

Spring 2026 | Lecture 14

Kidney Exchange

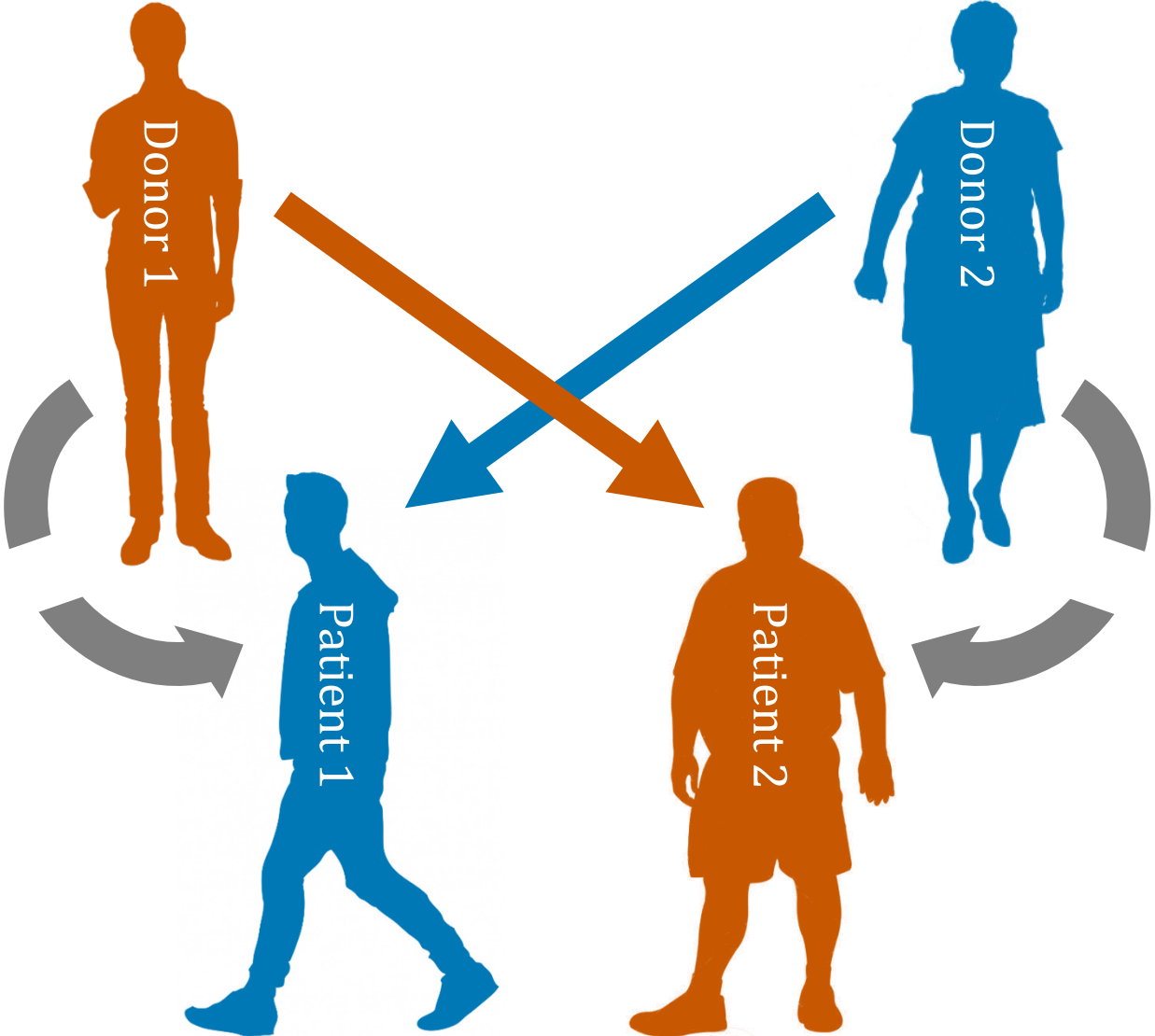
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KIDNEY EXCHANGE

- Kidney failure is a serious medical problem
- The preferred treatment is kidney transplant, from a deceased or live donor
- Must be blood-type and tissue-type compatible
- On March 22, 2026, there were 94,474 patients waiting for kidney transplant in the United States

