Fall 2021 | Lecture 8

Game Theory

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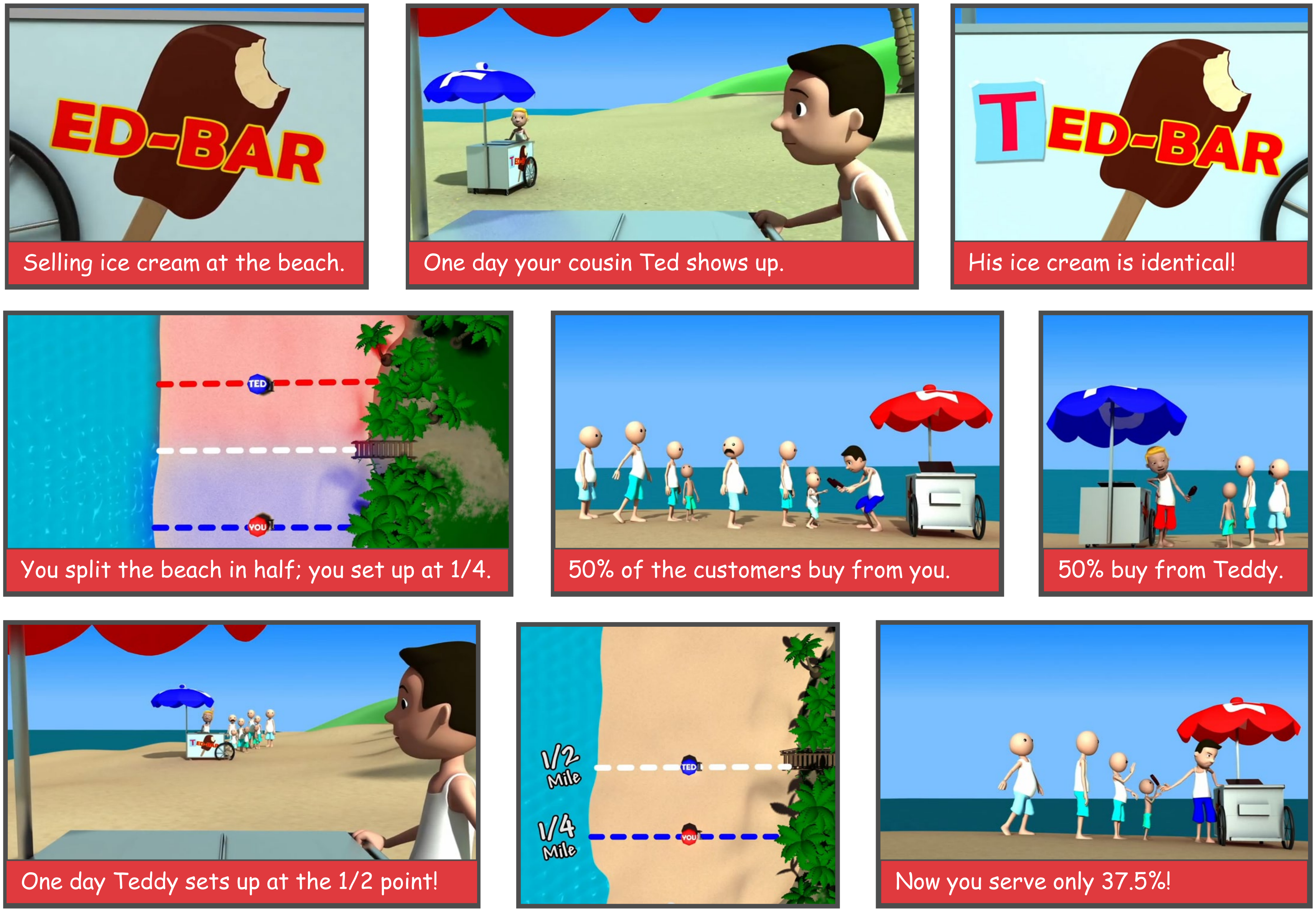
NORMAL-FORM GAME

- A game in normal form consists of:
  - Set of players $N = \{1, \ldots, n\}$
  - Strategy set S
  - For each $i \in N$, utility function $u_i: S^n \rightarrow \mathbb{R}$, which gives the utility of player i, $u_i(s_1, \ldots, s_n)$, when each j $\in N$ plays the strategy $s_j \in S$
- Next example created by taking screenshots of `http://youtu.be/jILgxeNBK_8`

THE ICE CREAM WARS



THE ICE CREAM WARS

- $N = \{1,2\}$
- $S = [0,1]$
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i + s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i + s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases}$

- To be continued…

THE PRISONER'S DILEMMA

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year

THE PRISONER'S DILEMMA

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?

