CS 1360 Spring 2025
Final Exam
— Solutions —

Problem 1: Social Choice

1. **[5 pts]** Describe in words the concepts *Condorcet winner* and *Condorcet consistency*.

   Solution: A Condorcet winner is an alternative that defeats every other alternative in a head-to-head comparison. A rule is Condorcet consistent if it always selects a Condorcet winner whenever it is presented with a profile that contains one

2. **[15 pts]** Define the *score* of an alternative $x \in A$ in a preference profile $\boldsymbol{\sigma}$ to be $\min_{y \in A \setminus \{x\}} |\{i \in N : x \succ_{\sigma_i} y\}|$. In words, the score of an alternative is the result of its worst head-to-head comparison with another alternative. The *maximin rule* chooses an alternative with maximum score (breaking ties if needed). Prove that the maximin rule is Condorcet consistent.

   Solution:

   We wish to show that if a Condorcet winner exists in a profile, then the maximin rule selects it. Assume that c is a Condorcet winner in the profile σ. Since c beats all other alternatives in a head-to-head comparison, we have that for all $x \in A$

   $$|\{i \in N : c \succ_{\sigma_i} x\}| > |\{i \in N : x \succ_{\sigma_i} c\}|$$

   But since $|\{i \in N : c \succ_{\sigma_i} x\}| + |\{i \in N : x \succ_{\sigma_i} c\}| = N$, we conclude that

   $$|\{i \in N : x \succ_{\sigma_i} y\}| < \frac{N}{2}$$

   and

   $$|\{i \in N : c \succ_{\sigma_i} x\}| > \frac{N}{2} \implies s(c) = \min_{y \in A} |\{i \in N : c \succ_{\sigma_i} y\}| > \frac{N}{2}$$

   Now, for any $x \in A$, we have that

$$s(x) = \min_{y \in A} |\{i \in N : x \succ_{\sigma_i} y\}| \leq |\{i \in N : x \succ_{\sigma_i} c\}| < \frac{N}{2} < s(c)$$

Thus, the Condorcet winner c has the higher score and is therefore chosen by the maximin rule.

Problem 2: The VCG Mechanism

1. [**5 pts**] Define (using mathematical notation) the *VCG Mechanism.*

   Solution: The VCG mechanism is defined by a welfare-maximizing choice rule,

   $$f(v) \;\in\; \arg\max_{x \in A} \sum_{i \in N} v_i(x)$$

   and a payment rule p, where A^{-i} is the set of alternatives available when i is not present, and

   $$p_i(v) \;=\; \max_{x \in A^{-i}} \sum_{j \neq i} v_j(x) \;-\; \sum_{j \neq i} v_j\big(f(v)\big).$$

2. [**15 pts**] Prove that under the VCG Mechanism, when $A = A^{-i}$ for all players $i \in N$, the payment of each player is always non-negative.

   Solution: We see that in the first term of the payment rule, x can take on the value of $f(v)$. This is because $f(v) \in A = A^{-i}$ is still a possible alternative even if player i is not there, we just do not sum over their utility $v_i(x)$. Thus, for any i, we get that

   $$\max_{x \in A^{-i}} \sum_{j \neq i} v_j(x) \geq \sum_{j \neq i} v_j(f(v))$$

   and so it immediately follows that

   $$p_i(v) \;=\; \max_{x \in A^{-i}} \sum_{j \neq i} v_j(x) \;-\; \sum_{j \neq i} v_j\big(f(v)\big) \geq 0$$

Problem 3: Rent Division

1. [**5 pts**] In the context of rent division with quasi-linear utilities, define (using mathematical notation) the concept of *envy-free solution* $(\pi, \mathbf{p})$.

   Solution: A solution (π, p) is envy-free if and only if

   $$\forall\, i, j \in N, \quad v_{i,\pi(i)} - p_{\pi(i)} \;\geq\; v_{i,\pi(j)} - p_{\pi(j)}$$

2. **[15 pts]** Prove that if $(\pi, \mathbf{p})$ is an envy-free solution, then π is a welfare-maximizing assignment.

   Note: This was done in class.

   Solution: Let (π, p) be an envy-free solution, and let σ be any other assignment. By envy-freeness, for all $i \in N$,

$$v_{i,\pi(i)} - p_{\pi(i)} \;\geq\; v_{i,\sigma(i)} - p_{\sigma(i)}.$$

   Summing over $i \in N$ yields

$$\sum_{i \in N} v_{i,\pi(i)} - \sum_{i \in N} p_{\pi(i)} \;\geq\; \sum_{i \in N} v_{i,\sigma(i)} - \sum_{i \in N} p_{\sigma(i)}$$

   Since $\sum_j p_j = R$, it follows that

$$\sum_{i \in N} v_{i,\pi(i)} \;\geq\; \sum_{i \in N} v_{i,\sigma(i)}$$

   and so π is a welfare-maximizing assignment.

Problem 4: Stable Matching

1. **[5 pts]** Define (using mathematical notation) the concept of *student-optimal* stable matching in bipartite matching with two-sided preferences. (There is no need to define stable matching.)

   Solution: Student-optimal stable matching in bipartite matching with two-sided preferences is $\bar{\pi}$ such that $\bar{\pi} \geq_S \pi$ for every stable matching π

2. **[15 pts]** Prove that a given instance of bipartite matching with two-sided preferences has a unique stable matching if and only if the student-proposing deferred acceptance algorithm and the course-proposing deferred acceptance algorithm output the same matching.