<https://optn.transplant.hrsa.gov/data>

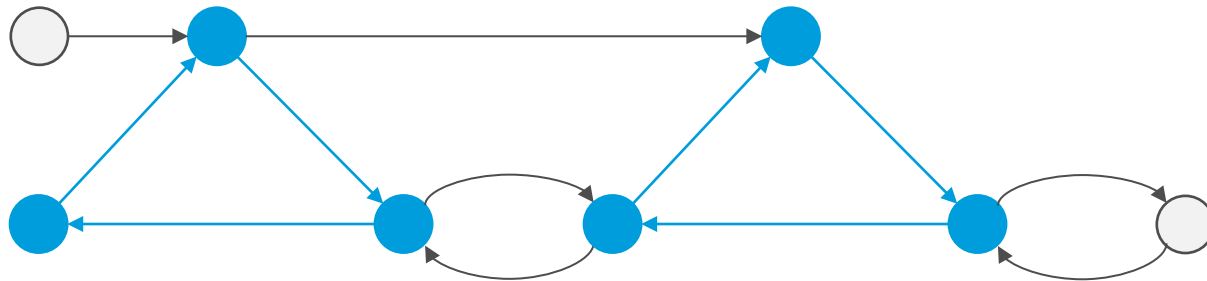
KIDNEY EXCHANGE



KIDNEY EXCHANGE GRAPHS

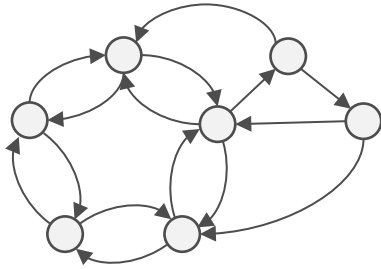
- More generally, we could have exchanges along longer cycles, although length 3 is usually the practical limit
- Model as a directed graph $G = (V, E)$ where V is a set of **donor-patient pairs** and there is an edge from u to v if the donor of u is compatible with the patient of v

CYCLE COVER

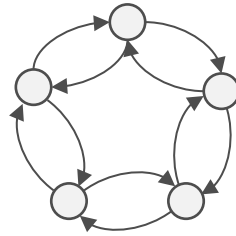


- **CYCLE-COVER** problem: Given a directed graph G and $L \in \mathbb{N}$, find a collection of disjoint cycles of length $\leq L$ in G that maximizes the number of covered vertices
- **Theorem:** For any constant $L \geq 3$, **CYCLE-COVER** is NP-complete

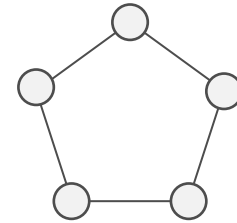
CYCLE-COVER WITH $L = 2$



Given a graph
with cycles of
any length



Focus on
cycles of
length 2



Now it's a
max matching
problem!

KIDNEY EXCHANGE AS IP

- For $L \geq 3$, CYCLE-COVER is solved in practice via integer programming
- Variables: For each cycle c of length $\ell_c \leq L$, variable $x_c \in \{0,1\}$, $x_c = 1$ iff cycle c is included in the cover

$$\begin{aligned} \max \quad & \sum_c x_c \ell_c \\ \text{s.t.} \quad & \forall v \in V, \sum_{c:v \in c} x_c \leq 1 \\ & \forall c, x_c \in \{0,1\} \end{aligned}$$

APPLICATION: UNOS

UNOS

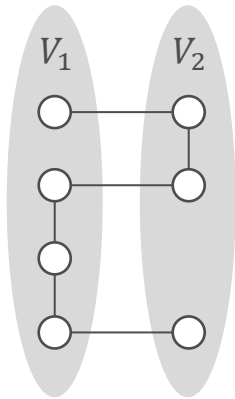
INCENTIVES

- In the past kidney exchanges were carried out by individual hospitals
- Today there are nationally organized exchanges; participating hospitals have little other interaction
- It was observed that hospitals match easy-to-match pairs internally, and enroll only hard-to-match pairs into larger exchanges
- Goal: incentivize hospitals to enroll all their pairs

