Fall 2022 | Lecture 17

Decision Trees

Ariel Procaccia | Harvard University

SUPERVISED LEARNING

- We are given a training set of n examples $\left(x^{(1)}, y^{(1)}\right), \dots, \left(x^{(n)}, y^{(n)}\right)$ where each pair was generated by an unknown function $y = f(x)$
- The goal is to find a hypothesis $h \in \mathcal{H}$ that approximates f, where $\mathcal{H}$ is called the hypothesis space
- h is chosen to be a best-fit function for which each $h\left(x^{(i)}\right)$ is “close” to $y^{(i)}$
- h generalizes well if it gives accurate predictions on a fresh test set

CLASSIFICATION

- Classification is the task of learning f whose range is a discrete, finite set
- Such a function is called a classifier
- When the cardinality of the range is 2 then the task is known as binary classification, otherwise it's called multi-class classification

EXAMPLE: IMAGE CLASSIFICATION

$(\boldsymbol{x}^{(1)}, dog)$

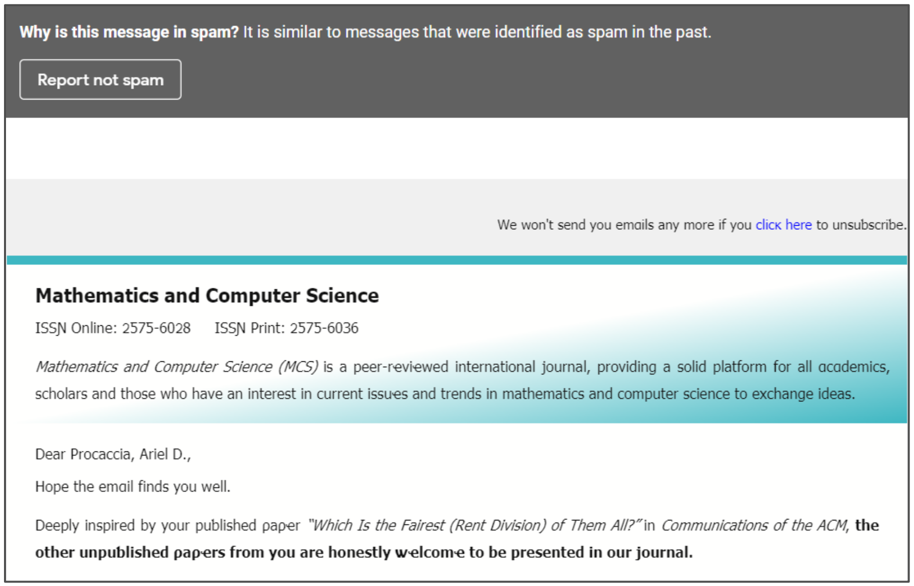
$(\boldsymbol{x}^{(2)}, cat)$

$(\boldsymbol{x}^{(3)}, dog)$

$(\boldsymbol{x}^{(4)}, dog)$

$f(\boldsymbol{x}) = ?$

EXAMPLE: SPAM FILTER


Why is this message in spam? It is similar to messages that were identified as spam in the past.

Report not spam

We won't send you emails any more if you click here to unsubscribe.

Mathematics and Computer Science

ISSN Online: 2575-6028 ISSN Print: 2575-6036

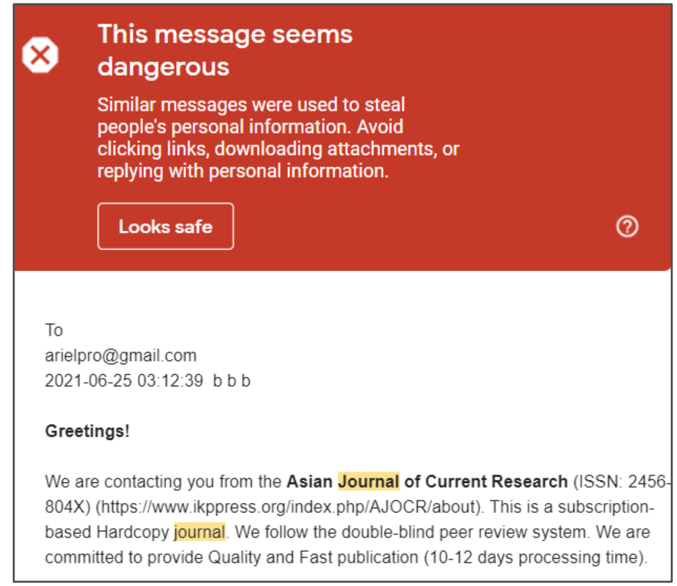
Mathematics and Computer Science (MCS) is a peer-reviewed international journal, providing a solid platform for all academics, scholars and those who have an interest in current issues and trends in mathematics and computer science to exchange ideas.

