



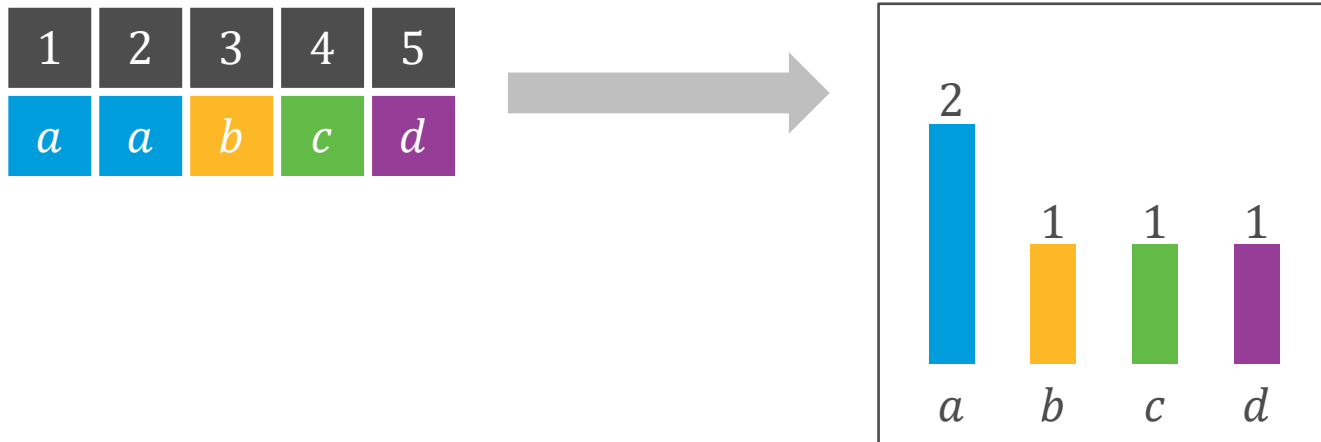
Spring 2026 | Lecture 5

Voting Rules

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PLURALITY

- Each person votes for a single alternative, and the alternative with most points wins
- A highly problematic voting rule!



SOME BALLOT TYPES



Rankings



Approvals



Scores/stars

We will focus on rankings!



Jean-Charles de Borda

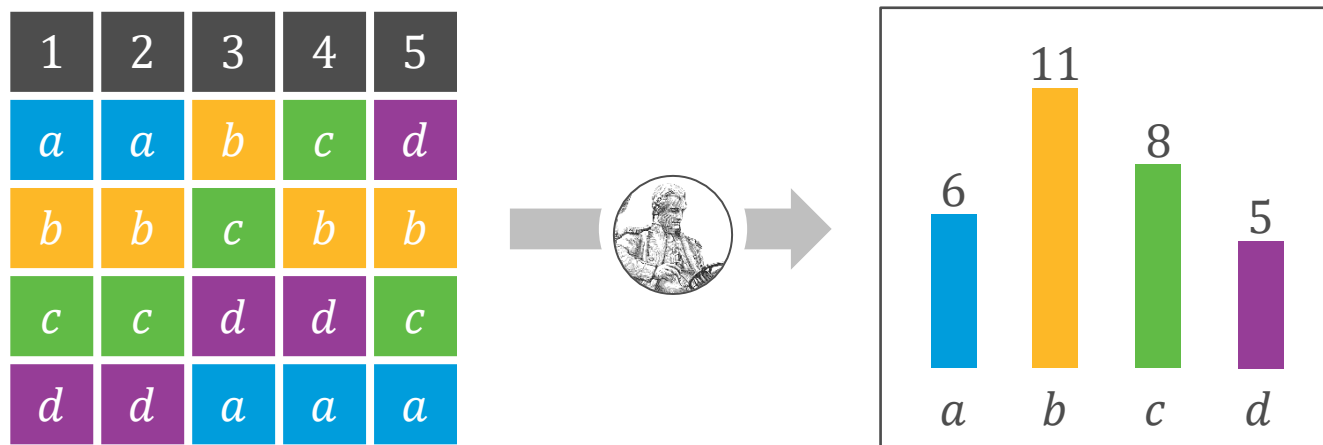
1733–1799

Mathematician, engineer, and naval officer. Also remembered as an instigator of the metric system.