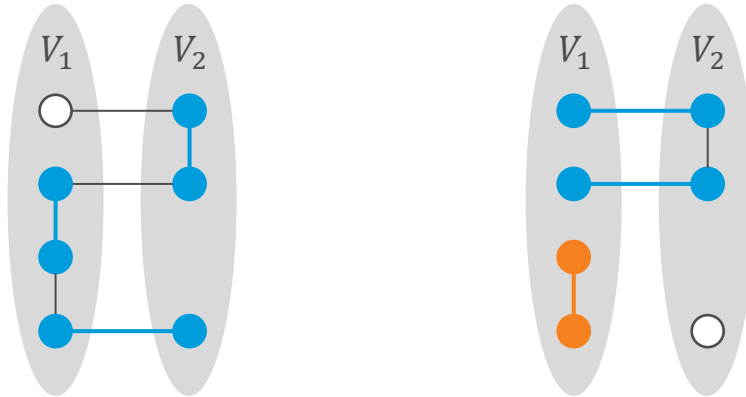
THE STRATEGIC MODEL

- Undirected graph (**only pairwise matches!**)
- Each player controls subset of vertices
- Mechanism receives a graph and returns a matching
- Utility of player = # its matched vertices
- Target: # matched vertices (util. social welfare)
- Strategy: subset of revealed vertices (but edges are public knowledge)
- Mechanism is strategyproof (SP) if it is a dominant strategy to reveal all vertices

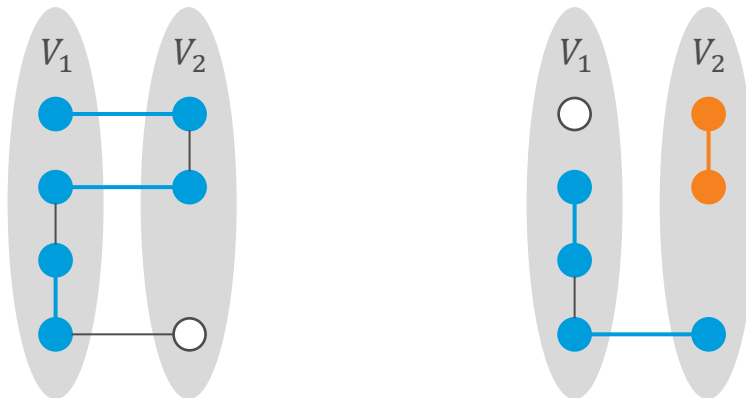
OPT IS MANIPULABLE



Player 1 has unmatched vertex

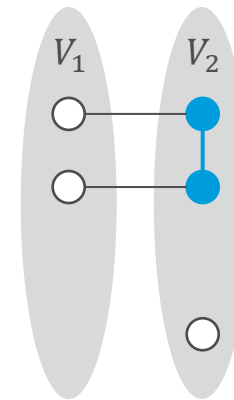
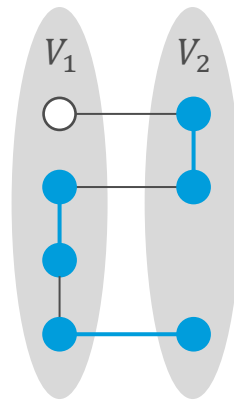
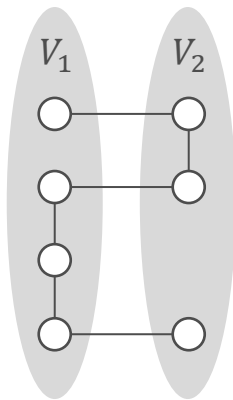


Player 2 has unmatched vertex



SP MECHANISM: TAKE 1

- Assume two players
- The $\text{MATCH}_{\{\{1\},\{2\}\}}$ mechanism:
 - Consider matchings that maximize the number of “internal edges”
 - Among these return a matching with max cardinality