Dear Procaccia, Ariel D.,

Hope the email finds you well.

Deeply inspired by your published paper *"Which Is the Fairest (Rent Division) of Them All?"* in *Communications of the ACM*, **the other unpublished papers from you are honestly welcome to be presented in our journal.**

$(x^{(1)}, spam)$


This message seems dangerous

Similar messages were used to steal people's personal information. Avoid clicking links, downloading attachments, or replying with personal information.

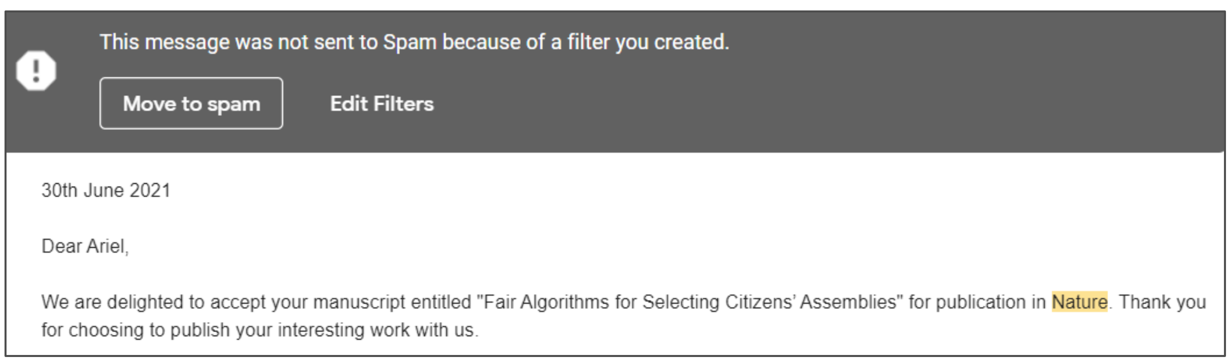
Looks safe

To
arielpro@gmail.com
2021-06-25 03:12:39 b b b

Greetings!

We are contacting you from the **Asian Journal of Current Research** (ISSN: 2456-804X) (https://www.ikppress.org/index.php/AJOCR/about). This is a subscription-based Hardcopy journal. We follow the double-blind peer review system. We are committed to provide Quality and Fast publication (10-12 days processing time).

$(x^{(2)}, spam)$


This message was not sent to Spam because of a filter you created.