BORDA COUNT

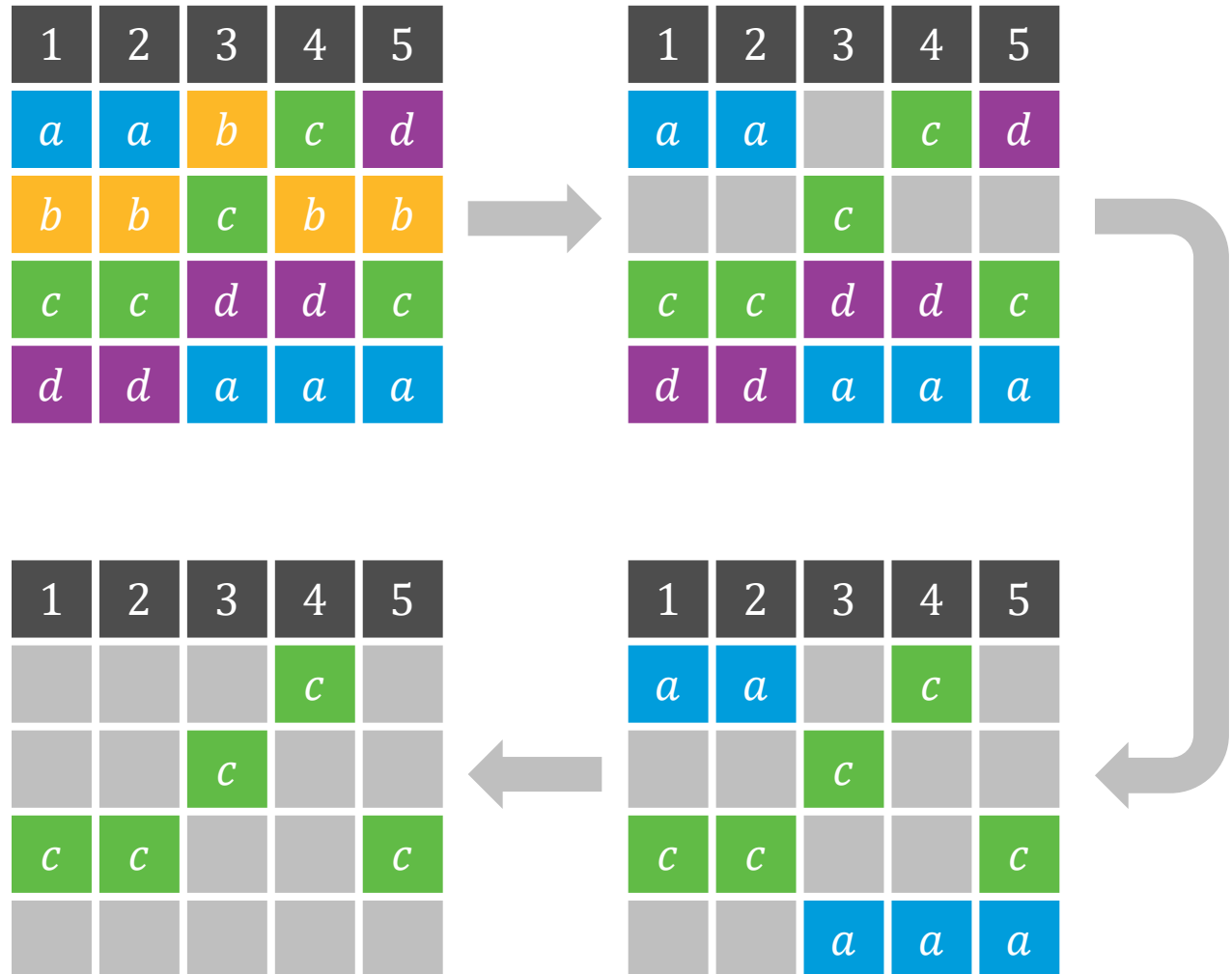
- Each voter awards $m - k$ points to the alternative placed in the k 'th position, where m is the number of alternatives



INSTANT-RUNOFF VOTING

- Also known as “alternative vote” and (misleadingly) “ranked-choice voting”
- Votes are tabulated in rounds, where in each round the alternative with the lowest plurality score is eliminated; last alternative left standing is the winner

