Fall 2021 | Lecture 8

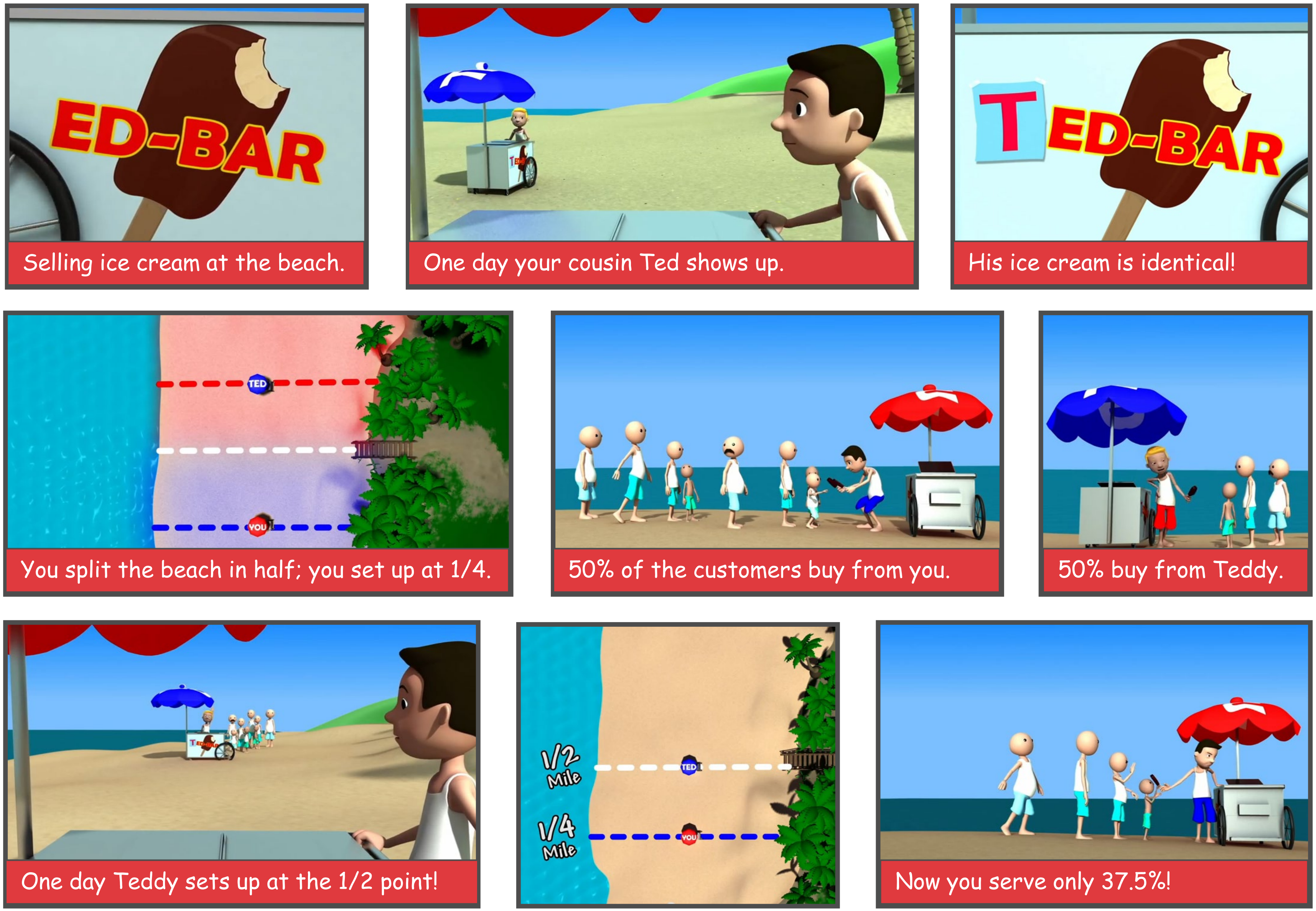
Game Theory

Ariel Procaccia | Harvard University

NORMAL-FORM GAME

- A game in normal form consists of:
  - Set of players $N = \{1, \ldots, n\}$
  - Strategy set S
  - For each $i \in N$, utility function $u_i: S^n \rightarrow \mathbb{R}$, which gives the utility of player i, $u_i(s_1, \ldots, s_n)$, when each $j \in N$ plays the strategy $s_j \in S$
- Next example created by taking screenshots of `http://youtu.be/jILgxeNBK_8`