Move to spam Edit Filters

30th June 2021

Dear Ariel,

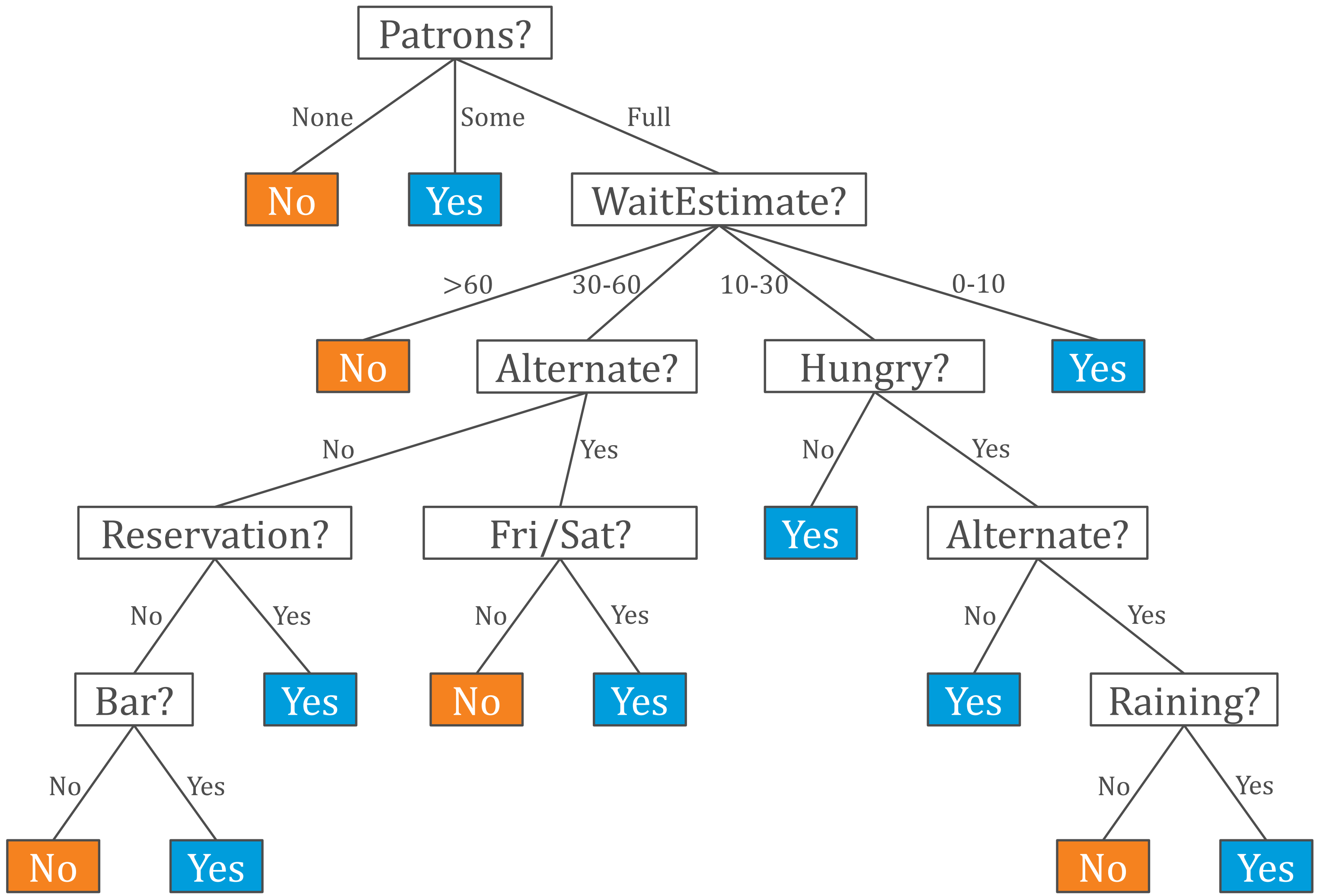
We are delighted to accept your manuscript entitled "Fair Algorithms for Selecting Citizens' Assemblies" for publication in Nature. Thank you for choosing to publish your interesting work with us.

$f(x) = ?$

EXAMPLE: RESTAURANT WAITING

Example	Input Features										Output
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait*
$\boldsymbol{x}^{(1)}$	*Y*	*N*	*N*	*Y*	*Some*	$$$	*N*	*Y*	*French*	0-10	$y^{(1)} = Y$
$\boldsymbol{x}^{(2)}$	*Y*	*N*	*N*	*Y*	*Full*	$	*N*	*N*	*Thai*	30-60	$y^{(2)} = N$
$\boldsymbol{x}^{(3)}$	*N*	*Y*	*N*	*N*	*Some*	$	*N*	*N*	*Burger*	0-10	$y^{(3)} = Y$
$\boldsymbol{x}^{(4)}$	*Y*	*N*	*Y*	*Y*	*Full*	$	*Y*	*N*	*Thai*	10-30	$y^{(4)} = Y$
$\boldsymbol{x}^{(5)}$	*Y*	*N*	*Y*	*N*	*Full*	$$$	*N*	*Y*	*French*	>60	$y^{(5)} = N$
$\boldsymbol{x}^{(6)}$	*N*	*Y*	*N*	*Y*	*Some*	$$	*Y*	*Y*	*Italian*	0-10	$y^{(6)} = Y$
$\boldsymbol{x}^{(7)}$	*N*	*Y*	*N*	*N*	*None*	$	*Y*	*N*	*Burger*	0-10	$y^{(7)} = N$
$\boldsymbol{x}^{(8)}$	*N*	*N*	*N*	*Y*	*Some*	$$	*Y*	*Y*	*Thai*	0-10	$y^{(8)} = Y$
$\boldsymbol{x}^{(9)}$	*N*	*Y*	*Y*	*N*	*Full*	$	*Y*	*N*	*Burger*	>60	$y^{(9)} = N$
$\boldsymbol{x}^{(10)}$	*Y*	*Y*	*Y*	*Y*	*Full*	$$$	*N*	*Y*	*Italian*	0-30	$y^{(10)} = N$
$\boldsymbol{x}^{(11)}$	*N*	*N*	*N*	*N*	*None*	$	*N*	*N*	*Thai*	0-10	$y^{(11)} = N$
$\boldsymbol{x}^{(12)}$	*Y*	*Y*	*Y*	*Y*	*Full*	$	*N*	*N*	*Burger*	30-60	$y^{(12)} = Y$

