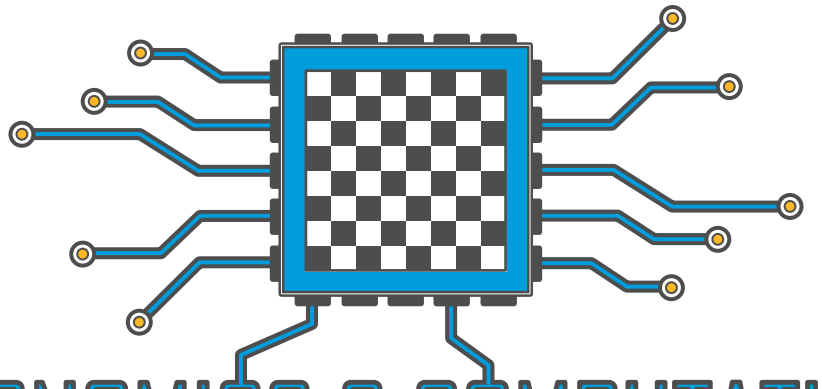


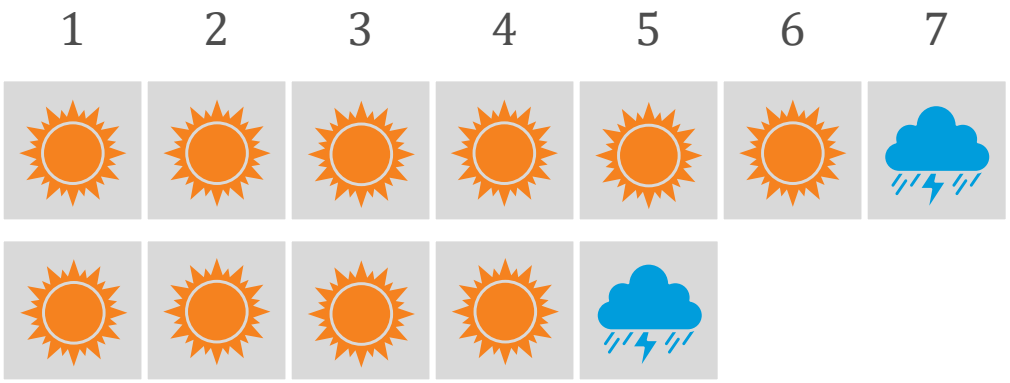
Spring 2026 | Lecture 13

Online Matching Algorithms

Ariel Procaccia | Harvard University

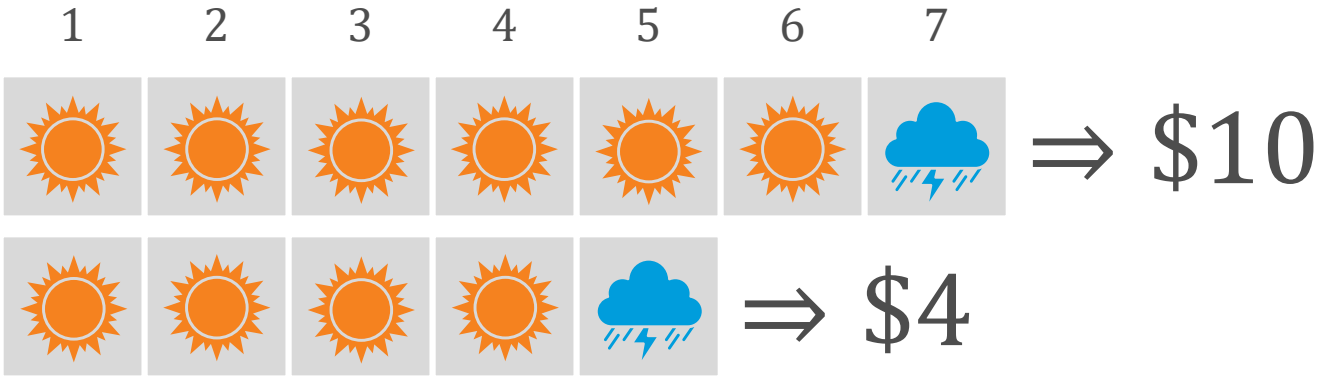
ONLINE ALGORITHMS

- You are on a ski vacation; you can buy skis for $\$B$ or rent for \$1/day
- You're very spoiled: You'll go home when it's not sunny
- Rent or buy when $B = 5$?



