Eight_queens_puzzle

Problem 2 *Constraint Satisfaction Problem*

Imagine that Harvard again rewrote their CS concentration requirements. Clare just began her senior year at Harvard and wants to graduate as soon as possible. She decided to leave all of her CS requirements for her last year and all she has left to take are CS requirements. Model the problem of her trying to find a set of classes to satisfy all requirements as a CSP. What are the decision variables? What are the initial domains for each of your variables? Propose at least one selection of courses that would satisfy Clare's requirements.

Suppose the updated Harvard CS department offers the following courses:

- CS20: Logic and Discrete Math
- CS50: Intro to Programming
- CS107: Systems Development for Computational Science
- CS109a: Data Science 1: Introduction to Data Science
- CS109b: Data Science 2: Advanced Topics in Data Science
- CS124: Data Structures and Algorithms
- CS181: Machine Learning
- CS182: Artificial Intelligence
- CS184: Reinforcement Learning
- CS600: Graph Theory

In order to graduate from the CS degree program, a student must fulfill the following coursework requirements:

Algorithms Requirement: (CS50 AND CS124) OR (CS124 AND CS182) OR (CS181 AND CS600)

Machine Learning Requirement: (CS182 AND CS181) OR (CS109a AND CS109b)

Reinforcement Learning Requirement: CS181 OR CS184

Graphical Models Requirement: CS124 OR CS600

Software Development Requirement: CS107 OR CS124

Course Count Requirement: Students must take at least 8 classes in CS

Single Usage Requirement: Courses cannot be used to count toward multiple graduation requirements e.g. if you use CS182 in part to fulfill part of the algorithms requirement, then it cannot count toward either the Machine Learning Requirement.

Assume that a student can only take 4 classes per semester and that all classes are offered each semester so it does not matter which order they are taken in.

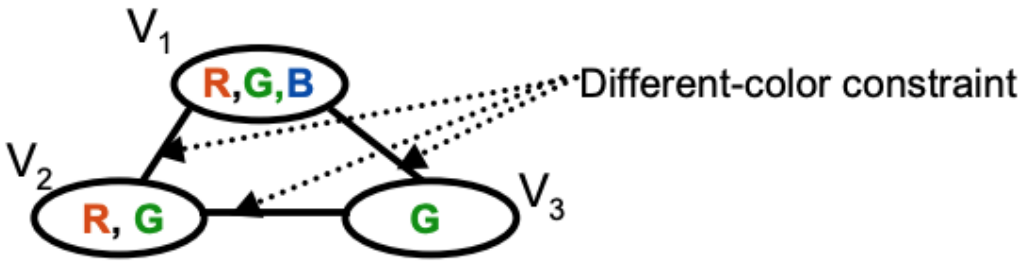
Problem and solution adapted from: https://www.cs.cmu.edu/afs/cs/academic/class/15381-s07/www/hw2/homework2_solutions.pdf

Arc Consistency

Arc consistency is an important tool for reducing the possible choices we have for each decision variable and operates by logically ruling out certain assignments that would violate the constraints.

- Definition: A variable X_i is arc-consistent with respect to variable X_j if for every value in D_i there exists a value in D_j that satisfies the binary constraint on (X_i, X_j).
- In other words, for each decision variable X_i, we should consider each value V_t remaining in its domain D_i and ask ourselves what would happen if we assigned V_t to X_i. If such an assignment is made, would there be a valid assignment for every other variable in the problem? If so, then we can move on to the next value, if not, we should remove V_t from D_i since we have learned that such an assignment is infeasible i.e. it would make us unable to satisfy one or more binary constraints.
- A CSP is said to be arc-consistent if every variable is arc-consistent with respect to every other variable.
- Enforcing arc consistency will sometimes reduce the domains to such an extent that there remains only 1 value for each decision variable, this will then have solved the CSP. Even if enforcing arc-consistency does not result in only 1 remaining value for each decision variable, it can still provide a significant reduction in the possible values to consider and can be a very useful pre-processing step for a search algorithm by considerably reducing the size of the search space.
- In certain cases, the constraints will not be restrictive enough to make arc consistency lead to any reduction in the variable domains, but we are no worse off by trying to apply it as a first step.

Problem 3 (Arc Consistency) *Is V_1 arc-consistent with V_2 and V_3? Is V_2 arc-consistent with V_3? Finally, is this CSP arc-consistent?*



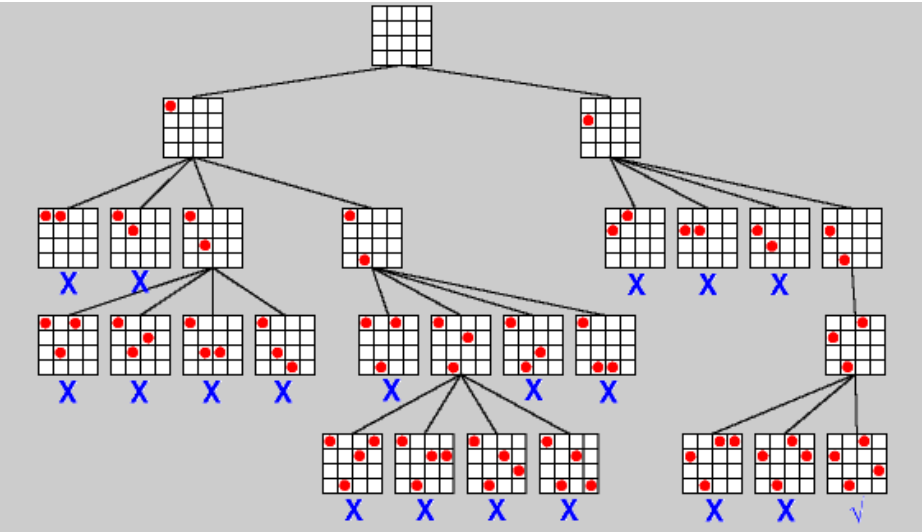
Source: https://cs.fit.edu/ dmitra/ArtInt/lectures/constraint.pdf

