

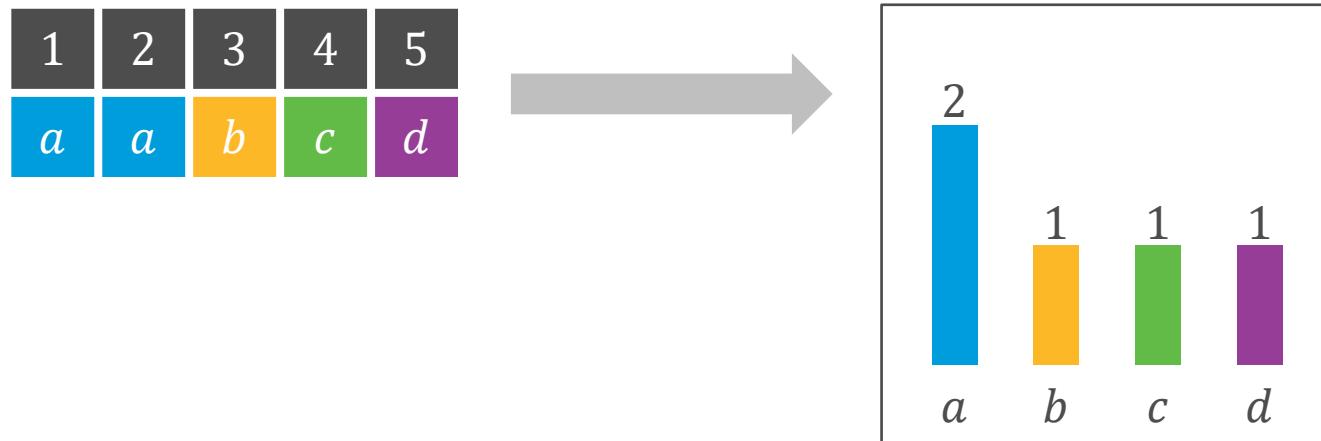
# Spring 2026 | Lecture 5

## Voting Rules

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# PLURALITY

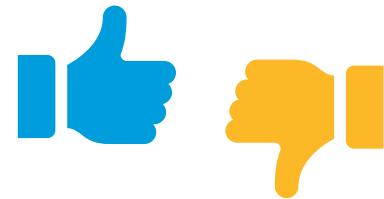
- Each person votes for a single alternative, and the alternative with most points wins
- A highly problematic voting rule!



# SOME BALLOT TYPES



Rankings



Approvals



Scores/stars

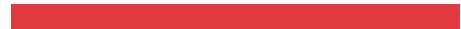
We will focus on rankings!



## Jean-Charles de Borda

1733–1799

Mathematician, engineer, and naval officer. Also remembered as an instigator of the metric system.



# BORDA COUNT

- Each voter awards  $m - k$  points to the alternative placed in the  $k$ 'th position, where  $m$  is the number of alternatives



# INSTANT-RUNOFF VOTING

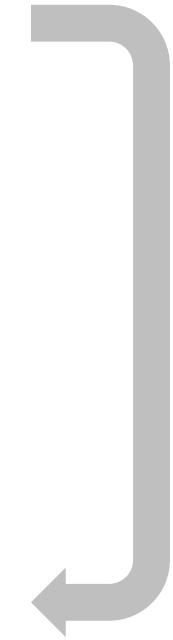
- Also known as “alternative vote” and (misleadingly) “ranked-choice voting”
- Votes are tabulated in rounds, where in each round the alternative with the lowest plurality score is eliminated; last alternative left standing is the winner

# INSTANT-RUNOFF VOTING

1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a



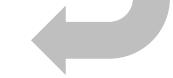
1	2	3	4	5
a	a		c	d
		c		
c	c	d	d	c
d	d	a	a	a