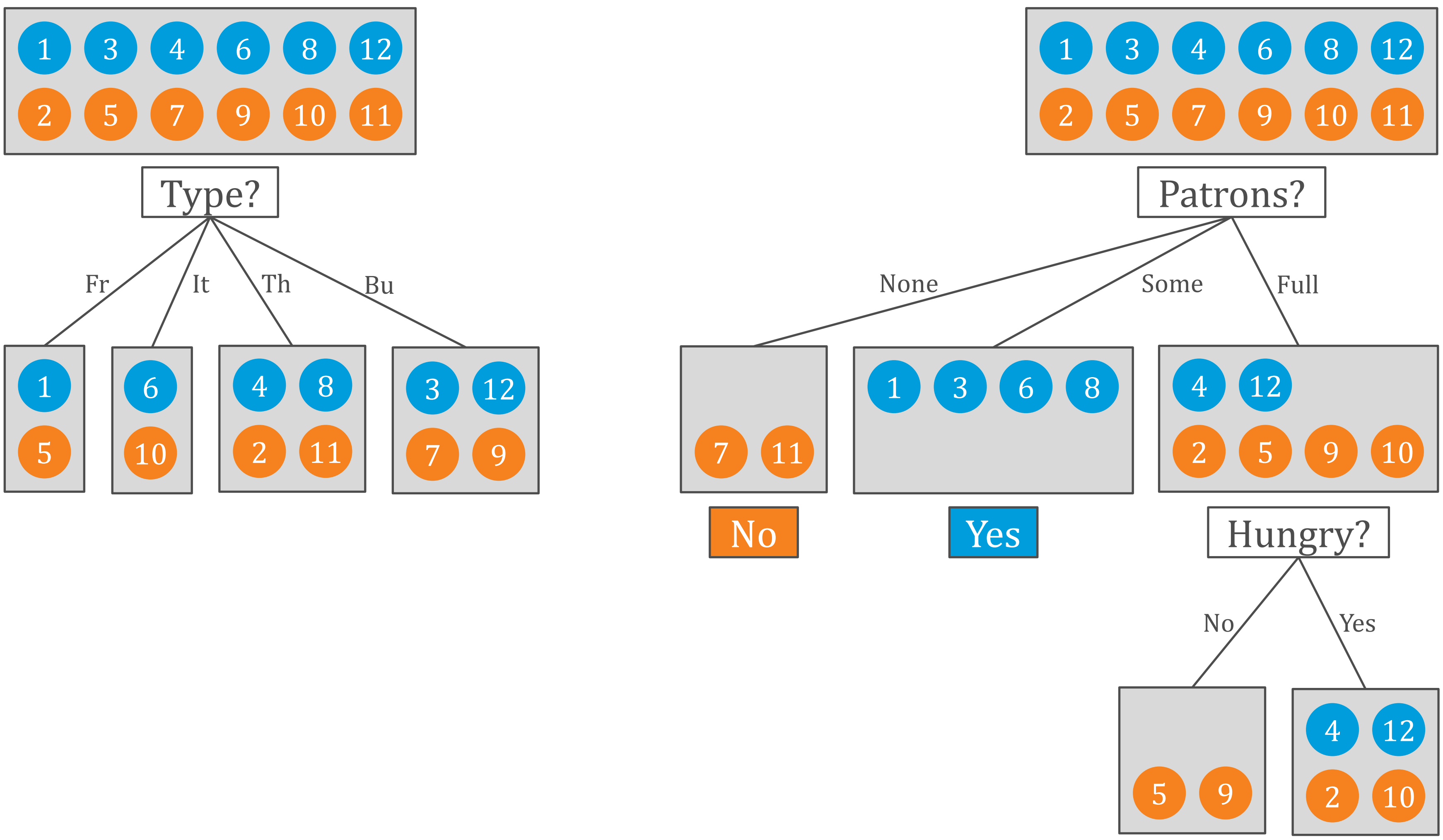
DECISION TREES



DECISION TREES

- A decision tree reaches an output (in the leaves) through a sequence of tests on the input attributes (in internal nodes)
- Decision trees can represent any classifier, but some may require a large tree
- Poll 1: Which of the following Boolean functions can be represented via a tree of size linear in the number of features?
    - Unanimity
    - Parity
    - Majority
    - None of the above

SPLITTING ON FEATURES



LEARNING DECISION TREES

```
function LEARN-DT(examples, features, parent_examples)
    if examples = ∅ then return PLURALITY-VALUE(parent_examples)
    else if all examples have the same label then return
    that label
    else if features = ∅ then return PLURALITY-VALUE(examples)
    else
        A ← argmax_{a∈features} IMPORTANCE(a, examples)
        tree ← new decision tree with root test A
        for each value v of A do
            new_examples ← {e ∈ examples: e.A = v}
            subtree ←LEARN-DT(new_examples, features \ {A}, examples)
            add branch to tree with label A = v and subtree
        return tree
```

Claude Shannon

1916-2001

Mathematician and electrical engineer, father of information theory.