UNDERSTANDING THE DILEMMA

- Defection is a dominant strategy
- But the players can do much better by cooperating
- Related to the tragedy of the commons

THE PROFESSOR'S DILEMMA

		Class	
		Listen	Sleep
Professor	Make effort	$10^6,10^6$	-10,0
	Slack off	0,-10	0,0

Dominant strategies?

John Forbes Nash

1928–2015

Mathematician and Nobel laureate in economics. Also remembered as the protagonist in “A Beautiful Mind.”

NASH EQUILIBRIUM

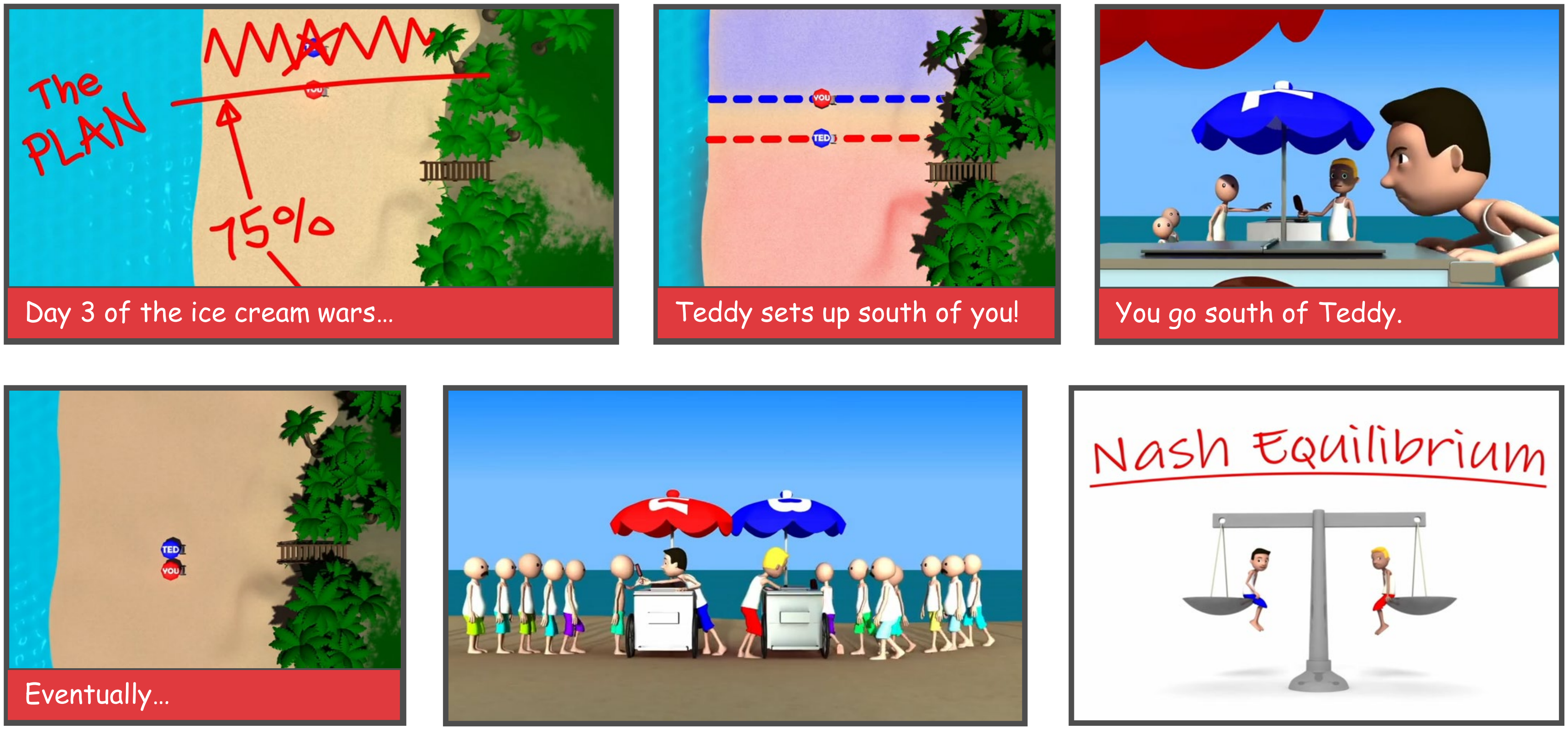
- In a Nash equilibrium, no player wants to unilaterally deviate
- Each player's strategy is a best response to strategies of others
- Formally, a Nash equilibrium is a vector of strategies $\boldsymbol{s} = (s_1 \dots, s_n) \in S^n$ such that for all $i \in N, s_i' \in S$,
$u_i(\boldsymbol{s}) \geq u_i(s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n)$