1	2	3	4	5
			c	
		c		
c	c			c



1	2	3	4	5
a	a		c	
		c		
c	c			c
		a	a	a



# IRV AROUND THE WORLD



## ■ Ireland

Used for all public elections

## ■ Canada

Used in Ontario for municipal elections

## ■ Australia

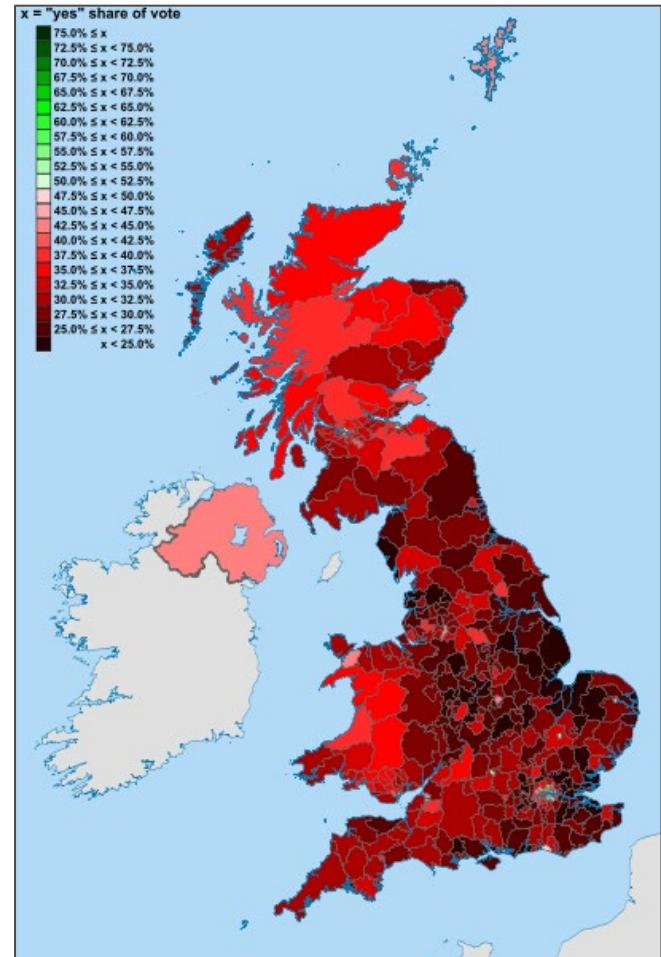
Used for parliamentary elections

## ■ USA

Used for statewide elections in ME and AK, and in cities like NYC and Cambridge

# BARRIERS TO ADOPTION

- UK referendum (2011): Choose between plurality and IRV as a method for electing MPs
- Academics agreed IRV is better
- But IRV was seen as beneficial to a particular politician

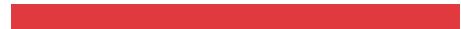




# Marquis de Condorcet

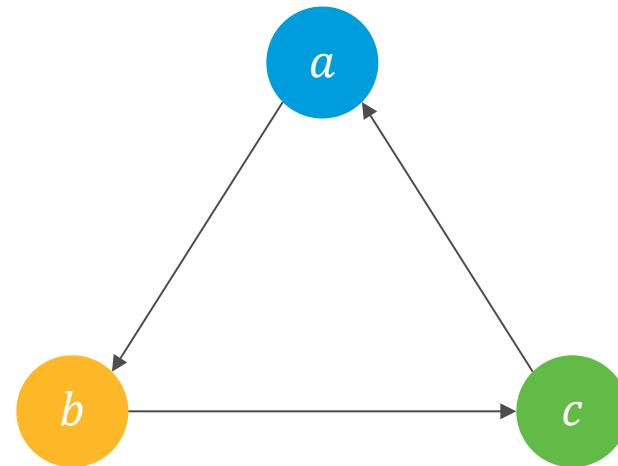
1743–1794

Philosopher, mathematician,  
enlightened nobleman. Also known  
for dying mysteriously in prison.



# THE CONDORCET PARADOX

1	2	3
<i>a</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>b</i>	<i>a</i>



The preferences of the majority may be cyclical!

# CONDORCET CONSISTENT RULES

- A **Condorcet winner** is an alternative that defeats every other alternative in a head-to-head comparison
- A rule is **Condorcet consistent** if it always selects a Condorcet winner whenever it is presented with a profile that contains one

## Poll 1

Which rule is Condorcet consistent?