INSTANT-RUNOFF VOTING



IRV AROUND THE WORLD



■ Ireland

Used for all public elections

■ Canada

Used in Ontario for municipal elections

■ Australia

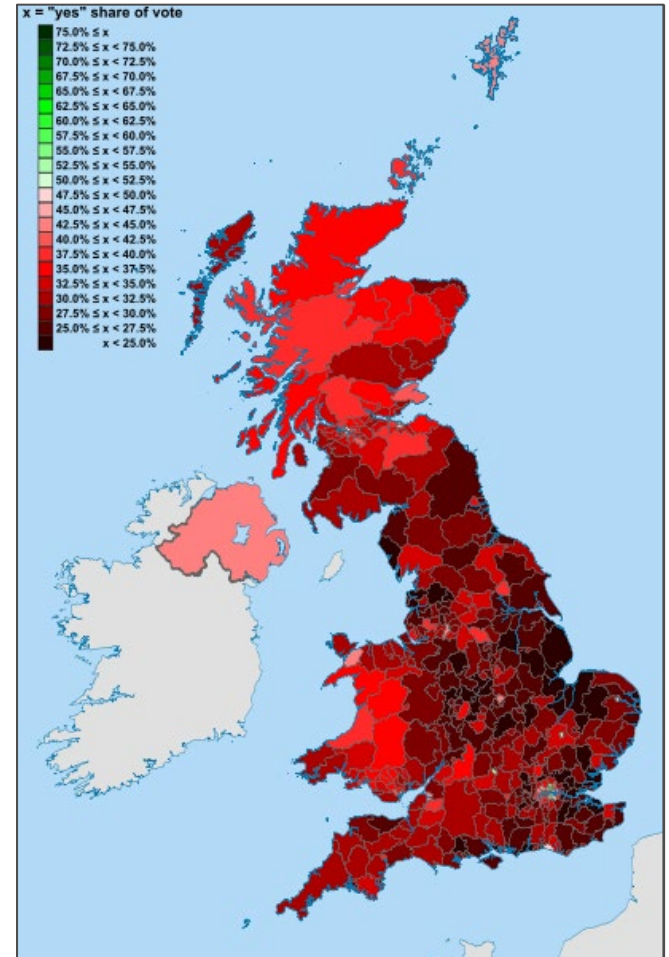
Used for parliamentary elections

■ USA

Used for statewide elections in ME and AK, and in cities like NYC and Cambridge

BARRIERS TO ADOPTION

- UK referendum (2011): Choose between plurality and IRV as a method for electing MPs
- Academics agreed IRV is better
- But IRV was seen as beneficial to a particular politician





Marquis de Condorcet

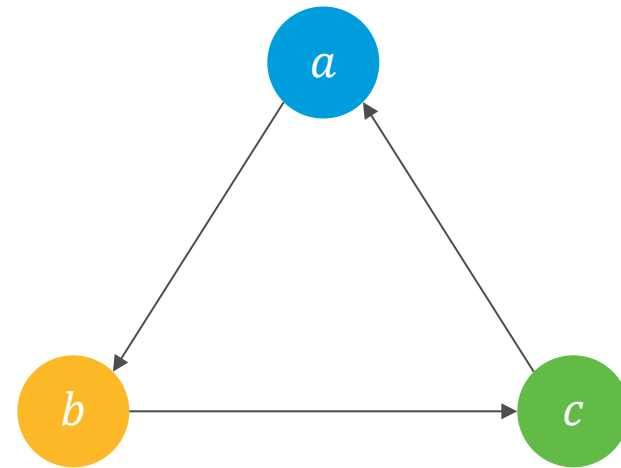
1743–1794

Philosopher, mathematician,
enlightened nobleman. Also known
for dying mysteriously in prison.



THE CONDORCET PARADOX

1	2	3
<i>a</i>	<i>c</i>	<i>b</i>
<i>b</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>b</i>	<i>a</i>



The preferences of the majority may be cyclical!

CONDORCET CONSISTENT RULES

- A **Condorcet winner** is an alternative that defeats every other alternative in a head-to-head comparison
- A rule is **Condorcet consistent** if it always selects a Condorcet winner whenever it is presented with a profile that contains one

Poll 1

Which rule is Condorcet consistent?

