Harvard CS 1360, Spring 2026
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Voting Rules

Lecture 5

Definition 1 (Plurality)**.** Each person votes for a single alternative, and the alternative with the most points wins.

This is a problematic voting rule. Consider the following example.

Example 1 (An Issue with Plurality)**.** Consider plurality among the following ranking preference profile:

1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a

If everyone were to vote for their favorite alternative, a would win because it would have 2 votes from voters 1 and 2, while the other three alternatives would have 1 vote from each of the remaining voters. However, the latter three voters also rank a last, and alternative b is arguably a better alternative as b is ranked first or second by every single voter.

Different types of ballots include:

- Rankings: each voter submits a ranking of alternatives
- Approvals: voters either approve or disapprove alternatives, and they can approve as many alternatives as they'd like
- Scores/stars: voters attribute a score or some number of stars to alternatives.

This lecture focuses on rankings. This is the most common way to think about voter preferences.

Definition 2 (Borda Count)**.** Each voter awards $m - k$ points to the alternative placed in the k-th position, where m is the number of alternatives.

This system was invented by Jean-Charles de Borda (1733 - 1799), who was a mathematician, engineer, and naval officer. He was also remembered as an instigator of the metric system.

Example 2 (Borda Count)**.** Consider running Borda Count on the following preference profile, resulting in the points per alternative shown to the right.

1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a

$\rightarrow$

alternative	points
a	6
b	11
c	8
d	5

Note that alternative c gets 1 point from voter 1, 1 point from voter 2, 2 points from voter 3, 3 points from voter 4, and 1 point from voter 5, resulting in a total of 8 points.

Definition 3 (Instant-Runoff Voting (IRV))**.** Also known as "alternative vote" and (misleadingly) "ranked-choice voting." Votes are tabulated in rounds, where in each round, the alternative with the lowest plurality score is eliminated. The last alternative left standing is the winner.

Example 3 (IRV)**.** Consider starting off with the following preference profile:

1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a

Breaking ties by alphabetical order, b has the lowest plurality score here. So, eliminating b, we are left with the following preference profile:

1	2	3	4	5
a	a	c	c	d
c	c	d	d	c
d	d	a	a	a

Now, d has the lowest plurality score, and so we eliminate d to get

1	2	3	4	5
a	a	c	c	c
c	c	a	a	a

Since a has the lowest plurality score, we eliminate a leaving c as the winner of the election.

Example 4 (IRV Around the World)**.** Examples of countries and regions using IRV include:

- Ireland: Used for all public elections.
- Canada: Used in Ontario for municipal elections.
- Australia: Used for parliamentary elections.
- USA: Used for statewide elections in Maine (ME) and Alaska (AK), and in cities like NYC and Cambridge.

There are many barriers to the adoption of IRV around the world. For example, the UK referendum (2011) presented a choice between plurality and IRV for electing MPs. Academics agreed that IRV was better, but it was seen as benefiting a particular politician, and so it was not adopted.

Marquis de Condorcet (1743 - 1794) was a philosopher, mathematician, and enlightened nobleman. He was also known for dying mysteriously in prison. He did a lot of work on voting, and we will see him more in-depth in the following lectures. He presented the following paradox:

Definition 4 (Condorcet Paradox)**.** The preferences of the majority may be cyclical, meaning no clear winner.

Example 5 (Condorcet Paradox)**.** Consider the following preference profile among 3 voters for 3 alternatives.

1	2	3
a	c	b
b	a	c
c	b	a

In this example, a is preferred over b, b is preferred over c, and c is preferred over a. Thus, although individual rankings are acyclical, the preferences of the majority are cyclical.

This may be trivial to us now, but at the time this was a big discovery and a large criticism of democracy. Condorcet characterized a clearly winning alternative as the following.

Definition 5 (Condorcet Winner)**.** A Condorcet winner is an alternative that defeats every other alternative in a head-to-head comparison.

Definition 6 (Condorcet Consistent Rules)**.** A rule is Condorcet consistent if it always selects a Condorcet winner whenever it is presented with a profile that contains one.

Example 6 (Condorcet Consistent Rules)**.** Plurality is not Condorcet consistent. Consider plurality on the following preference profile (we have already seen this example above):

1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a

Here, a is the plurality winner, but b is the Condorcet winner! This is because a majority of voters (3, 4, and 5) rank b above a, a majority of voters (1, 2, 3, and 5) rank b above c, and a majority of voters (1, 2, 3, and 4) rank b above d. Thus, plurality is not Condorcet consistent.