INFORMATION GAIN

- To instantiate the IMPORTANCE function we will use the notion of entropy, which is measured in bits
- The entropy of random variable V that takes each value v with probability $P(v)$ is
$$H(V) = \sum_{v} P(v) \log \frac{1}{P(v)} = -\sum_{v} P(v) \log P(v)$$
- The entropy of a fair coin flip is 1 bit:
$$H(Fair) = -(0.5 \log 0.5 + 0.5 \log 0.5) = 1$$
- The entropy of a biased coin with 99% heads is:
$$H(Biased) = -(0.99 \log 0.99 + 0.01 \log 0.01) \approx 0.08$$
- Denote the entropy of a Bernoulli random variable that is true with probability q by
$$B(q) = -(q \log q + (1-q) \log(1-q))$$

INFORMATION GAIN

- If a training set contains p positive examples and n negative examples, the entropy of the output variable is

$$H(Output) = B\left(\frac{p}{p+n}\right)$$

- Feature A with d values divides the training set into d subsets, each with p_k positive examples and n_k negative examples
- The entropy after testing A is

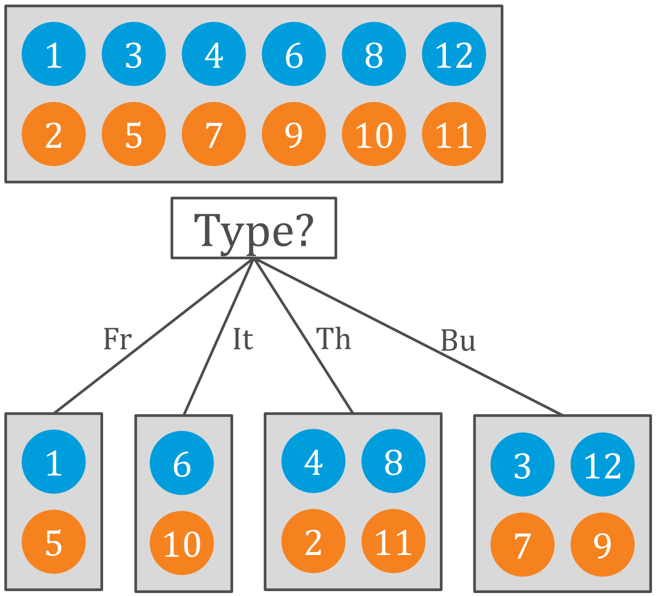
$$Remainder(A) = \sum_{k=1}^{d} \frac{p_k + n_k}{p+n} B\left(\frac{p_k}{p_k + n_k}\right)$$

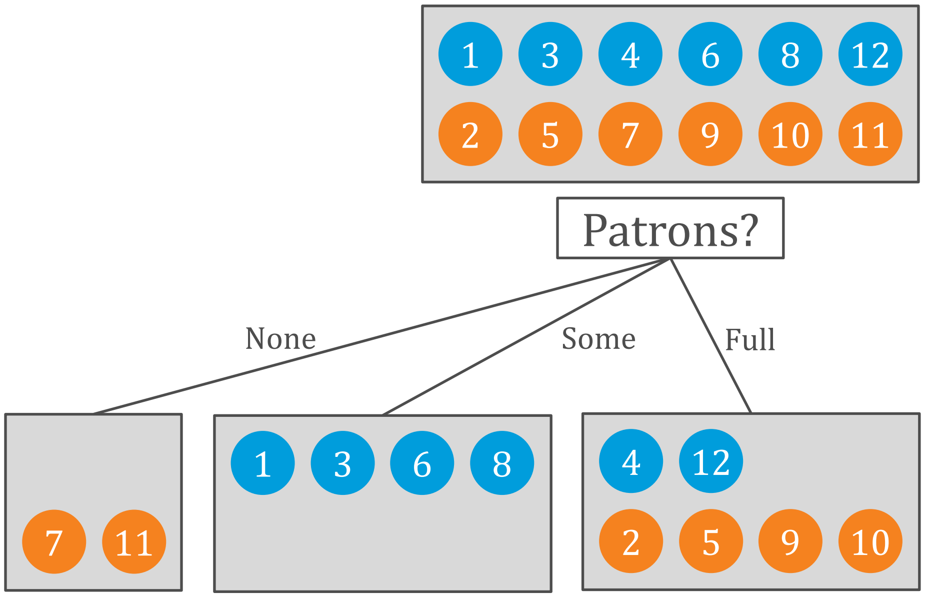
- The information gain from testing A is