- Plurality
- Borda Count
- Both rules
- Neither one





Ramon Llull

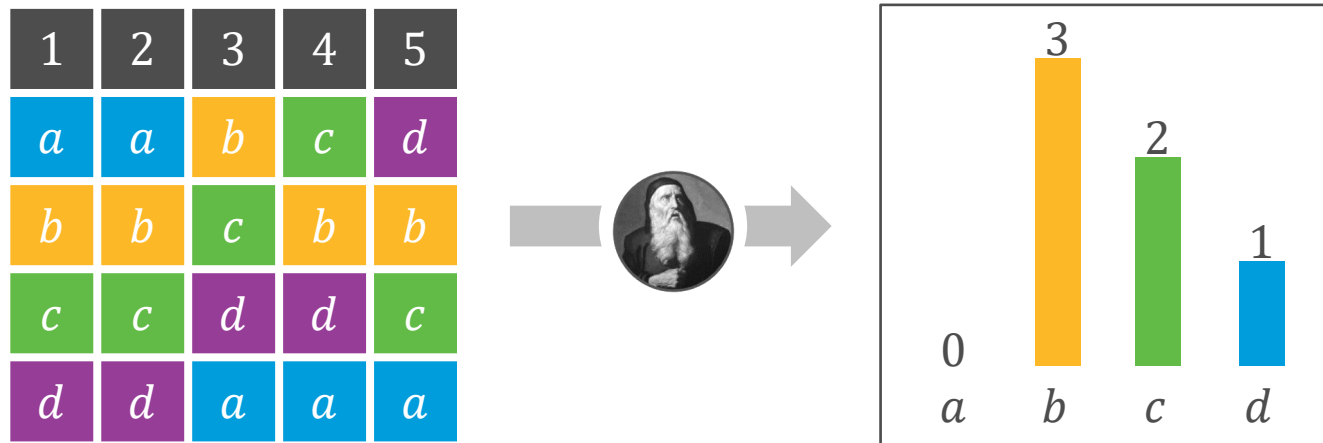
c. 1232–1315

Monk, missionary, and philosopher; one of the most influential intellectuals of his time. Also remembered for publishing a medieval parenting guide.



LLULL'S RULE

- Each alternative receives one point for each head-to-head comparison it wins (as well as for tied comparisons)



- Llull's rule is Condorcet consistent — **why?**



Charles Lutwidge Dodgson

1832–1898

Professor of mathematics at Oxford,
pioneer photographer, and beloved
author. Also known for not plagiarizing
Condorcet's work.