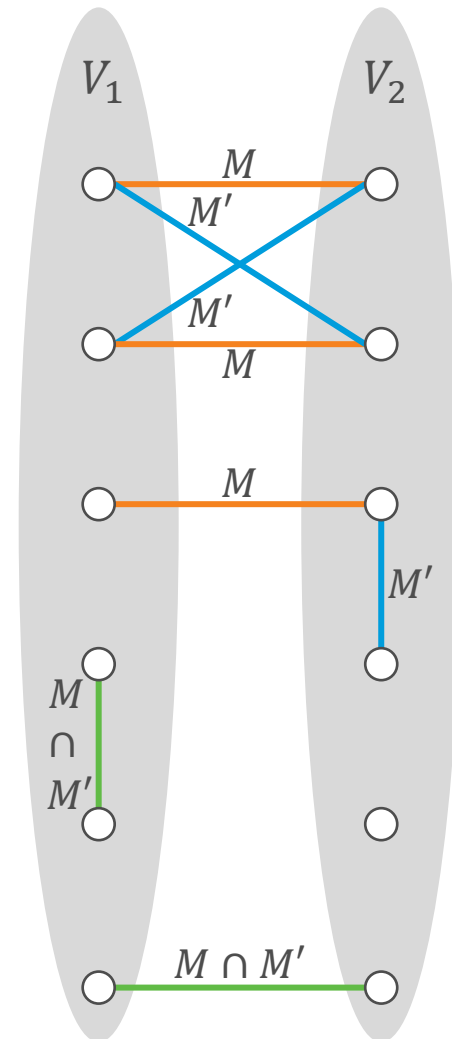


GUARANTEES

- $\text{MATCH}_{\{\{1\},\{2\}\}}$ gives a $1/2$ -approximation
 - Cannot add more edges to matching
 - For each edge in optimal matching, one of the two vertices is in mechanism's matching
- **Theorem (special case):** $\text{MATCH}_{\{\{1\},\{2\}\}}$ is strategyproof for two players

PROOF OF THEOREM

- M = matching when player 1 is honest, M' = matching when player 1 hides vertices
- $M \Delta M'$ consists of paths and cycles, each consisting of alternating M, M' edges
- In a cycle, M and M' both match all vertices, so player 1 is indifferent
- We will show that for every path, player 1 has at least as many matched vertices under M as M'