Borda Count is also not Condorcet consistent! Consider the following preference profile:

1	2	3
a	a	b
b	b	c
c	c	d
d	d	a

$\rightarrow$

alternative	points
a	6
b	7
c	4
d	1

Here, b is the winner under the Borda Count with 7 points, however, a is the Condorcet winner. This is because a majority of voters (1 and 2) rank a above all other alternatives. Thus, the Borda Count is not Condorcet consistent.

Ramon Llull (1232 - 1315) was a Monk, missionary, and philosopher. He was one of the most influential intellectuals of his time. He was also remembered for publishing a medieval parenting guide. He presented the following rule:

Definition 7 (Llull's Rule)**.** Each alternative receives one point for each head-to-head comparison it wins, as well as for tied comparisons.

Example 7 (Llull's Rule)**.** Consider Lull's Rule on the following preference profile:

1	2	3	4	5
a	a	b	c	d
b	b	c	b	b
c	c	d	d	c
d	d	a	a	a

$\rightarrow$

alternative	points
a	0
b	3
c	2
d	1

Note that c gets 2 points because c beats d (majority from voters 1, 2, 3, and 4) and a (majority from voters 3, 4, and 5) in head-to-head comparisons. Here, alternative b is the winner, which we have also seen is the Condorcet winner.

Note that Llull's rule is Condorcet consistent. This is because if there exists a Condorcet winner, it will achieve a score of $m - 1$ under Lull's rule, which is the maximum score achievable. Further, this score is necessarily unique because there can only be one winner of a head-to-head comparison, so if alternative a beats all other alternatives in a head-to-head comparison, the maximum score for any other alternative is $m - 2$ because they lose to alternative a.

Charles Lutwidge Dodgson (1832–1898 was a professor of mathematics at Oxford, a pioneer photographer, and a beloved author. He was also known for not plagiarizing Condorcet's work.

Definition 8 (Dodgson's Rule)**.** The Dodgson score of an alternative x is the minimum number of swaps between adjacent alternatives needed to make x a Condorcet winner. We then select an alternative with the minimum score.

Note that Dodgson's rule is trivially Condorcet consistent because a Condorcet winner would have a score of 0 and all other alternatives would have a positive score, as they are not Condorcet winners. Additionally, note that Dodgson's rule is NP-hard to compute.

Example 8 (Dodgson's Rule)**.** We will calculate the Dodgson score of b in the following preference profile:

1	2	3	4	5
a	a	d	d	d
b	b	c	c	c
c	c	a	b	b
d	d	b	a	a

Here, b is losing in head-to-head comparisons with all other preferences. We start by flipping b and c in voter 4's preferences to get

1	2	3	4	5
a	a	d	d	d
b	b	c	b	c
c	c	a	c	b
d	d	b	a	a

Now, b is beating c in a pairwise comparison, but b is still losing to a and d. We will now swap b and d in voter 4's preferences to get

1	2	3	4	5
a	a	d	b	d
b	b	c	d	c
c	c	a	c	b
d	d	b	a	a

Now, b is beating c and d in pairwise comparisons but is still losing to a. So, we swap a and b in voter 1's rankings to get

1	2	3	4	5
b	a	d	b	d
a	b	c	d	c
c	c	a	c	b
d	d	b	a	a

Now, b is beating all other alternatives according to the preferences of the majority, and so is a Condorcet winner in this profile. As this took 3 swaps (which is the minimum number of swaps) the Dodgson score of b is 3.

Definition 9 (Schulze's Rule)**.** Define $P(x, y)$ as the number of voters who prefer x to y. A path from x to y of strength p is a sequence of alternatives $x = a_1, \ldots, a_k = y$ such that for all $i = 1, \ldots, k-1$:

$$P(a_i, a_{i+1}) > P(a_{i+1}, a_i) \quad \text{and} \quad P(a_i, a_{i+1}) \geq p.$$

Let $S(x, y)$ be the strength of the strongest path from x to y (it's 0 if no path exists). It can be checked that if $S(x, y) > S(y, x)$ and $S(y, z) > S(z, y)$ then $S(x, z) > S(z, x)$. Therefore, a winning alternative x^* exists that satisfies

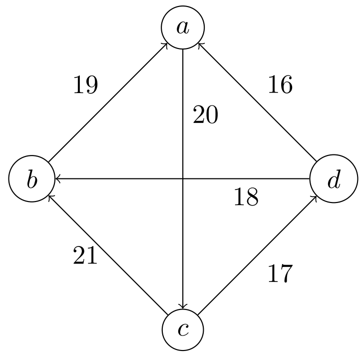
$$S(x^*, y) \geq S(y, x^*) \quad \text{for all } y.$$

Schulze's rule is Condorcet consistent.