THE ICE CREAM WARS



THE ICE CREAM WARS

- $N = \{1,2\}$
- $S = [0,1]$
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i + s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i + s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases}$

- To be continued…

THE PRISONER'S DILEMMA

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year

THE PRISONER'S DILEMMA

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?

UNDERSTANDING THE DILEMMA

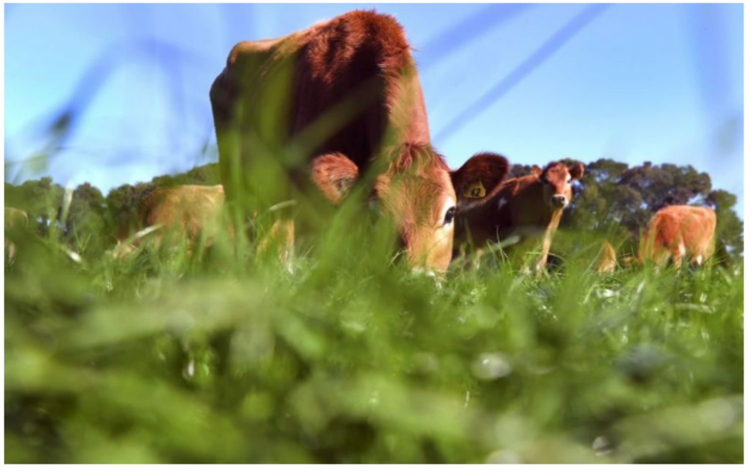
- Defection is a dominant strategy
- But the players can do much better by cooperating
- Related to the tragedy of the commons

THE TRAGEDY OF THE COMMONS

Opinion
Ariel Procaccia

Tech Giants, Gorging on AI Professors Is Bad for You

If industry keeps hiring the cutting-edge scholars, who will train the next generation of innovators in artificial intelligence?



Eat too much and there won't be grass for anyone. *Photographer: William West/AFP/Getty Images*

By Ariel Procaccia
January 7, 2019 at 6:00 AM EST

THE PROFESSOR'S DILEMMA

Professor \ Class	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

Dominant strategies?

John Forbes Nash

1928–2015

Mathematician and Nobel laureate in economics. Also remembered as the protagonist in “A Beautiful Mind.”