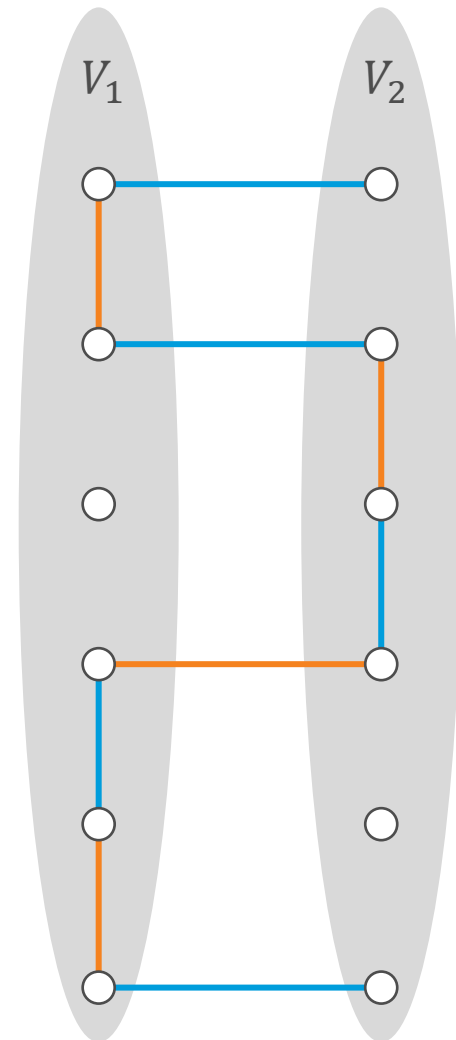


PROOF OF THEOREM

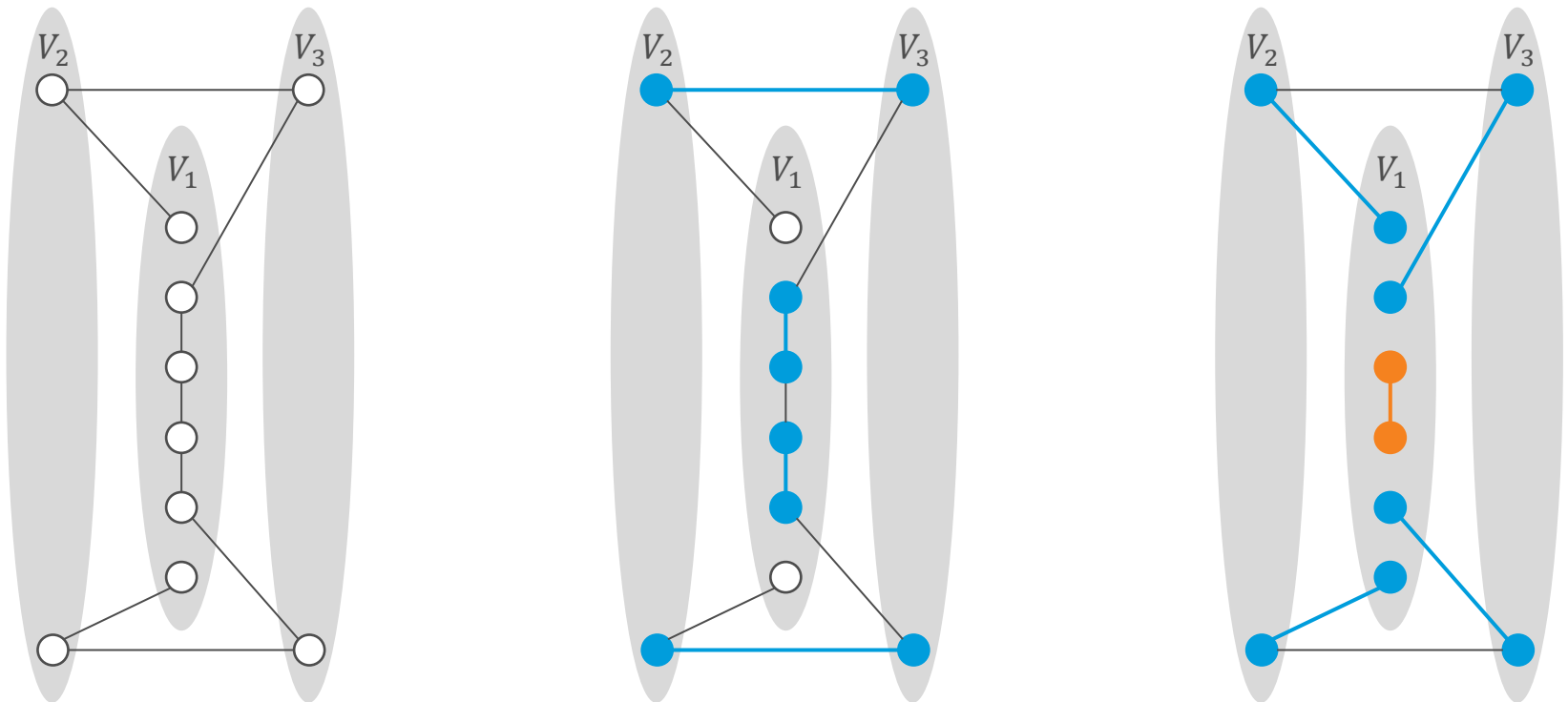
- Consider a path in $M\Delta M'$, denote its edges in M by P and its edges in M' by P'
- For $i, j \in \{1, 2\}$,
$$P_{ij} = \{(u, v) \in P : u \in V_i, v \in V_j\}$$
$$P'_{ij} = \{(u, v) \in P' : u \in V_i, v \in V_j\}$$
- $|P_{11}| \geq |P'_{11}|$, suppose $|P_{11}| = |P'_{11}|$
- It holds that $|P_{22}| = |P'_{22}|$
- M is max cardinality $\Rightarrow |P_{12}| \geq |P'_{12}|$
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2|P'_{11}| + |P'_{12}| = U_1(P')$

PROOF OF THEOREM

- Suppose $|P_{11}| > |P'_{11}|$
- $|P_{12}| \geq |P'_{12}| - 2$
 - Every subpath within V_2 is of even length
 - We can pair the edges of P_{12} and P'_{12} , except maybe the first and the last
- $$U_1(P) = 2|P_{11}| + |P_{12}| \geq 2(|P'_{11}| + 1) + |P'_{12}| - 2 = 2|P'_{11}| + |P'_{12}| = U_1(P') \blacksquare$$



THE CASE OF 3 PLAYERS



Maximizing internal edges is no longer SP

SP MECHANISM: TAKE 2

- Let $\Pi = (\Pi_1, \Pi_2)$ be a bipartition of the players
- The MATCH_{Π} mechanism:
 - Consider matchings that maximize the number of “internal edges” **and do not have any edges between different players on the same side of the partition**
 - Among these return a matching with max cardinality

EUREKA?

- **Theorem:** MATCH_{Π} is strategyproof for any number of players and any partition Π
- Recall: for $n = 2$, $\text{MATCH}_{\{\{1\},\{2\}\}}$ guarantees a $1/2$ -approximation

Poll

Approximation guarantees given by MATCH_{Π} for $n = 3$ and $\Pi = \{\{1\}, \{2,3\}\}$?

- $1/2$ -approx
- $1/3$ -approx
- $1/4$ -approx
- Less than $1/4$



MIX AND MATCH

- The MIX-AND-MATCH mechanism:
 - Mix: Choose a random partition Π
 - Match: Execute MATCH_{Π}
- **Theorem:** MIX-AND-MATCH is strategyproof and guarantees a $1/2$ -approximation