Lecture 8: Game Theory

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In this lecture we cover concepts in *Game theory* and will start covering multi-agent systems, which seeks to understand and accurately model the interactions between several distinct parties. As compared to the problems and formulations we saw in Search, Constraint Satisfaction Problems (CSPs), etc, we will talk about multiple agents in either a cooperative or in an adversarial situation. Here, we present game theory as a main tool for thinking about such interactions.

1 Normal-Form Game

1.1 Definition and Concepts

The normal form formulation of a game can be defined as:

- Set of players $N = \{1, \ldots, n\}$
- Strategy set S: The strategies that the players can take
- Utility function of player i: $u_i : S^n : R$, the utility player i will have when each $j \in N$ plays the strategy $s_j \in S$

For example, in the *Ice Cream War* game, each of the two players on the beach wants to choose the best location to sell the ice cream so that they can have as many customers as possible. Suppose the decisions of customers buying ice cream are completely dependent on the distance between them and the players, then the players will try their best to take up the places on beach and try to avoid losing customers. Therefore, the problem can be formalized as a two-player game where the strategies will be the point they choose at the beach, ranging from 0 to 1. Therefore, the problem can be described as:

- $N = \{1, 2\}$
- $S = [0, 1]$ The point on the beach, ranging from 0 to 1
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2} & \text{if } s_i < s_j \\ 1 - \frac{s_i+s_j}{2} & \text{if } s_i > s_j \\ \frac{1}{2} & \text{if } s_i = s_j \end{cases}$, since the customers tend to buy ice cream from nearest shop.

Definition 1 (Dominance) *Dominance refers to a strategy $s = (s_1, \ldots, s_n)$ where each player i chooses a strategy s_i that maximizes their utility u_i regardless of the strategy s_j used by any other player $j \in N \setminus \{i\}$.*

1.2 Examples of Games

Here, we present two typical games with different sets of strategies. Generally, we will demonstrate the game with a matrix where the row and column represent the strategies of two players separately. And the entries are the utilities they can get.

The Prisoner's Dilemma

For both prisoners, the dominant strategy, i.e., the strategy that will give them most utility than others no matter what the other player chooses, will be to defect. In this case, there is a dominant strategy because it will give them highest utility regardless of what the other player choose.

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

Figure 1: The Prisoner's Dilemma

The Professor's Dilemma

The best strategy of class of professor will depend another player's strategy. For example, when the class chooses to listen, then the professor should choose to make effort. But when the class chooses to sleep, then the professor should choose to slack off. This will be different from the situation in Prisoner's dilemma, where the dominant strategy will always be to defect for both prisoners. In this case, there is no dominant strategy since the strategy that will bring better pay-off will depend on the strategies of other players.

1.3 Example

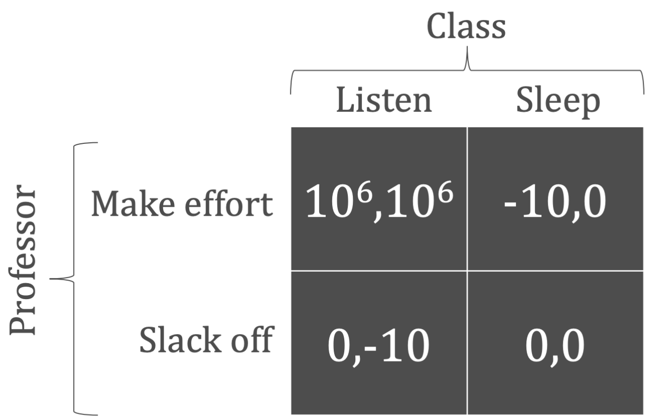

Figure 2: The Professor's Dilemma

1.4 Nash Equilibrium

For the second example mentioned above, the professor dilemma, we need a notion to deal with such type of game. Here comes the Nash Equilibrium

Nash Equilibrium appears when no player wants to unilaterally deviate, since each player's strategy is the best given the strategies of other players, i.e. their utility will decrease if they change their current strategies given that other's strategies will not change.

It can be described in a mathematical way: A Nash equilibrium is a vector of strategies $s = (s_1, ..., s_n) \in S^n$ such that for all $i \in N, s_i' \in S$, $u_i(s) \geq u_i(s_1, ..., s_{i-1}, s_i', s_{i+1}, ..., x_n)$

For example, in the professor's dilemma, listening for class and make effort for professor; sleep for class and slack off for professor, will be both the Nash Equilibrium in this game. And for the ice cream war, the 1/2 point will be the Nash Equilibrium.

2 Mixed Strategies

There is no Nash equilibrium for the Rock-Paper-Scissors game with what we have learned so far, i.e., the pure strategies, since one player can always switch to another strategy to win. Therefore, we need to introduce a refined notion of strategies, i.e., the Mixed Strategies.

2.1 Definition and Concepts

A mixed strategy is a probability distribution over (pure) strategies.

- Distribution of strategies:

$$x_i(s_i) = P(i \, plays \, s_i)$$

- Utility of player $i \in N$:

$$u_i(x_1, ..., x_n) = \sum_{(s_1,...,s_n) \in S^n} u_i(s_1, ..., s_n) \prod_{j=1}^{n} x_j(s_j)$$

which is the expectation over the distribution of mixed strategies.

2.2 Example

Figure 3: Rock-Paper-Scissors

For the Rock-Paper-Scissor problem, we can have a Nash Equilibrium with the current new definition of strategy. Consider when the other player plays (1/3, 1/3, 1/3) as his/her mixed strategy, then no matter what we do, we will get zero pay off. For example, if we choose rock, then we will get $0 * 1/3 + 1 * 1/3 + (-1) * 1/3 = 0$. So the best thing we could do is also playing (1/3, 1/3, 1/3), which is at least as good as anything we could do. And when we choose (1/3, 1/3, 1/3), the other player will have same situation. However, if the other player's policy can make us get a positive pay-off, then the other player can always switch his/her policy to change the situation.

1,1	3,0
0,0	2,1

Figure 4: Stackelberg Game

3 Commitment

Commitment will make the game theory more interesting. In a Stackelberg game, the *leader* can make a commitment, and the *follower* observes the commitment and chooses the strategy accordingly. Then the leader will have the opportunity to win more utility.

For example, in the game shown in the picture, there is a dominant strategy for the row player to to choose above. So the only NE is at the top left. But the leader (row player) can commit to down, then the follower will choose right correspondingly so that the leader can gain more.

And similar reasoning holds for mixed strategies. Suppose the leader can commit to play a mixed strategy, and without loss of generality the follower responds with a pure strategy. We will break tie by making the assumption that column player chooses the strategy that is better for the leader. Then the leader can commit to the mixed strategy (1/2, 1/2). Observing this (and breaking ties as above), the follower will respond by playing right. This gives the leader a utility of 2.5.