$$Gain(A) = H(Output) - Remainder(A)$$

- In LEARN-DT, we can measure IMPORTANCE based on information gain

INFORMATION GAIN: EXAMPLE

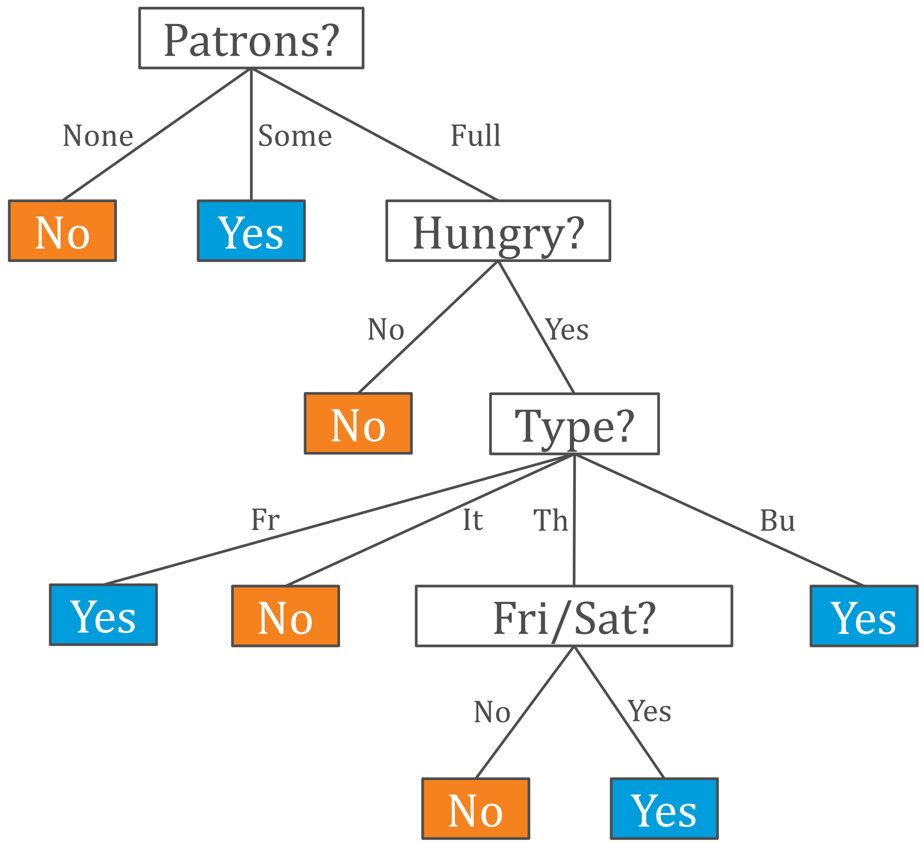




$$Gain(Type) = 1 - \left[\frac{2}{12}B\left(\frac{1}{2}\right) + \frac{2}{12}B\left(\frac{1}{2}\right) + \frac{4}{12}B\left(\frac{2}{4}\right) + \frac{4}{12}B\left(\frac{2}{4}\right)\right] = 0$$

$$Gain(Patrons) = 1 - \left[\frac{2}{12}B\left(\frac{0}{2}\right) + \frac{4}{12}B\left(\frac{4}{4}\right) + \frac{6}{12}B\left(\frac{2}{6}\right)\right] \approx 0.541$$

