# Economics and Computation (Spring 2026)
# Assignment #2

Due: 2/26/2026 11:59pm ET

## Problem 1: The price of anarchy

Consider the following scheduling game. The players $N = \{1, \ldots, n\}$ are associated with tasks, each with weight $w_i$. There is also a set $M$ of $m$ machines. Each player chooses a machine to place their task on, that is, the strategy space of each player is $M$. A strategy profile induces an assignment $A : N \to M$ of players (or tasks) to machines; the *cost* of player $i$ is the total load on the machine to which $i$ is assigned: $\ell_{A(i)} = \sum_{j \in N:\ A(j)=A(i)} w_j$. Our objective function is the *makespan*, which is the maximum load on any machine: $\text{cost}(A) = \max_{\mu \in M} \ell_\mu$. It is known that scheduling games always have pure Nash equilibria.

1. **[20 points]** Let $G$ be a scheduling game with $n$ tasks of weight $w_1, \ldots, w_n$, and $m$ machines. Let $A : N \to M$ be a Nash equilibrium assignment. Prove that

$$\text{cost}(A) \leq \left(2 - \frac{2}{m+1}\right) \cdot \text{opt}(G).$$

   That is, the price of anarchy is at most $2 - 2/(m+1)$.

2. **[15 points]** Prove that the upper bound of part (a) is tight, by constructing an appropriate family of scheduling games for each $m \in \mathbb{N}$.

## Problem 2: Voting rules

**[15 points]** When the number of alternatives is $m$, a *positional scoring rule* is defined by a score vector $(s_1, \ldots, s_m)$ such that $s_k \geq s_{k+1}$ for all $k = 1, \ldots, m-1$. Each voter gives $s_k$ points to the alternative they rank in position $k$, and the points are summed over all voters. We discussed two examples of positional scoring rules: plurality, defined by the vector $(1, 0, \ldots, 0)$, and Borda, defined by the vector $(m-1, m-2, \ldots, 0)$. Another common example is *veto*, defined by the vector $(1, \ldots, 1, 0)$.

For the case of $m = 3$, prove that any positional scoring vector with $s_2 > s_3$ is *not* Condorcet consistent.

**Hint**: It is possible to do this via a single preference profile that includes 7 voters.

## Problem 3: The epistemic approach to voting

**[15 points]** Suppose that there is a true ranking of $m$ alternatives, each of $n$ voters evaluates all pairs of alternatives according to the Condorcet noise model (Lecture 6, slide 5) with $p > 1/2$, and these comparisons are aggregated into a voting matrix. Prove that the output of the Kemeny rule applied to this voting matrix coincides with the true ranking with probability that goes to 1 as $n$ goes to infinity.

**Hint:** Use the Condorcet Jury Theorem (or the law of large numbers).

## Problem 4: Strategic manipulation in elections

We saw in class a proof sketch of the Gibbard-Satterthwaite Theorem for the special case of strategyproof and neutral voting rules with $m \geq 3$ and $m \geq n$. That proof relied on two key lemmas. In this problem, you will prove the two lemmas and formalize the theorem's proof for this special case.

Prove the following statements.

1. **[10 points]** Let $f$ be a strategyproof voting rule, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a preference profile, and $f(\boldsymbol{\sigma}) = a$. If $\boldsymbol{\sigma}'$ is a profile such that $[a \succ_{\sigma_i} x \;\Rightarrow\; a \succ_{\sigma'_i} x]$ for all $x \in A$ and $i \in N$, then $f(\boldsymbol{\sigma}') = a$.

2. **[10 points]** Let $f$ be a strategyproof and onto voting rule. Furthermore, let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a preference profile and $a, b \in A$ such that $a \succ_{\sigma_i} b$ for all $i \in N$. Then $f(\boldsymbol{\sigma}) \neq b$.

   **Hint**: use part (a).

3. **[15 points]** Let $m$ be the number of alternatives and $n$ be the number of voters, and assume that $m \geq 3$ and $m \geq n$. Furthermore, let $f$ be a strategyproof and neutral voting rule. Then $f$ is dictatorial.

   **Important note:** There are many proofs of the full version of the Gibbard-Satterthwaite Theorem; *here the task is specifically to formalize the proof sketch we did in class.*