THE PROFESSOR'S DILEMMA

Professor \ Class	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

Nash equilibria?

END OF THE ICE CREAM WARS



NASH IN REAL LIFE

Washington Street, Newton

ROCK-PAPER-SCISSORS

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Nash equilibria?

MIXED STRATEGIES

- A mixed strategy is a probability distribution over (pure) strategies
- The mixed strategy of player $i \in N$ is x_i, where

$$x_i(s_i) = \Pr[i \text{ plays } s_i]$$

- The utility of player $i \in N$ is

$$u_i(x_1, \dots, x_n) = \sum_{(s_1,\dots,s_n)\in S^n} u_i(s_1, \dots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)$$

EXERCISE: MIXED NE

- **Exercise:** player 1 plays $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, player 2 plays $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. What is u_1?
- **Exercise:** Both players play $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. What is u_1?

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

EXERCISE: MIXED NE

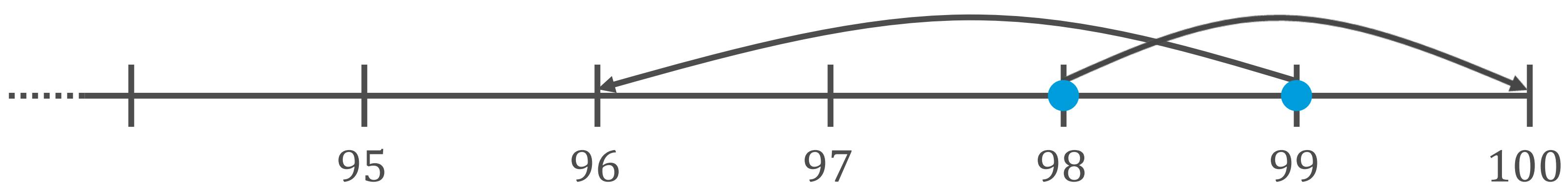
- Poll 1: Which is a NE?
  1. $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right)$
  2. $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right)$
  3. $\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)$ ✓
  4. $\left(\left(\frac{1}{3},\frac{2}{3},0\right),\left(\frac{2}{3},0,\frac{1}{3}\right)\right)$

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Theorem [Nash, 1950]: In any (finite) game there exists at least one (possibly mixed) Nash equilibrium

DOES NE MAKE SENSE?

- Two players, strategies are $\{2, \ldots, 100\}$
- If both choose the same number, that is what they get
- If one chooses s, the other t, and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$
- Poll 2: What would you choose?



COMMITMENT

http://youtu.be/S0qjK3TWZE8

STACKELBERG GAMES

- Playing up is a dominant strategy for row player
- So column player would play left
- Therefore, (1,1) is the only Nash equilibrium outcome

1,1	3,0
0,0	2,1

STACKELBERG GAMES

- A Stackelberg game is played as follows:
  - Row player (the leader) commits to playing a row
  - Column player (the follower) observes the commitment and chooses column
- The leader can commit to playing down!

1,1	3,0
0,0	2,1

STACKELBERG GAMES

- Poll 3: What reward can the leader get by committing to a mixed strategy? (Assume the follower breaks ties in favor of the leader)
  - 1
  - 1.5
  - 2
  - 2.5 ✓

1,1	3,0
0,0	2,1