ONLINE ALGORITHMS

- Now assume you don't know in advance how many days of sunshine there are
- Every day of sunshine you need to decide whether to rent or buy
- Algorithm: Rent for B days, then buy



ONLINE ALGORITHMS

- The competitive ratio of an online algorithm defined identically to approximation ratio
- The difference is that the online algorithm is competing with the offline optimum — the difficulty stems from lack of information

Poll 1

For $B \geq 8$, what is the competitive ratio of the "rent for B days, then buy" algorithm?

○ 2 ○ 3 ○ $B/2$ ○ B

ONLINE ALGORITHMS

- Renting for $B-1$ days has a competitive ratio of $(2B-1)/B$
- Theorem: No online ski rental algorithm has a lower competitive ratio
- Proof:
  - Algorithm is defined by renting for K days and buying on day $K+1$
  - Adversary makes it rain on day $K+2$
  - $K \geq B: OPT(I) = B, ALG(I) = K + B \geq 2B$
  - $K \leq B-2: OPT(I) = K+1,$
    $ALG(I) = K + B \geq 2K + 2$ ■

DISPLAY ADVERTISING

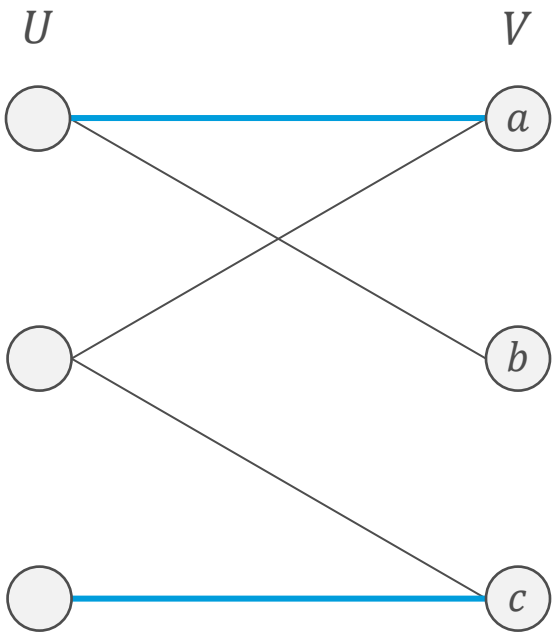
- Largest matching problem in the world
- Bipartite graph with advertisers and impressions
- Advertisers specify which impressions are acceptable — this defines the edges
- Impressions arrive online

THE (SIMPLEST) MODEL

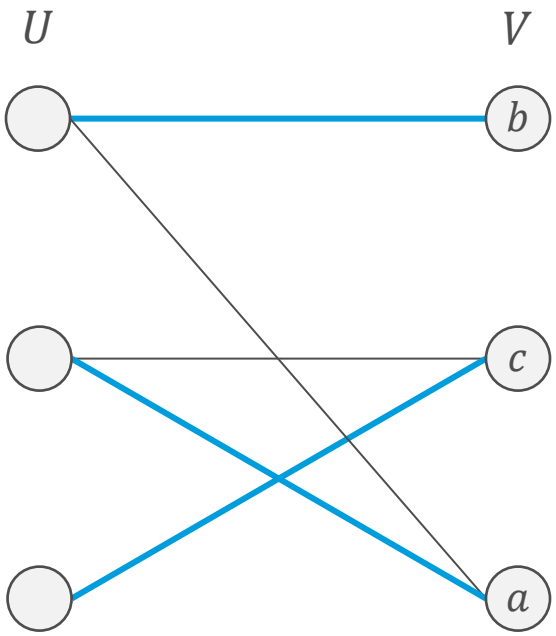
- Bipartite graph $G = (U, V, E)$ with $|U| = n$
- U is known "offline," the vertices of V arrive online (with their incident edges)
- Online vertices can only be matched when they arrive
- Objective: maximize size of matching
- ALG has competitive ratio $\alpha \leq 1$ if for every graph G and every input order π of V,

$$\frac{ALG(G, \pi)}{OPT(G)} \geq \alpha$$

EXAMPLE



Graph G, order (a, b, c)