Example 9 (Schulze's Rule)**.** Consider the following preference profile among 30 voters and 4 alternatives, where the number of voters who share that ranking is shown in the first line.

5 voters	2 voters	3 voters	4 voters	3 voters	3 voters	1 voter	5 voters	4 voters
a	a	a	b	c	c	d	d	d
c	c	d	a	b	d	a	b	c
b	d	c	c	d	b	c	a	b
d	b	b	d	a	a	b	c	a

This generates the following graph of pairwise comparisons, where a directed edge from x to y labeled ℓ represents that a majority of voters prefer x over y and that majority is ℓ voters.



For example, $b \to a$ is labeled with 19 because a is preferred to b by $4 + 3 + 3 + 5 + 4 = 19$ voters, which is greater than $30/2 = 15$ and so this preference is supported by a majority. Thus, we get that the strength of the paths $S(x, y)$ for all x and y is given by the following matrix:

	a	b	c	d
a	–	20	20	17
b	19	–	19	17
c	19	21	–	17
d	18	18	18	–

Note that $S(a, b) = 20$ because the strongest path from a to b is $a \to c \to b$ because each of these preferences is supported by the majority of voters and this path achieves a strength of 20, because $P(a, c) = 20$ and $P(c, b) = 21$. Further, $S(b, a) = 19$ because the only viable path from b to a is just $b \to a$ and $P(b, a) = 19$.

Thus, we get that d is the winner, because for all $x = a, b, c$,

$$18 = S(d, x) > S(x, d) = 17$$

Definition 10 (Independence of Clones)**.** A subset S of alternatives are clones in a given preference profile if no voter ranks any alternative $x \notin S$ between two alternatives in S. A voting rule is independent of clones if, when deleting alternatives from S:

- If the winner was in S, it remains in S.
- If the winner was $x \notin S$, it remains x.

Example 10 (Clones)**.** Consider the following preference profile:

	1	2	3	4	5
a	a	a	d	c	d
b	b	b	c	a	c
c	c	c	b	b	b
d	d	d	a	d	a

Here, alternatives a and b are clones because no voter ranks c or d between a and b in their preferences.

Example 11 (Rules that are Independent of Clones)**.** Borda Count is not independent of clones. Consider the following preference profile:

1	2	3
b	b	a
a	a	c
c	c	b

$\rightarrow$

alternative	points
a	4
b	4
c	1

Here, a is the winner (breaking ties by alphabetical order), and also note that alternatives a and c are clones. Now, remove alternative c to get

1	2	3
b	b	a
a	a	b

$\rightarrow$

alternative	points
a	1
b	2

and so the winner switches to b which was not in our set of clones. Thus, the Borda Count is not independent of clones.

Further, plurality is not independent of clones either. Consider the following preference profile:

1	2	3	4	5
a	a	a	b	b
c	d	e	a	a
e	c	d	c	c
d	e	c	d	d
b	b	b	e	e

Here, a is the winner, and further, alternatives a, c, d, e are all clones. If we remove the alternative a, we get the preference profile

1	2	3	4	5
c	d	e	b	b
e	c	d	c	c
d	e	c	d	d
b	b	b	e	e

and so b is the winner. As b was not in our set of clones, plurality is not independent of clones!

However, it turns out that Schulze's rule is independent of clones. The intuition behind this is that within our graph of pairwise comparisons (from the Schulze's rule example), we can group the nodes that represent our set of clones, and if we remove clones, the in and out-edges of this group of nodes will stay the same.

Example 12 (An Awesome Example)**.** Consider the following preference profile among 100 voters

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Plurality tells us that alternative a is the winner. Borda Count tells us that alternative b is the winner. c is a Condorcet winner here, so Llull's Rule, Dodgson's Rule, and Schulze's Rule will tell us c is the winner. Finally, IRV will select d as the winner.

So how do we choose between voting rules? Is there a single rule that rules them all? We will explore this question in subsequent lectures.