Lecture 9: Game Theory

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In this lecture we cover concepts in *Game theory* and will start covering multi-agent systems, which seeks to understand and accurately model the interactions between several distinct parties. As compared to the problems and formulations we saw in Search, Constraint Satisfaction Problems (CSPs), etc, we will talk about multiple agents in either a cooperative or in an adversarial situation. Here, we present game theory as a main tool for thinking about such interactions.

1 Normal-Form Game

1.1 Definition and Concepts

The normal form formulation of a game can be defined as:

- Set of players $N = \{1, \ldots, n\}$
- Strategy set S: The strategies that the players can take
- Utility function of player i: $u_i : S^n : R$, the utility player i will have when each $j \in N$ plays the strategy $s_j \in S$

For example, in the *Ice Cream War* game, each of the two players on the beach wants to choose the best location to sell the ice cream so that they can have as many customers as possible. Suppose the decisions of customers buying ice cream are completely dependent on the distance between them and the players, then the players will try their best to take up the places on beach and try to avoid losing customers. Therefore, the problem can be formalized as a two-player game where the strategies will be the point they choose at the beach, ranging from 0 to 1. Therefore, the problem can be described as:

- $N = \{1, 2\}$
- $S = [0, 1]$ The point on the beach, ranging from 0 to 1
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2} & \text{if } s_i < s_j \\ 1 - \frac{s_i+s_j}{2} & \text{if } s_i > s_j \\ \frac{1}{2} & \text{if } s_i = s_j \end{cases}$, since the customers tend to buy ice cream from nearest shop.

Definition 1 (Dominance) *Dominance refers to a strategy $s = (s_1, \ldots, s_n)$ where each player i chooses a strategy s_i that maximizes their utility u_i regardless of the strategy s_j used by any other player $j \in N \setminus \{i\}$.*

1.2 Examples of Games

Here, we present two typical games with different sets of strategies. Generally, we will demonstrate the game with a matrix where the row and column represent the strategies of two players separately. And the entries are the utilities they can get.

The Prisoner's Dilemma

For both prisoners, the dominant strategy, i.e., the strategy that will give them most utility than others no matter what the other player chooses, will be to defect. In this case, there is a dominant strategy because it will give them highest utility regardless of what the other player choose.

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

Figure 1: The Prisoner's Dilemma

The Professor's Dilemma

The best strategy of class of professor will depend another player's strategy. For example, when the class chooses to listen, then the professor should choose to make effort. But when the class chooses to sleep, then the professor should choose to slack off. This will be different from the situation in Prisoner's dilemma, where the dominant strategy will always be to defect for both prisoners. In this case, there is no dominant strategy since the strategy that will bring better pay-off will depend on the strategies of other players.

1.3 Example

Professor \ Class	Listen	Sleep
Make effort	10^6,10^6	-10,0
Slack off	0,-10	0,0

Figure 2: The Professor's Dilemma

1.4 Nash Equilibrium

For the second example mentioned above, the professor dilemma, we need a notion to deal with such type of game. Here comes the Nash Equilibrium

Nash Equilibrium appears when no player wants to unilaterally deviate, since each player's strategy is the best given the strategies of other players, i.e. their utility will decrease if they change their current strategies given that other's strategies will not change.

It can be described in a mathematical way: A Nash equilibrium is a vector of strategies $s = (s_1, ..., s_n) \in S^n$ such that for all $i \in N, s'_i \in S$, $u_i(s) \geq u_i(s_1, ..., s_{i-1}, s'_i, s_{i+1}, ..., x_n)$

For example, in the professor's dilemma, listening for class and make effort for professor; sleep for class and slack off for professor, will be both the Nash Equilibrium in this game. And for the ice cream war, the 1/2 point will be the Nash Equilibrium.

2 The Hotelling Model

Imagine that we can visualize political spectrum on the real line below.

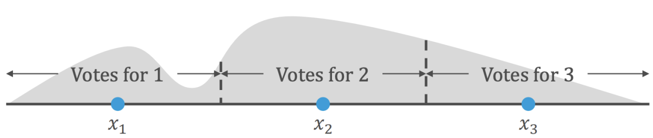

Figure 3: Political Spectrum

There is a nonatomic distribution of voters with each voter taking a point on the real line. Similarly, candidates are players who take positions on the spectrum, $x_1, x_2,$ and x_3 in the example above. Each candidate attracts voters with positions closest to them on the real line with ties being split equally.

Consider the case with two players/candidates x_1 and x_2:

1. Let their individual utilities be 1 if they win, 0 if they lose, and $\frac{1}{2}$ if they tie.
2. Let m represent the median *voter* position (for simplicity assume that it is unique.)

