

Economics and Computation (Spring 2026)
Assignment #1
— Solutions —

Due: 2/12/2026 11:59pm ET

Problem 1: Nash equilibrium

Recall the ice cream wars setting from Lecture 1. Suppose that there are n ice cream sellers on a beach which can be modeled as the interval $[0, 1]$ and that each ice cream seller chooses a real number within that interval where they will set up their cart to sell ice cream. The ice cream being sold from all carts is identical, that is, customers are indifferent about which cart's ice cream they buy and will always buy from the seller closest to them. Customers are uniformly distributed along the beach (assume distance only matters to customers in the horizontal direction along the beach's length, the width of the beach is negligible). We saw that when $n = 2$, there is a Nash equilibrium in pure strategies where both sellers are located at $1/2$.

1. [15 points] Prove that when $n = 3$, a Nash equilibrium in pure strategies does not exist.

Solution: For $n = 3$, there is no pure strategy Nash equilibrium. Denote the seller positions by x_1, x_2, x_3 , and assume w.l.o.g. that $x_1 \leq x_2 \leq x_3$.

If $x_1 = x_2 = x_3$ then all three sellers are getting utility $1/3$. If w.l.o.g. their common position is at most $1/2$ (i.e., $x_1 = x_2 = x_3 \leq 1/2$), then any of the sellers can deviate to 0.51 and get utility greater than 0.49 .

Otherwise, $x_1 < x_3$. If $x_1 < x_2$, then x_1 can move closer to x_2 and obtain higher utility. Otherwise it holds that $x_1 = x_2 < x_3$, and x_3 will gain by moving closer to x_2 .

2. [15 points] Prove that when n is even, a Nash equilibrium in pure strategies exists.

Solution: When n is even, a pure strategy Nash equilibrium is achieved when exactly two players choose each of the positions $\frac{1}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}$. Note that for the two players at position $\frac{k}{n}$, where k is an odd integer such that $1 \leq k \leq n-1$, these two players capture all customers in the interval $(\frac{k-1}{n}, \frac{k+1}{n})$, which has a length of $\frac{2}{n}$. Since two players share this interval, each receives a payoff of $\frac{1}{n}$.

Consider a deviation to another position already occupied by two players. In this case, three players would share the interval of length $\frac{2}{n}$, resulting in a payoff of $\frac{2}{3n}$ for the player, which is less than their original $\frac{1}{n}$.

Now, consider a deviation to an unoccupied position outside the range of existing positions, such as $x < \frac{1}{n}$. The deviating player would serve customers from 0 to x and up to the midpoint between x and $\frac{1}{n}$. However, the length of this interval is less than $\frac{1}{n}$. A similar argument holds for deviations to positions $x > \frac{n-1}{n}$.

Finally, consider a deviation to an unoccupied position between two occupied positions, such as y between $\frac{k}{n}$ and $\frac{k+2}{n}$ for some odd integer k such that $1 \leq k \leq n-3$. The deviating player would serve customers from the midpoint between $\frac{k}{n}$ and y to the midpoint between y and $\frac{k+2}{n}$. The total length of this interval is $\frac{1}{n}$, which is equal to the deviator's payoff before the deviation.

In all cases, a unilateral deviation results in a payoff less than or equal to $\frac{1}{n}$. Therefore, no player has an incentive to deviate, confirming that the proposed configuration is a pure strategy Nash equilibrium.

Problem 2: Equilibrium computation

In a normal-form game, we say that strategy s_i is *strictly dominated* for player i if there exists a mixed strategy x_i of player i such that $u_i(x_i, \mathbf{s}_{-i}) > u_i(\mathbf{s})$ for all pure strategy profiles \mathbf{s}_{-i} of the other players.

For example, in the following game, M is dominated by $(0.5, 0, 0.5)$:

U	2,0	-1,0
M	0,0	0,0
D	-1,0	2,0

A process of *iterated elimination of strictly dominated strategies* starts by identifying a strictly dominated strategy, if one exists. That strategy is then removed from the strategy set of the relevant player, and the process is repeated for the smaller game. The process terminates when strictly dominated strategies no longer exist.

1. [25 points] Prove that if \mathbf{x} is a mixed strategy Nash equilibrium, and $x_i(s_i) > 0$, then s_i cannot be eliminated in a process of iterated elimination of strictly dominated strategies.

Guidance: Use induction on the order of elimination. If s_i was eliminated, it is strictly dominated by a mixed strategy x'_i ; in x_i , transfer the probability weight of s_i to x'_i , thus obtaining a strictly better mixed strategy that violates the assumption that \mathbf{x} is a Nash equilibrium.

Solution: Following the guidance, we proceed by induction on the order of elimination.