Graph G, order (b, c, a)

A GREEDY ALGORITHM

- Algorithm GREEDY: match to an arbitrary unmatched neighbor (if one exists)

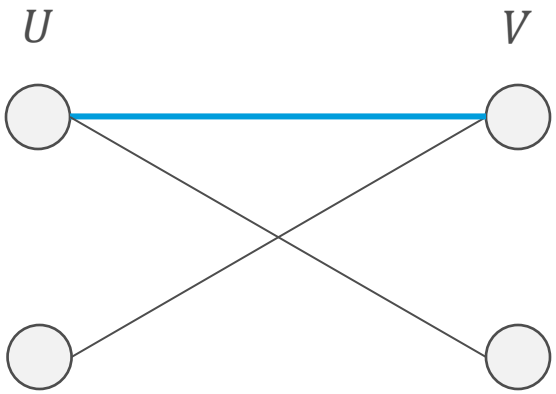
Poll 2

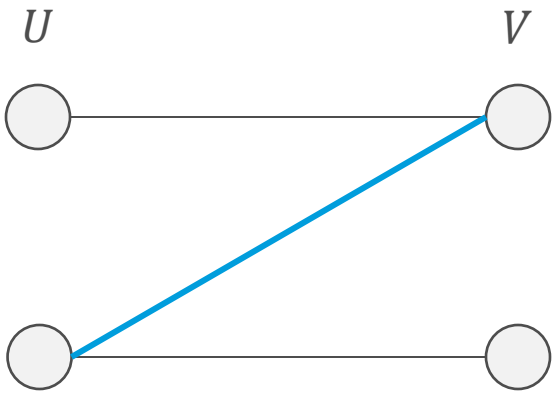
Competitive ratio of GREEDY?

- ○ $1/n$
- ○ $1/\log n$
- ○ $1/\sqrt{n}$
- ○ $1/2$

UPPER BOUND

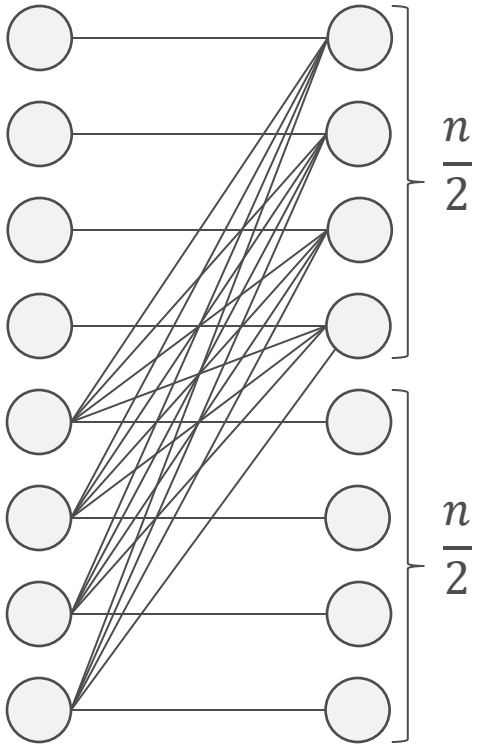
Observation: The competitive ratio of any deterministic algorithm is at most 1/2





TAKE 2: ALGORITHM RANDOM

- Obvious idea: randomness
- Algorithm RANDOM: Match to an unmatched neighbor uniformly at random
- Achieves ¾ on previous example



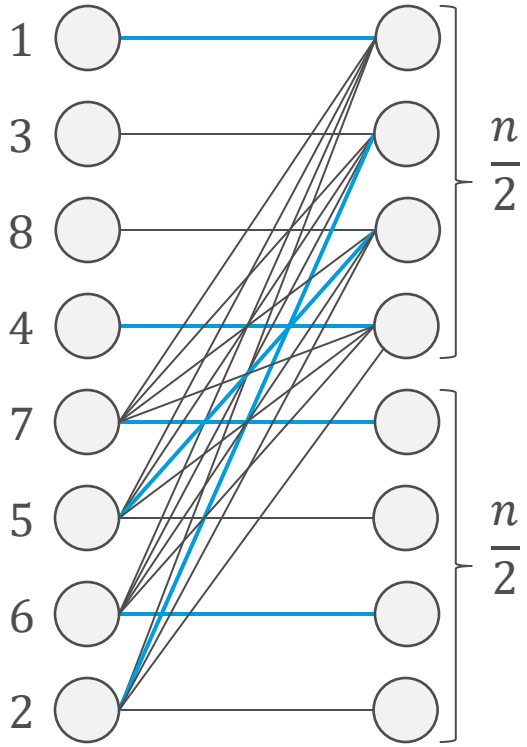
Poll 3

Competitive ratio of RANDOM on current graph?