Backtracking Search

Backtracking search is a search algorithm used to solve CSPs in a fast and efficient way, which exhaustively searches through all combinations of possible assignments to the decision variables until it finds a set of values that satisfies all constraints; however, unlike brute-force guess-and-check, backtracking search utilizes heuristics and forward-checking to greatly reduce the average search time required to locate a valid assignment.

Backtracking search is very similar to DFS, but with early pruning of search tree branches and heuristics to help decide which value assignments are most likely to lead to a valid solution. Below is a simplified conceptual outline of the steps backtracking search uses to find a solution if one exists:

1. Use a heuristic to select which of the remaining unassigned variables to assign a value to next (e.g. if we have unassigned variables A, B, and C perhaps we select decision variable C)
2. Use another heuristic to select which of the potential remaining values for the selected unassigned variable to assign to it and remove it from the set of unassigned variables (e.g. out of the options (1,2,3) assign value 3 to variable C). Also check that this is a valid assignment based on the constraints
3. Repeat steps 1 and 2 until there are no more remaining unassigned variables or we reach an impasse where we have an unassigned variable with no possible values to assign it
    (a) If there are no remaining unassigned variables and all constraints are satisfied, then this is a valid solution and our search process ends!
    (b) If we reach an impasse or if the assignment of values does not satisfy the constraints, then backtrack to the last assignment made, and make a different assignment and then continue again
    (c) After each assignment, we can also use forward checking to enforce arc-consistency in order to limit the number of decision variables to consider and also check if there is a foreseeable impasse along this search branch, and if so, to backtrack early before spending unnecessary iterations to come to the same conclusion later.



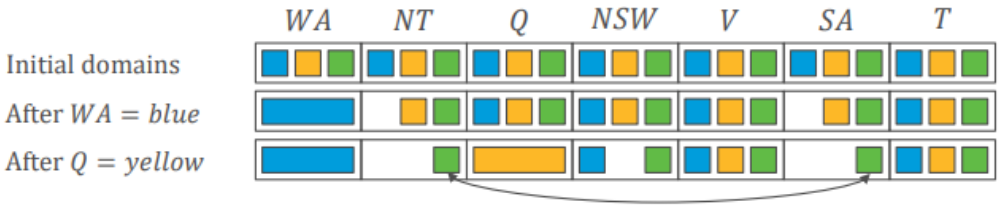
Backtracking search being used to solve the N=4 queens problem

Source: https://ivoroshilin.wordpress.com/

As mentioned above, there are 3 main components of backtracking search which differentiate it from brute force guess-and-check, they are as follows:

1. Variable ordering heuristic - This heuristic tells us which unassigned variable we should assign next in order to find a solution faster than randomly selecting.
   (a) The minimum remaining value heuristic chooses the variable with the fewest legal values
   (b) For example: $A : \{1, 2, 3\}, B : \{1, 2, 3, 4, 5\}, C : \{1, 8\}$. This heuristic would select variable C since it has the smallest remaining legal candidate values to try
2. Domain ordering heuristic - This heuristic tells us which value within the unassigned variable domain to try next in our iterative assignment process
   (a) The least constraining value heuristic chooses the value that rules out the fewest choices for other variables i.e. the value that limits the domains of other decision variables based on the constraints the least
   (b) For example: Select $C : \{1, 8\}$. This heuristic might choose 2 for instance if 8 reduced the possible candidates for all other variables the least among the options $\{1, 8\}$.
3. Forward checking - Allows us to interleave inference in our search process by enforcing arc-consistency after each variable assignment. That is, when a new assignment is made, we should go through and reduce the list of potential candidate values for the other unassigned variables based on the constraints that such an assignment would rule out.

Example 2 *An example of forward checking from the map coloring CSP mentioned in lecture:*



1. *In the above example, we make our first assignment to WA, we assign it the color blue. This then removes blue from the domains of NT and SA by enforcing-arc consistency.*

2. *Then we make our next assignment of yellow to Q. This then removes yellow from the domains of NT, NSW, and SA by enforcing-arc consistency.*

3. *At this point, we should stop our assignment process even though we still have unassigned variables NT, NSW, and SA since we can see an obvious impasse. NT and SA both now only have green as their only remaining candidate value, which we know cannot work since these are neighboring regions and therefore cannot have the same color. Arc-consistency would fail when it examines NT and SA since there does not exist a valid assignment to SA if NT is assigned the color green. Therefore we should backtrack early and change our assignment to Q.*