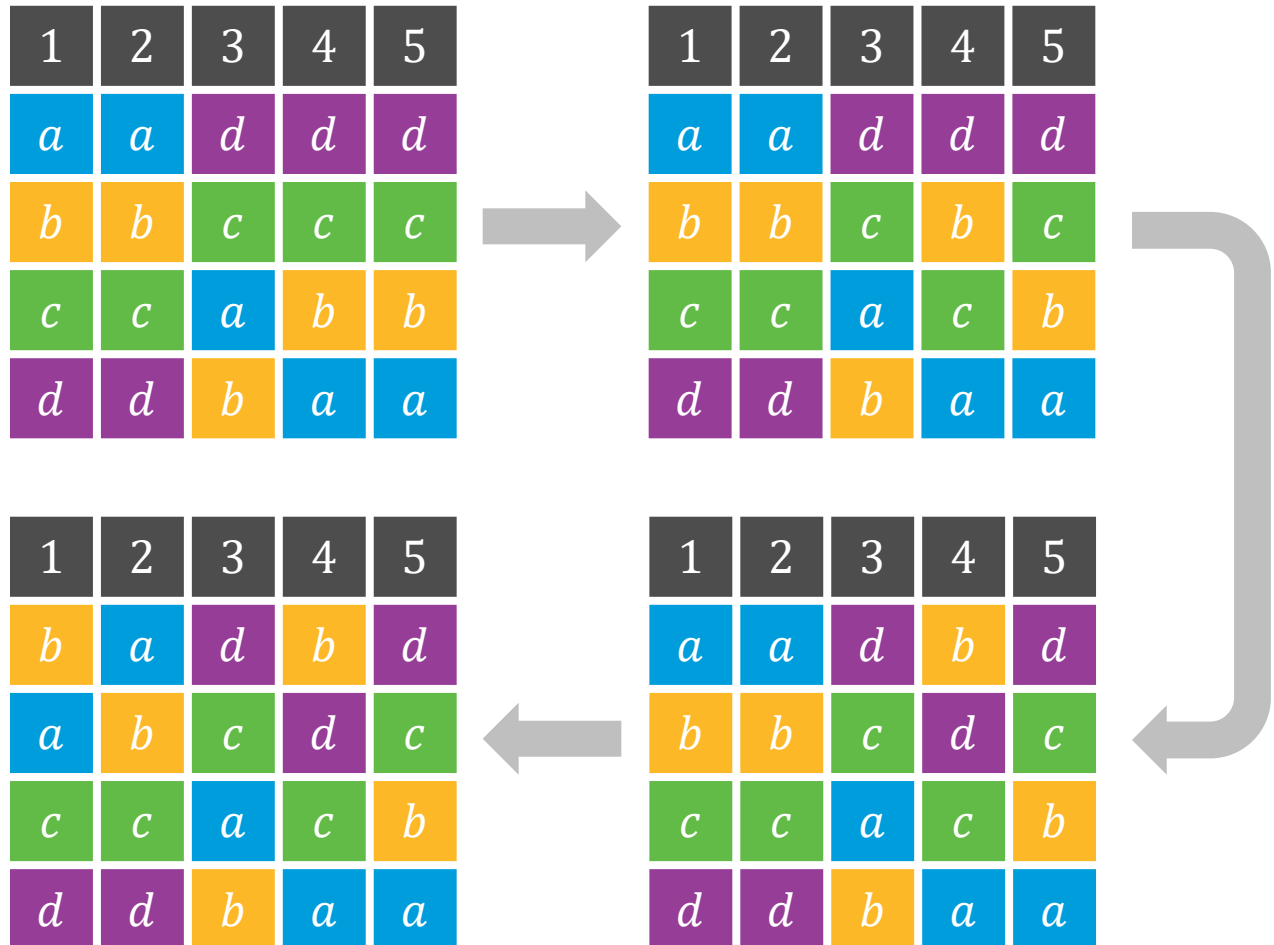


DODGSON'S RULE

- The **Dodgson score** of an alternative x is the minimum number of swaps between adjacent alternatives needed to make x a Condorcet winner; select an alternative with minimum score
- Dodgson's rule is Condorcet consistent
- Dodgson's rule is NP-hard to compute!

DODGSON'S RULE

What is the Dodgson score of b ?

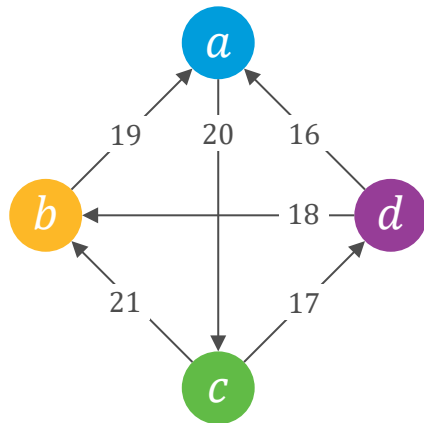


SCHULZE'S RULE

- Let $P(x, y)$ denote the number of voters who prefer x to y
- A path from x to y of strength p is a sequence of alternatives $x = a_1, \dots, a_k = y$ such that for all $i = 1, \dots, k - 1$, $P(a_i, a_{i+1}) > P(a_{i+1}, a_i)$ and $P(a_i, a_{i+1}) \geq p$
- Let $S(x, y)$ be the strength of the strongest path from x to y — it's 0 if there's no path
- **Exercise:** If $S(x, y) > S(y, x)$ and $S(y, z) > S(z, y)$ then $S(x, z) > S(z, x)$
- Therefore there exists a winning alternative x^* such that $S(x^*, y) \geq S(y, x^*)$ for all y
- Schulze's rule is Condorcet consistent

SCHULZE'S RULE

5 voters	2 voters	3 voters	4 voters	3 voters	3 voters	1 voter	5 voters	4 voters
<i>a</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>d</i>	<i>d</i>
<i>c</i>	<i>c</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>d</i>	<i>c</i>	<i>c</i>	<i>d</i>	<i>b</i>	<i>c</i>	<i>a</i>	<i>b</i>
<i>d</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>a</i>



Pairwise comparisons

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>a</i>	—	20	20	17
<i>b</i>	19	—	19	17
<i>c</i>	19	21	—	17
<i>d</i>	18	18	18	—

Strength of paths $S(x, y)$

INDEPENDENCE OF CLONES

A subset of alternatives S is called **clones** in a given preference profile if no voter ranks any alternative $x \notin S$ between two alternatives in S