For the base of the induction, initially all strategies are available to all players. Suppose for contradiction that $x_i(s_i^*) > 0$ but s_i^* is eliminated in the first round of the iterated elimination process. Let x'_i be the (possibly) mixed strategy that strictly dominates s_i^* . We construct a new mixed strategy x''_i , defined for any pure strategy $s_i \neq s_i^*$ by

$$x''_i(s_i) = x_i(s_i) + x_i(s_i^*) \cdot x'_i(s_i),$$

and for s_i^* by

$$x''_i(s_i^*) = x_i(s_i^*) \cdot x'_i(s_i^*).$$

Note that x_i'' can be seen as transferring the probability weight of $x_i(s_i^*)$ to x_i' . Therefore, for all pure strategy profiles \mathbf{s}_{-i} ,

$$u_i(x_i'', \mathbf{s}_{-i}) - u_i(x_i, \mathbf{s}_{-i}) = x_i(s_i^*) \cdot (u_i(x_i', \mathbf{s}_{-i}) - u_i(s_i^*, \mathbf{s}_{-i})) > 0,$$

where the inequality holds because $x_i(s_i^*) > 0$ and s_i^* is strictly dominated by x_i' . Now, since the inequality holds pointwise for each \mathbf{s}_{-i} , we conclude that

$$u_i(x_i'', \mathbf{x}_{-i}) - u_i(x_i, \mathbf{x}_{-i}) > 0,$$

in contradiction to the assumption that \mathbf{x} is a Nash equilibrium.

For the induction step, assume that k strategies have already been eliminated, and we want to show that elimination step $k + 1$ does not eliminate a strategy in the support of \mathbf{x} . By the induction assumption, the first k elimination steps have not eliminated any strategies in the support of \mathbf{x} . Therefore, \mathbf{x} is still a Nash equilibrium in the reduced game, and we can repeat the exact same analysis as in the base case.

2. **[15 points]** For the case of two players, design an algorithm that performs iterated elimination of strictly dominated strategies in time that is polynomial in the size of the strategy set.

Hint: Use linear programming. You may assume that LPs can be solved in polynomial time and, although this is a bit messy, you may include strict inequalities in your LP.

Solution: The following feasibility LP determines whether s_1^* is strictly dominated by identifying a dominating mixed strategy x_1 ; the variables are $x_1(s_1)$ for each pure strategy $s_1 \in S$. (Assume for ease of notation that both players share the same strategy set S .)

$$\sum_{s_1 \in S} x_1(s_1) u_1(s_1, s_2) > u_1(s_1^*, s_2) \quad \forall s_2 \in S$$

$$x_1(s_1) \geq 0 \quad \forall s_1 \in S$$

$$\sum_{s_1 \in S} x_1(s_1) = 1$$

If such an \mathbf{x}_1 is feasible, then s_1^* is strictly dominated, and an analogous LP works for player 2.

For running time, there are at most $2|S|$ elimination steps, and each elimination step involves checking at most $2|S|$ strategies using an LP that can be solved in polynomial time.

Taken together, these two facts suggest that iterated elimination of dominated strategies can be a useful *preprocessing* step when attempting to compute a mixed Nash equilibrium.

Problem 3: Extensive-form games

1. **[15 points]** Consider a 2-player game in normal form. Let (x_1, x_2) be a (possibly mixed) Nash equilibrium. Show that if (x_1^*, s_2) is a strong Stackelberg equilibrium (where player 1 is the leader) then $u_1(x_1^*, s_2) \geq u_1(x_1, x_2)$.

Solution: If (x_1^*, s_2) is a strong Stackelberg equilibrium, then x_1^* maximizes the quantity $\max_{s_2 \in B_2(x)} u_1(x, s_2)$, considering all mixed strategies x . By the definition of a strong Stackelberg equilibrium:

$$u_1(x_1^*, s_2) = \max_{s_2 \in B_2(x_1^*)} u_1(x_1^*, s_2) \geq \max_{s_2 \in B_2(x_1)} u_1(x_1, s_2)$$

so it suffices to show that $u_1(x_1, x_2) \leq \max_{s_2 \in B_2(x_1)} u_1(x_1, s_2)$. Since (x_1, x_2) is a NE, we claim that x_2 must have support entirely in $B_2(x_1)$. Assume for contradiction that x_2 contains in its support a strategy that itself is not a best response to x_1 . Then, player 2 would be better off decreasing the attributed probability of that strategy and increasing the attributed probability to a best response to x_1 , which is a contradiction because (x_1, x_2) is a NE.

Thus, we get:

$$u_1(x_1, x_2) = \sum_{s_2 \in B_2(x_1)} x_2(s_2) u_1(x_1, s_2) \leq \max_{s_2 \in B_2(x_1)} u_1(x_1, s_2),$$

as desired.

2. [15 points] Prove that in the game of chess, precisely one of the following holds: white has a strategy that guarantees a win no matter how black plays, black has a strategy that guarantees a win no matter how white plays, or each player has a strategy that guarantees a tie no matter how the other plays.

Hint: Backward induction.

Solution: Solve for the subgame-perfect equilibrium for the game of chess using backward induction. Since the payoffs at the leaves are -1 (black wins), 0 (draw) or 1 (white wins), these payoffs propagate upwards through the backward induction process until the root is labeled with precisely one of them.

Assume that the value at the root is 1, meaning that there is a subgame-perfect equilibrium where white wins. It follows that white has a winning strategy: given white's strategy, no matter what strategy black plays, they cannot achieve a better outcome. Similar reasoning applies to the cases where the value at the root is 0 (each player can guarantee a tie) and -1 (black has a winning strategy).