Depending on x_2's position, x_1's best position is as follows:

- If $x_2 < m$, the best response for player 1 is to take any position x_1 such that:

$$x_1 > x_2 \text{ and } \frac{x_1 + x_2}{2} < m$$

- Similarly, if $x_2 > m$, the best response for player 1 is to take any position x_1 such that:

$$x_1 < x_2 \text{ and } \frac{x_1 + x_2}{2} > m$$

- Finally, if $x_2 = m$, the best response for player 1 is to take any position x_1 such that:

$$x_1 = m$$

The only point where the best response sets of the two players intersect is (m, m). Therefore, the unique Nash equilibrium is at (m, m).

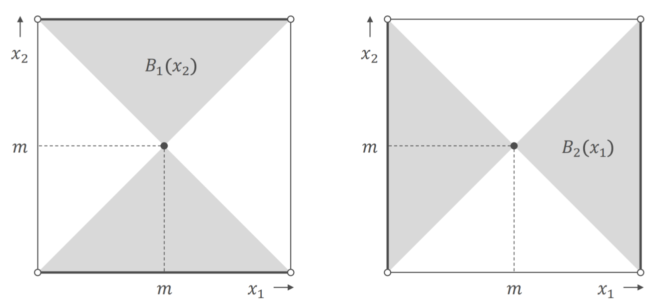

Figure 4: The best response sets of player 1 (left) and player 2 (right).

3 Mixed Strategies

3.1 Definition and Concepts

A mixed strategy is a probability distribution over (pure) strategies.

- Distribution of strategies:
$$x_i(s_i) = P(i \text{ plays } s_i)$$
- Utility of player $i \in N$:
$$u_i(x_1, ..., x_n) = \sum_{(s_1,...,s_n)\in S^n} u_i(s_1, ..., s_n) \prod_{j=1}^{n} x_j(s_j)$$
which is the expectation over the distribution of mixed strategies.

3.2 Example

Figure 5: Rock-Paper-Scissors

Consider the game of Rock-Paper-Scissors. In the above payoff matrix, we assign payoffs of 1, 0, and -1 to wins, ties, and losses respectively.

Under this construction, is there a pure Nash Equilibrium?
We see that there is not a pure Nash Equilibrium because for any strategy profile, there is always at least one player who is not winning. This player can always deviate to a winning strategy, and thus there is no pure Nash Equilibrium. Can we find a mixed Nash Equilibrium?

Exercise 1: Using the payoff matrix displayed in Figure 5, if player 1 plays the mixed strategy $(\frac{1}{2}, \frac{1}{2}, 0)$ and player 2 plays $(0, \frac{1}{2}, \frac{1}{2})$, what is u_1?

Referring back to figure 5, the four strategy profiles in the top right are each played with probability $\frac{1}{4}$. The payoffs for player one in these four cases are -1, 1, -1, and 0. Thus the expected utility for player 1 is: $-1 * \frac{1}{4} + 1 * \frac{1}{4} + -1 * \frac{1}{4} + 0 * \frac{1}{4} = -\frac{1}{4}$

Exercise 2: What is u_1 if both players play $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$?

Intuitively, each of the nine strategy profiles has the same probability of being played, and because rock-paper-scissors is a zero-sum game, if both players play the same strategy, we expected u_1 to be 0 by symmetry.

Exercise 3: Which of the following mixed strategies for (player 1, player 2) is a Nash Equilibrium?

- $((\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 0))$
- $((\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}))$
- $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$
- $((\frac{1}{3}, \frac{2}{3}, 0), (\frac{2}{3}, 0, \frac{1}{3}))$

We can prove that third option is a Nash Equilibrium by showing that no player wishes to unilaterally deviate from the strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. When both players are playing this strategy, the expected utility for both of these players is 0 as shown in exercise 2. Thus, if either player wishes to deviate, there must be another strategy that yields an expected utility greater than 0. Let's assume that player 1 is considering deviating to strategy (a,b,c). Then, player 1's utility will be:

$$u_1 = \frac{1}{3} * a * 1 + \frac{1}{3} * a * 0 + \frac{1}{3} * a * (-1) + ... + \frac{1}{3} * c * 1 + \frac{1}{3} * c * 0 + \frac{1}{3} * c * (-1) = 0$$

Regardless of what new strategy player 1 chooses, there is no increase in utility, so player 1 will not deviate. The same argument holds for player 2 by symmetry, and thus option 3 is a Nash Equilibrium.