   Note: Prove both directions. You may rely on any result stated in class.

   Solution: First, we will prove that if the student-proposing deferred acceptance algorithm and the course-proposing deferred acceptance algorithm output the same matching, then the given instance of bipartite matching with two-sided preferences has a unique stable matching. Let's say that π_1 is the shared stable matching. For the sake of contradiction, let's say that there exists a different stable matching π_2. Since π_1 is student-optimal, all the students weakly prefer π_1 to π_2. Similarly, since π_1 is course-optimal, all the courses weakly prefer π_1 to π_2. Thus, all the "players" weakly prefer π_1 to π_2. Since $\pi_1 \neq \pi_2$, this means that there exists at least one student - course pairing that is different between the two. However, this would mean that the student and courses involved in this pair would still weakly prefer π_1 to π_2, making π_2 not a stable matching, so a contradiction has occurred.

Next, we will prove that if a given instance of bipartite matching with two-sided preferences has a unique stable matching, then the student-proposing deferred acceptance algorithm and the course-proposing deferred acceptance algorithm output the same matching. Let's denote the unique stable matching as π^*. We proved in class (Theorem 3) that the student-proposing DA terminates with a student-optimal stable matching and the course-proposing DA terminates with a course-optimal stable matching. This means that since π^* is the only stable matching, the student-proposing DA outputs π^* and the course-proposing also outputs π^*. Thus, the two algorithms output the same matching.

Problem 5: Submodular Functions

1. **[5 pts]** Define (using mathematical notation) the concept of a *submodular* set function.

   Solution: A set function $f : 2^V \to \mathbb{R}$ is *submodular* if for all $X \subseteq Y \subseteq V$ and any element $z \notin Y$,

$$f(X \cup \{z\}) - f(X) \ \geq\ f(Y \cup \{z\}) - f(Y).$$

2. **[15 pts]** For $B \geq 0$, $w_1, \ldots, w_n \geq 0$, and $S \subseteq \{1, \ldots, n\}$, define the following two functions:

   - $f_1(S) = \min\left\{B, \sum_{i \in S} w_i\right\}$.
   - $f_2(S) = \max\left\{B, \sum_{i \in S} w_i\right\}$.

   Which of these two functions is submodular? For each, prove submodularity or give a counterexample.

   Solution:

   - $f_1(S) = \min\left\{B, \sum_{i \in S} w_i\right\}$ is submodular. To prove this, there are 3 cases to consider.

     Let the first case be when $f(X \cup \{z\}) \leq B$ and $f(Y \cup \{z\}) \leq B$. In this case, $f(X \cup \{z\} - f(X) = \sum_{i \in X} w_i + w_z - \sum_{i \in X} w_i = w_z$. Similarly, $f(Y \cup \{z\} - f(Y)$ also equals w_z. Thus $f(X \cup \{z\} - f(X) = f(Y \cup \{z\} - f(Y)$ so submodularity holds in this case.

     In the next case, we will consider when $f(X \cup \{z\}) \leq B$ and $f(Y \cup \{z\}) > B$. Once again, $f(X \cup \{z\} - f(X) = w_z$. Then, $f(Y \cup \{z\} - f(Y) = B - \sum_{i \in Y} w_i$. Since $B \leq \sum_{i \in Y} w_i + w_z$, $B - \sum_{i \in Y} w_i \leq w_z$. Thus, $f(X \cup \{z\} - f(X) \geq f(Y \cup \{z\} - f(Y)$ so submodularity holds in this case.

     Finally, we consider the case when $f(X \cup \{z\}) > B$ and $f(Y \cup \{z\}) > B$. Using similar logic to the previous case, we have that $f(X \cup \{z\}) - f(X) = B - \sum_{i \in X} w_i$ and $f(Y \cup \{z\}) - f(Y) = B - \sum_{i \in Y} w_i$. Let us say that $R = Y\ X$. Then,

$f(Y \cup \{z\}) - f(Y) = B - \sum_{i \in R} w_i + \sum_{i \in X} w_i$. Thus, for f_1 to be submodular, then $B - \sum_{i \in Y} w_i \geq B - \sum_{i \in R} w_i + \sum_{i \in X} w_i$. This must be true since $\sum_{i \in R} w_i \geq 0$. Note that since $X \subseteq Y$, then $\sum_{i \in X} x_i \leq \sum_{i \in Y} x_i$, so we don't have to consider the case where $f(X \cup \{z\}) > B$ but $f(Y \cup \{z\}) \leq B$

- $f_2(S) = \max\left\{B, \sum_{i \in S} w_i\right\}$ is not submodular. We will show this using a counterexample. Let us set B $= 8$, $w_1 = 4$, $w_2 = 5$, $w_3 = 6$, $X = \{1\}$, $Y = \{1, 2\}$, and $z = 3$. Then, we have that $f(X \cup \{z\}) = \max\{8, 10\} = 10$, and $f(X) = \max\{8, 4\} = 8$. Next, we have that $f(Y \cup \{z\}) = 4 + 5 + 6 = \max\{8, 15\} = 15$ and $f(Y) = 4 + 5 = \max\{8, 9\} = 9$. This means that $f(X \cup \{z\} - f(X) = 10 - 8 = 2$ and $f(Y \cup \{z\} - f(Y) = 15 - 9 = 6$. This violates the submodularity inequality as 2 is not greater than or equal to 6, so f_2 is not submodular.