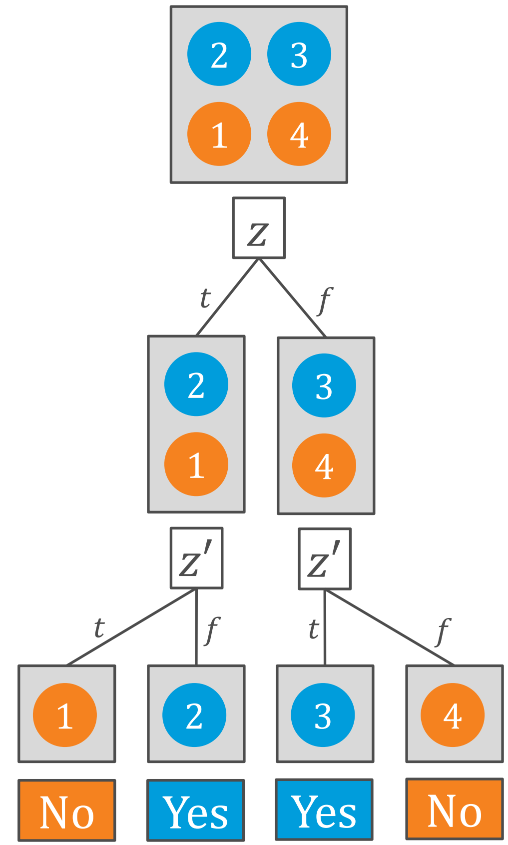
LEARNING DECISION TREES: EXAMPLE



The output of LEARN-DT is simpler than the original tree!

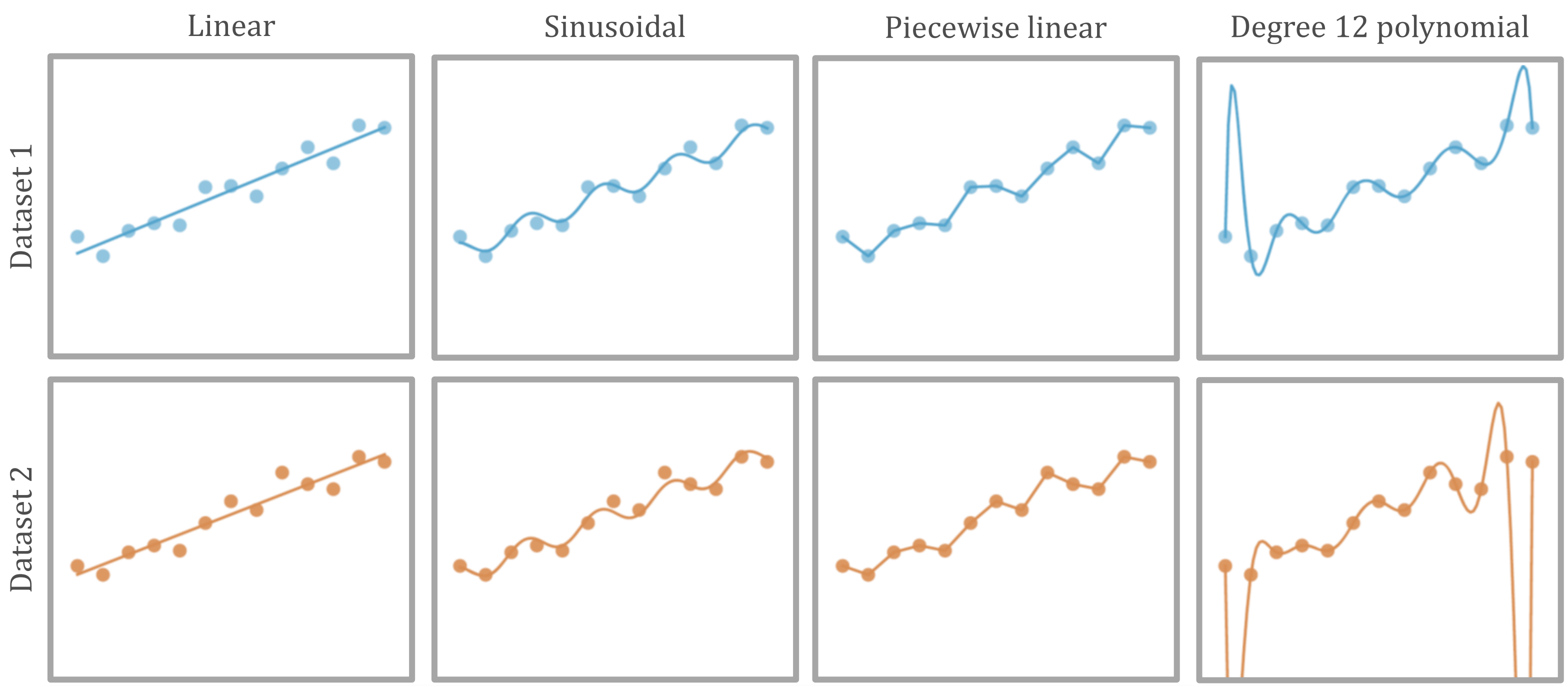
EARLY STOPPING

Example	Input features z	Input features z'	Output $z \oplus z'$
$\boldsymbol{x}^{(1)}$	t	t	f
$\boldsymbol{x}^{(2)}$	t	f	t
$\boldsymbol{x}^{(3)}$	f	t	t
$\boldsymbol{x}^{(4)}$	f	f	f



Should we stop LEARN-DT when the information gain is low? We may miss situations where combos of features are informative!