NASH EQUILIBRIUM

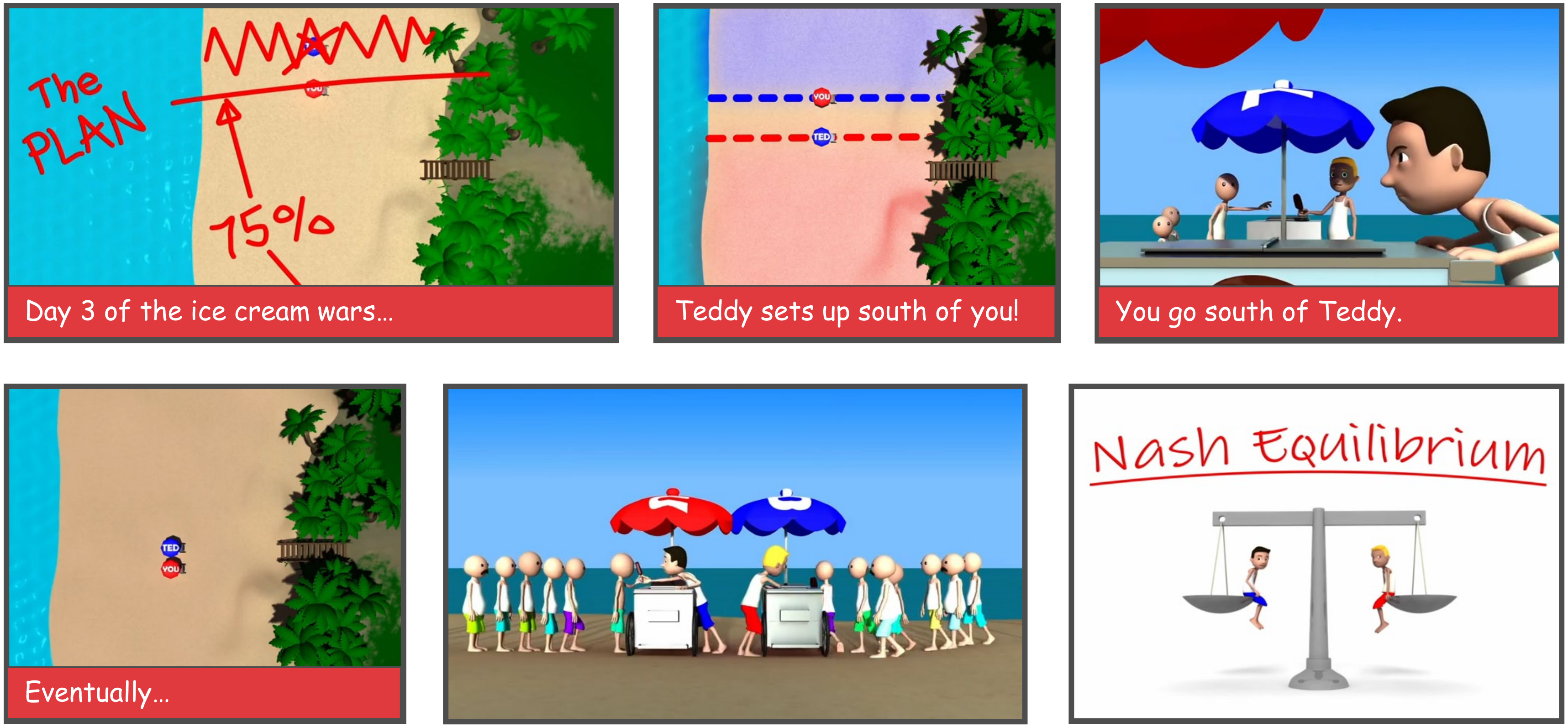
- In a Nash equilibrium, no player wants to unilaterally deviate
- Each player's strategy is a best response to strategies of others
- Formally, a Nash equilibrium is a vector of strategies $\boldsymbol{s} = (s_1 \dots, s_n) \in S^n$ such that for all $i \in N, s_i' \in S$,
  $u_i(\boldsymbol{s}) \geq u_i(s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n)$

THE PROFESSOR'S DILEMMA

Professor \ Class	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

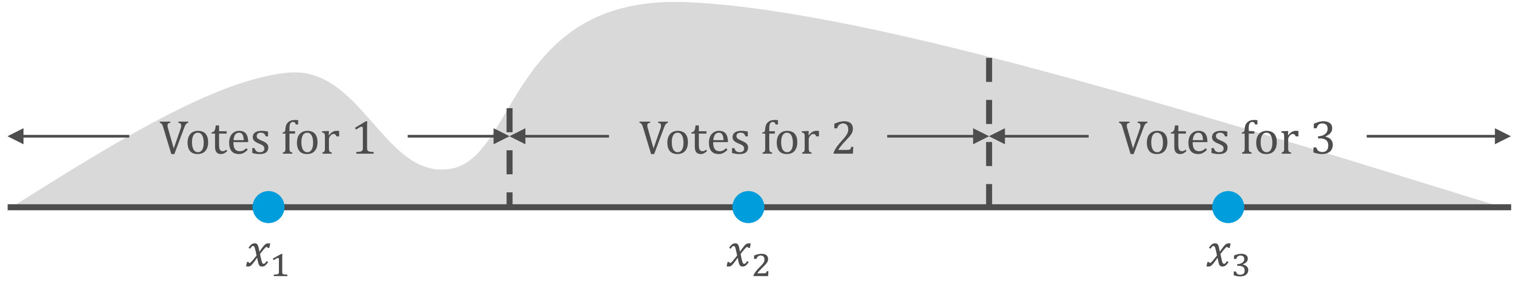
Nash equilibria?