- ~7/8
- ~5/8
- ~6/8
- ~4/8

TAKE 3: ALGORITHM RANKING

- Algorithm RANKING:
  - Choose a random permutation $\pi: U \to [n]$
  - Match each vertex to its unmatched neighbor u with the lowest $\pi(u)$
- Looks like this is doing better than RANDOM on previous example!
- Theorem: The competitive ratio of RANKING is $1 - 1/e \approx 0.63$, and this is the best possible

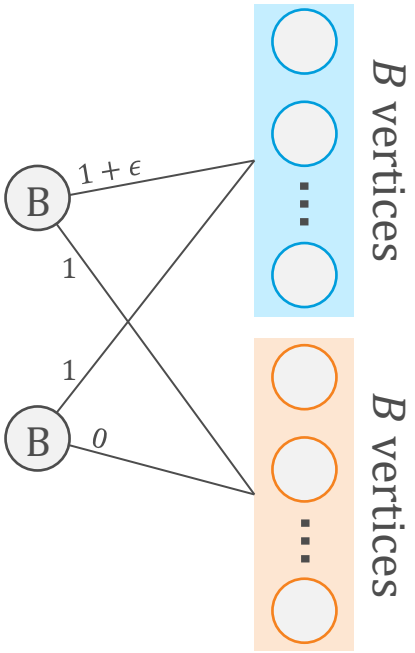


WEIGHTED MATCHING

- Let's augment the problem with the following features:
  - Each offline vertex u has a budget B_u
  - Each edge has a weight ("bid") and the goal is to maximize the weight of the matching
- Algorithm GREEDY' matches highest weight edge subject to budget
- Theorem: The competitive ratio of GREEDY' is 1/2

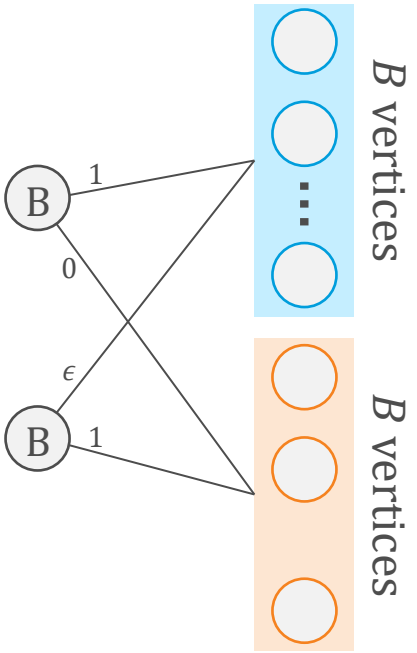
WEIGHTED MATCHING

- Let's make the realistic assumptions that for all v, $w_{uv} \ll B_u$
- The competitive ratio of GREEDY' is still 1/2



WEIGHTED MATCHING

- We need to take the remaining budget into account, but just allocating based on remaining budget is obviously a bad idea



THE MSVV ALGORITHM

- Denote by x_u the fraction of u's budget that has been spent
- Define $f(x) = 1 - e^{x-1}$
- In the MSVV Algorithm, each vertex v is matched with u that maximizes $w_{uv} \cdot f(x_u)$
- Theorem: MSVV has a competitive ratio that approaches $1 - 1/e$ as the budgets grows, and this is the best possible even among randomized algorithms

PRACTICAL CONSIDERATIONS

- The MSVV algorithm extends to advertisers arriving at different times, bidders paying only for clicks, and winning bidders paying the second-highest bid
- Significant impact on practice:

"The core problem of budget management remains important, and the core idea [of spending budget smoothly] remains impactful"

– Aranyak Mehta (Google Research), July 2024