1	2	3	4	5
<i>a</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>d</i>
<i>b</i>	<i>b</i>	<i>c</i>	<i>a</i>	<i>c</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>b</i>	<i>b</i>
<i>d</i>	<i>d</i>	<i>a</i>	<i>d</i>	<i>a</i>

a and b are clones

INDEPENDENCE OF CLONES

- A voting rule is independent of clones if when deleting alternatives from a set of clones S :
 - If the winner was in S , it is still in S
 - If the winner was $x \notin S$, it is still x

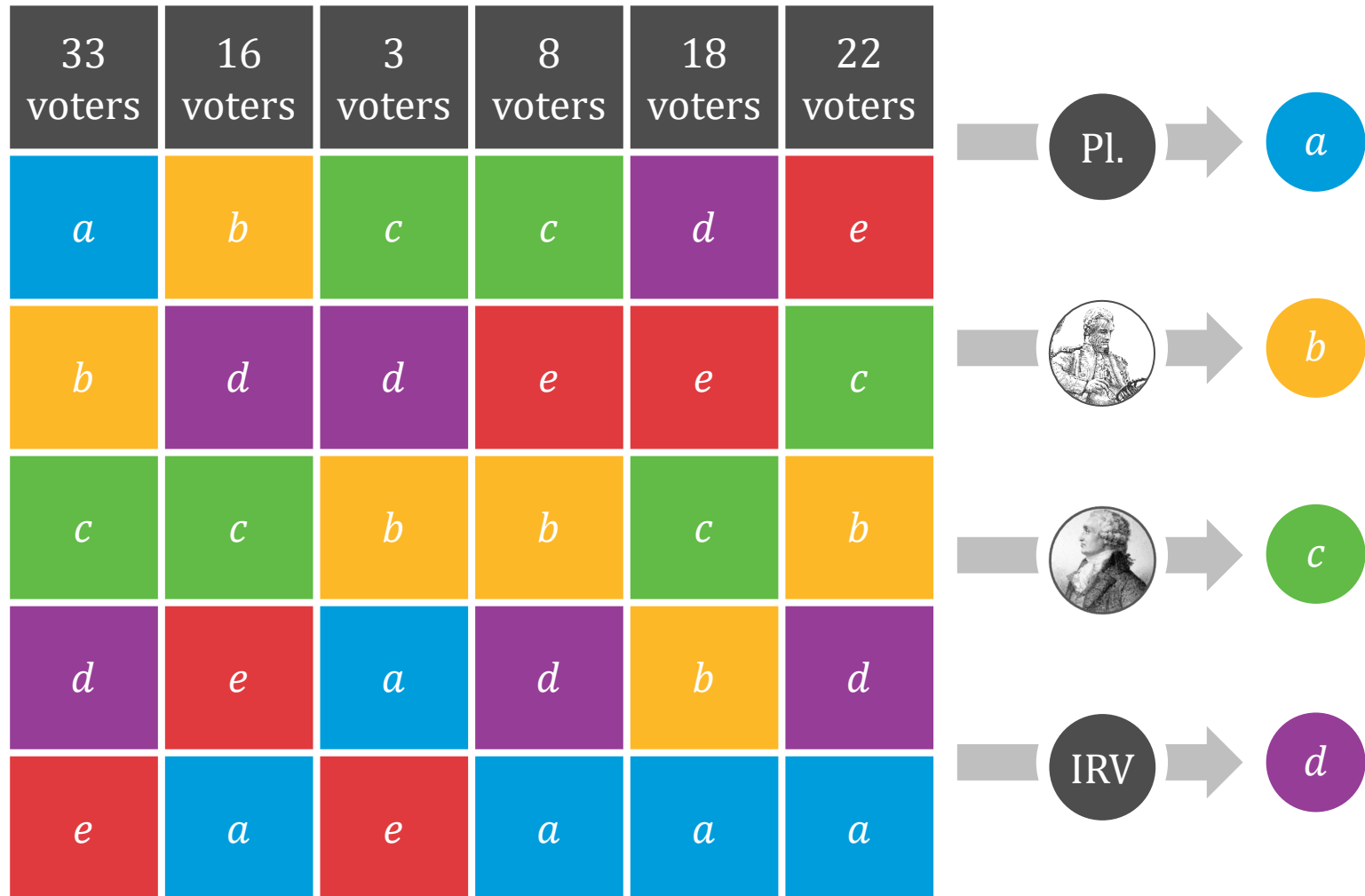
Poll 2

Which rule is **not** independent of clones?

- Borda count
- IRV
- Schulze
- None of the above



AWESOME EXAMPLE





One rule
to rule
them all?