END OF THE ICE CREAM WARS



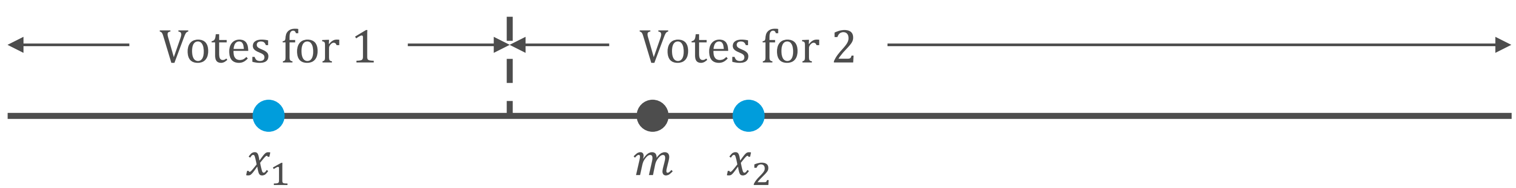
THE HOTELLING MODEL

- Political spectrum is $\mathbb{R}$
- There is a (nonatomic) distribution of voters, each with a position in $\mathbb{R}$
- Players are candidates, who strategically choose positions $x_1, \dots, x_n$
- Each candidate attracts the votes of voters who are closest to them, with votes being split equally in case of a tie



THE HOTELLING MODEL

- Two candidates seek to win a majority of votes
- The utility of each candidate is 1 if they win, 1/2 if they tie, and 0 if they lose
- Denote the median voter position by m (assume for simplicity that it's unique)