- Plurality
- Both rules
- Borda Count
- Neither one

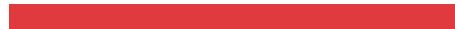




# Ramon Llull

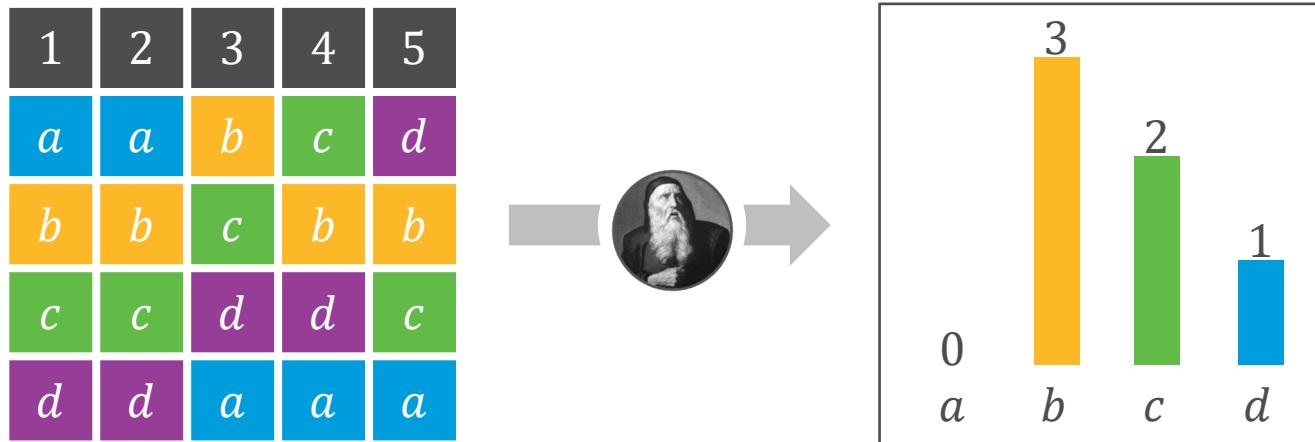
*c. 1232–1315*

Monk, missionary, and philosopher; one of the most influential intellectuals of his time. Also remembered for publishing a medieval parenting guide.



# LLULL'S RULE

- Each alternative receives one point for each head-to-head comparison it wins (as well as for tied comparisons)



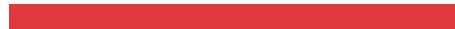
- Llull's rule is Condorcet consistent — why?



# Charles Lutwidge Dodgson

1832–1898

Professor of mathematics at Oxford,  
pioneer photographer, and beloved  
author. Also known for not plagiarizing  
Condorcet's work.



# DODGSON'S RULE

- The **Dodgson score** of an alternative  $x$  is the minimum number of swaps between adjacent alternatives needed to make  $x$  a Condorcet winner; select an alternative with minimum score
- Dodgson's rule is Condorcet consistent
- Dodgson's rule is NP-hard to compute!

# DODGSON'S RULE

What is the Dodgson score of  $b$ ?

1	2	3	4	5
$a$	$a$	$d$	$d$	$d$
$b$	$b$	$c$	$c$	$c$
$c$	$c$	$a$	$b$	$b$
$d$	$d$	$b$	$a$	$a$



1	2	3	4	5
$a$	$a$	$d$	$d$	$d$
$b$	$b$	$c$	$b$	$c$
$c$	$c$	$a$	$c$	$b$
$d$	$d$	$b$	$a$	$a$



1	2	3	4	5
$b$	$a$	$d$	$b$	$d$
$a$	$b$	$c$	$d$	$c$
$c$	$c$	$a$	$c$	$b$
$d$	$d$	$b$	$a$	$a$



1	2	3	4	5
$a$	$a$	$d$	$b$	$d$
$b$	$b$	$c$	$d$	$c$
$c$	$c$	$a$	$c$	$b$
$d$	$d$	$b$	$a$	$a$

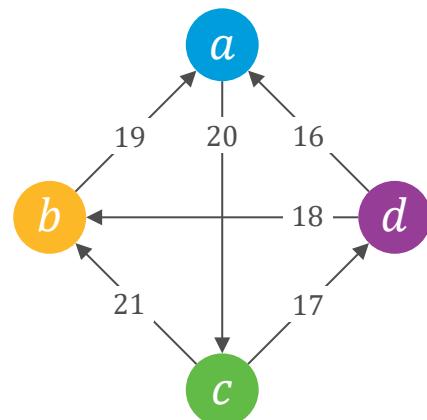


# SCHULZE'S RULE

- Let  $P(x, y)$  denote the number of voters who prefer  $x$  to  $y$
- A path from  $x$  to  $y$  of strength  $p$  is a sequence of alternatives  $x = a_1, \dots, a_k = y$  such that for all  $i = 1, \dots, k - 1$ ,  $P(a_i, a_{i+1}) > P(a_{i+1}, a_i)$  and  $P(a_i, a_{i+1}) \geq p$
- Let  $S(x, y)$  be the strength of the strongest path from  $x$  to  $y$  — it's 0 if there's no path
- **Exercise:** If  $S(x, y) > S(y, x)$  and  $S(y, z) > S(z, y)$  then  $S(x, z) > S(z, x)$
- Therefore there exists a winning alternative  $x^*$  such that  $S(x^*, y) \geq S(y, x^*)$  for all  $y$
- Schulze's rule is Condorcet consistent

# SCHULZE'S RULE

5 voters	2 voters	3 voters	4 voters	3 voters	3 voters	1 voter	5 voters	4 voters
<i>a</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>d</i>	<i>d</i>
<i>c</i>	<i>c</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>d</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>c</i>	<i>a</i>	<i>b</i>
<i>d</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>a</i>



Pairwise comparisons

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>a</i>	—	20	20	17
<i>b</i>	19	—	19	17
<i>c</i>	19	21	—	17
<i>d</i>	18	18	18	—

