CS 1360 Spring 2026
Midterm Exam

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Problem 1: Price of Anarchy [25 points]

We define the class of *anti-coordination games* as follows. Let $G = (V, E)$ be an undirected graph. There is one player for each vertex $i \in V$. Each player chooses a color $s_i \in \{\mathsf{Red}, \mathsf{Blue}\}$. Given a strategy profile s, an edge $\{i, j\} \in E$ is *cut* if $s_i \neq s_j$.

Player i's utility is the number of i's cut edges, i.e., the number of their neighbors whose color is different from that of i: $u_i(\mathbf{s}) = \left|\{j \in V \setminus \{i\} : \{i, j\} \in E \text{ and } s_i \neq s_j\}\right|$.

Prove that the price of anarchy of anti-coordination games — defined with respect to pure Nash equilibrium and the social welfare objective $\sum_{i \in V} u_i$ — is at least 1/2.

Solution: Fix a Nash equilibrium $\mathbf{s}$. For each player i, flipping color does not increase their utility, and therefore at least half of their neighbors have the opposite color, i.e., $u_i(\mathbf{s}) \geq \deg(i)/2$. Summing over all vertices:

$$\sum_{i \in V} u_i(\mathbf{s}) \geq \sum_{i \in V} \frac{\deg(i)}{2} = \frac{1}{2} \cdot (2|E|) = |E|.$$

Since each edge can contribute at most 2 to the social welfare, trivially $\text{OPT} \leq 2|E|$, and the ratio of at least 1/2 directly follows.

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Problem 2: Zero-Sum Games [25 points]

Let $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}', \mathbf{y}')$ be two (possibly) mixed Nash equilibria in a zero-sum game. Prove that $u_1(\mathbf{x}, \mathbf{y}) = u_1(\mathbf{x}', \mathbf{y}')$.

Note: Prove this from scratch. You may not rely on the Minimax Theorem stated in class.

Solution: Define

$$v := u_1(\mathbf{x}, \mathbf{y}).$$

Because $(\mathbf{x}, \mathbf{y})$ is an equilibrium,

$$u_1(\mathbf{x}', \mathbf{y}) \le v \qquad \text{and} \qquad u_1(\mathbf{x}, \mathbf{y}') \ge v.$$

Because $(\mathbf{x}', \mathbf{y}')$ is an equilibrium,

$$u_1(\mathbf{x}, \mathbf{y}') \le u_1(\mathbf{x}', \mathbf{y}') \qquad \text{and} \qquad u_1(\mathbf{x}', \mathbf{y}) \ge u_1(\mathbf{x}', \mathbf{y}').$$

Combining these inequalities,

$$v \le u_1(\mathbf{x}, \mathbf{y}') \le u_1(\mathbf{x}', \mathbf{y}') \le u_1(\mathbf{x}', \mathbf{y}) \le v.$$

Hence all inequalities are equalities and

$$u_1(\mathbf{x}, \mathbf{y}) = u_1(\mathbf{x}', \mathbf{y}').$$

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Problem 3: Equilibrium Computation [25 points]

Consider the following Stackelberg game:

	L	R
U	$2, 1$	$0, 2$
M	$0, 2$	$3, 0$
D	$1, 0$	$2, 1$

Note: This problem has two parts; part 2 is on the next page.

1. [**15 points**] Write down two linear programs that find the optimal mixed strategy for the leader under the constraint that the follower's best response is L or R.

   Solution: Let the leader commit to $\mathbf{x} = (p_U, p_M, p_D)$ with $p_U + p_M + p_D = 1$ and $p_U, p_M, p_D \geq 0$.

   LP (follower best-responds with L):

$$\begin{aligned} \max \quad & 2p_U + p_D \\ \text{s.t.} \quad & p_U + 2p_M \geq 2p_U + p_D \\ & p_U + p_M + p_D = 1 \\ & p_U, p_M, p_D \geq 0. \end{aligned}$$

   LP (follower best-responds with R):

$$\begin{aligned} \max \quad & 3p_M + 2p_D \\ \text{s.t.} \quad & 2p_U + p_D \geq p_U + 2p_M \\ & p_U + p_M + p_D = 1 \\ & p_U, p_M, p_D \geq 0. \end{aligned}$$

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2. **[10 points]** Compute the leader's utility at a strong Stackelberg equilibrium.

   Note: You do not need to *formally* derive the solutions to the LPs.

   Solution: The optimal solution to the LP for L is $p_U = 2/3, p_M = 1/3, p_D = 0$, giving utility $4/3$ to the leader. The optimal solution to the LP for R is $p_U = 0, p_M = 1/3, p_D = 2/3$, giving utility $7/3$ to the leader. The latter is higher and is therefore the answer.

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Problem 4: Strategic Manipulation in Elections [25 points]

Consider the following algorithm for the Borda-Manipulation problem: the manipulator i ranks the desired alternative p first, and then ranks the other alternatives in order of non-decreasing Borda score (based on the given votes σ_{-i} of the non-manipulators).

Prove that if there exists any vote for the manipulator that makes p the unique winner under Borda, then this algorithm also makes p the unique winner.

Note: Prove this from scratch. You may not rely on the result for the greedy manipulation algorithm established in class.

Solution: First, without loss of generality, restrict attention to manipulator votes that rank p first: if p is not first, swapping p upward increases p's score by 1 and decreases another alternative's score by 1, so p remains (and becomes even more strongly) the unique winner.

Denote by $S(x)$ the Borda score of x based on the votes of the non-manipulators. With p ranked first, the manipulator assigns the multiset of points $\{0, 1, \ldots, m-2\}$ to the other alternatives. Let $a(x)$ denote the points assigned to $x \neq p$. Final scores are

$$F(p) = S(p) + (m-1), \qquad F(x) = S(x) + a(x).$$

Thus p is the unique winner iff

$$\max_{x \neq p}(S(x) + a(x)) < S(p) + (m-1).$$

We claim that among all assignments $a(\cdot)$ of $\{0, \ldots, m-2\}$, the reverse-score rule (largest S gets 0, next gets 1, etc.) minimizes

$$\max_{x \neq p}(S(x) + a(x)).$$

Indeed, suppose $S(x) > S(y)$ but $a(x) > a(y)$. Swap their positions and assigned points. After the swap,

$$\max\{S(x) + a(y),\, S(y) + a(x)\} < \max\{S(x) + a(x),\, S(y) + a(y)\}.$$

So the maximum opponent score does not increase.

Repeating such swaps until no inversion remains yields exactly the reverse-score assignment, without increasing the maximum.

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Additional Space / Scratch Paper

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Additional Space / Scratch Paper