MODEL SELECTION



Degree 12 polynomials exhibit overfitting

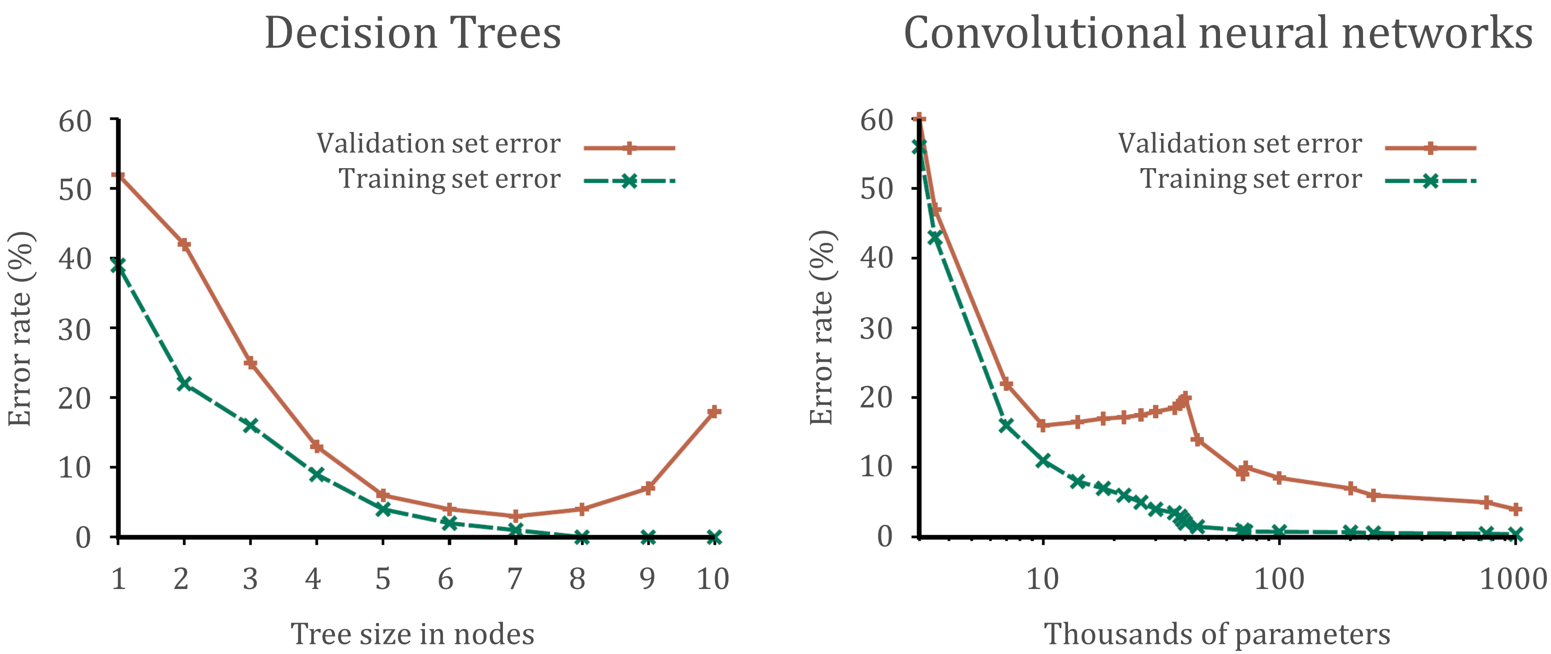
MODEL SELECTION

- Let us think of the quality of the “fit” of hypothesis h as error rate, i.e., the probability that $h(\boldsymbol{x}) \neq f(\boldsymbol{x})$
- We divide the data into three sets:
  1. Training set to train candidate models
  2. Validation set to choose among different models or hypothesis classes
  3. Test set to perform an unbiased evaluation of the best model

MODEL SELECTION

- We refer to the "complexity" of the hypothesis class (e.g., number of nodes in a decision tree) as the model size
- Poll 2: As the model size grows (check all possible options):
  - Training error decreases, validation error decreases
  - Training error decreases, validation error increases
  - Training error increases, validation error decreases
  - Training error increases, validation error increases

MODEL SELECTION



The graph on the right is an example of one of the great mysteries of deep learning!