Strength of paths  $S(x, y)$

# INDEPENDENCE OF CLONES

A subset of alternatives  $S$  is called **clones** in a given preference profile if no voter ranks any alternative  $x \notin S$  between two alternatives in  $S$

1	2	3	4	5
$a$	$a$	$d$	$c$	$d$
$b$	$b$	$c$	$a$	$c$
$c$	$c$	$b$	$b$	$b$
$d$	$d$	$a$	$d$	$a$

$a$  and  $b$  are clones

# INDEPENDENCE OF CLONES

- A voting rule is independent of clones if when deleting alternatives from a set of clones  $S$ :
  - If the winner was in  $S$ , it is still in  $S$
  - If the winner was  $x \notin S$ , it is still  $x$

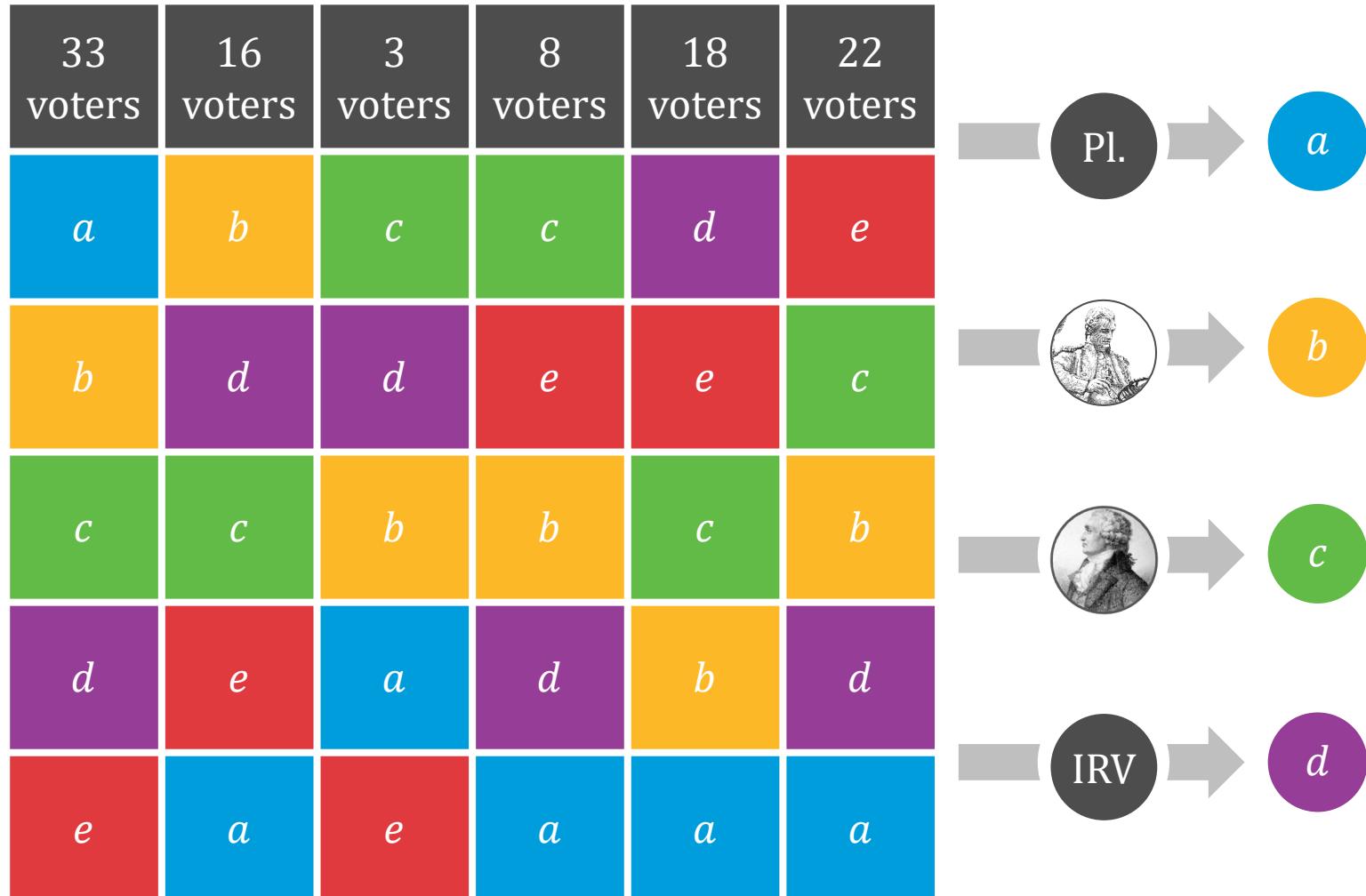
## Poll 2

Which rule is **not** independent of clones?

- Borda count
- IRV
- Schulze
- None of the above



# AWESOME EXAMPLE



One rule  
to rule  
them all?