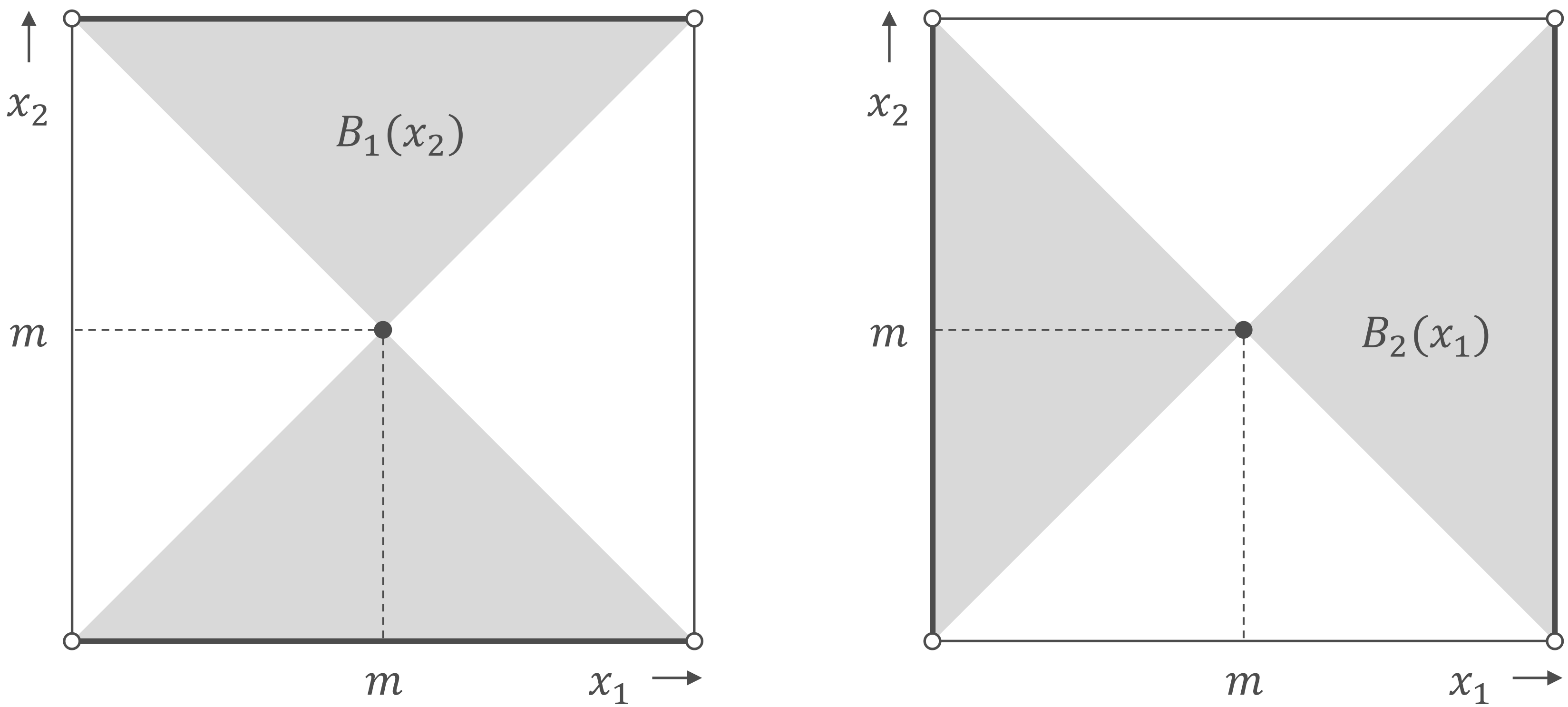


Who wins?

NE FOR TWO CANDIDATES

- If $x_2 < m$, the best response for 1 is all positions x_1 such that
$$x_1 > x_2 \quad \text{and} \quad \frac{x_1 + x_2}{2} < m$$
- A symmetric argument holds if $x_2 > m$
- If $x_2 = m$, the best response for 1 is m
- Therefore, it holds that
$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x_1 < 2m - x_2\} & x_2 < m \\ \{m\} & x_2 = m \\ \{x_1 : 2m - x_2 < x_1 < x_2\} & x_2 > m \end{cases}$$

NE FOR TWO CANDIDATES



The unique Nash equilibrium is at (m, m)

Harold Hotelling

1895–1973

“The competition for votes between the Republican and Democratic parties does not lead to a clear drawing of issues, an adoption of two strongly contrasted positions between which the voter may choose. Instead, each party strives to make its platform as much like the other's as possible.”

ROCK-PAPER-SCISSORS

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Nash equilibria?

MIXED STRATEGIES

- A mixed strategy is a probability distribution over (pure) strategies
- The mixed strategy of player $i \in N$ is x_i, where

$$x_i(s_i) = \Pr[i \text{ plays } s_i]$$

- The utility of player $i \in N$ is

$$u_i(x_1, \dots, x_n) = \sum_{(s_1,\dots,s_n) \in S^n} u_i(s_1, \dots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)$$

EXERCISE: MIXED NE

- **Exercise:** player 1 plays $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, player 2 plays $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. What is u_1?
- **Exercise:** Both players play $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. What is u_1?

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

EXERCISE: MIXED NE

- Poll 1: Which is a NE?
    1. $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right)$
    2. $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right)$
    3. $\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)$
    4. $\left(\left(\frac{1}{3},\frac{2}{3},0\right),\left(\frac{2}{3},0,\frac{1}{3}\right)\right)$

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Theorem [Nash, 1950]: In any (finite) game there exists at least one (possibly mixed) Nash equilibrium

DOES NE MAKE SENSE?

- Two players, strategies are $\{2, \dots, 100\}$
- If both choose the same number, that is what they get
- If one chooses s, the other t, and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$
- Poll 2: What would you choose?