Graduate AI

Lecture 19:
Game Theory I

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Normal-Form Game

- A game in normal form consists of:
  - Set of players $N = \{1, \dots, n\}$
  - Strategy set S
  - For each $i \in N$, utility function $u_i : S^n \to \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player i is $u_i(s_1, \dots, s_n)$

THE PRISONER'S DILEMMA

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year

THE PRISONER'S DILEMMA

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?

On TV

http://youtu.be/S0qjK3TWZE8

The professor's dilemma

		Class	
		Listen	Sleep
Professor	Make effort	$10^6, 10^6$	-10,0
	Slack off	0,-10	0,0

Dominant strategies?

Nash equilibrium

- Each player's strategy is a **best response** to strategies of others
- Formally, a **Nash equilibrium** is a vector of strategies $s = (s_1 \dots, s_n) \in S^n$ such that for all $i \in N, s_i' \in S$, $u_i(s) \geq u_i(s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n)$

The professor's dilemma

Professor \ Class	Listen	Sleep
Make effort	10^6,10^6	-10,0
Slack off	0,-10	0,0

Nash equilibria?

Rock-paper-scissors

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Nash equilibria?

Mixed strategies

- A mixed strategy is a probability distribution over (pure) strategies
- The mixed strategy of player $i \in N$ is x_i, where

$$x_i(s_i) = \Pr[i \text{ plays } s_i]$$

- The utility of player $i \in N$ is

$$u_i(x_1, \ldots, x_n) = \sum_{(s_1,\ldots,s_n) \in S^n} u_i(s_1, \ldots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)$$

Exercise: Mixed NE

- Exercise: player 1 plays $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, player 2 plays $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. What is u_1?
- Exercise: Both players play $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. What is u_1?

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Exercise: Mixed NE

- Poll 1: Which is a NE?
  1. $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right)$
  2. $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right)$
  3. $\left(\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right)$
  4. $\left(\left(\frac{1}{3},\frac{2}{3},0\right),\left(\frac{2}{3},0,\frac{1}{3}\right)\right)$

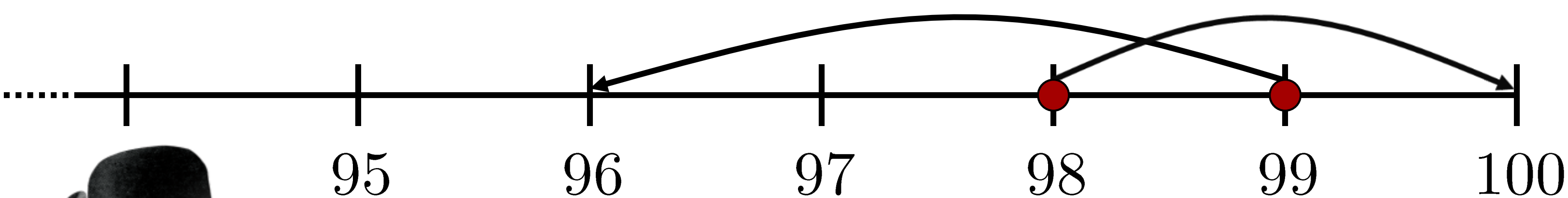
	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Nash's Theorem

- Theorem [Nash, 1950]: In any (finite) game there exists at least one (possibly mixed) Nash equilibrium

Does NE make sense?

- Two players, strategies are $\{2, \ldots, 100\}$
- If both choose the same number, that is what they get
- If one chooses s, the other t, and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$
- Poll 2: What would you choose?



Correlated equilibrium

- Let $N = \{1,2\}$ for simplicity
- A mediator chooses a pair of strategies (s_1, s_2) according to a distribution p over S^2
- Reveals s_1 to player 1 and s_2 to player 2
- When player 1 gets $s_1 \in S$, he knows that the distribution over strategies of 2 is

$$\Pr[s_2 | s_1] = \frac{\Pr[s_1 \wedge s_2]}{\Pr[s_1]} = \frac{p(s_1, s_2\ )}{\sum_{s_2' \in S} p(s_1, s_2')}$$

Correlated equilibrium

- Player 1 is best responding if for all $s_1' \in S$

$$\sum_{s_2 \in S} \Pr[s_2 | s_1]\, u_1(s_1, s_2) \geq \sum_{s_2 \in S} \Pr[s_2 | s_1]\, u_1(s_1', s_2)$$

- Equivalently,

$$\sum_{s_2 \in S} p(s_1, s_2) u_1(s_1, s_2) \geq \sum_{s_2 \in S} p(s_1, s_2) u_1(s_1', s_2)$$

- p is a correlated equilibrium (CE) if both players are best responding

Game of chicken

http://youtu.be/u7hZ9jKrwvo

Game of chicken

- Social welfare is the sum of utilities
- Pure NE: (C,D) and (D,C), social welfare = 5
- Mixed NE: both (1/2,1/2), social welfare = 4
- Optimal social welfare = 6

	Dare	Chicken
Dare	0,0	4,1
Chicken	1,4	3,3

Game of chicken

- Correlated equilibrium:
  - (D,D): 0
  - (D,C): $\frac{1}{3}$
  - (C,D): $\frac{1}{3}$
  - (C,C): $\frac{1}{3}$

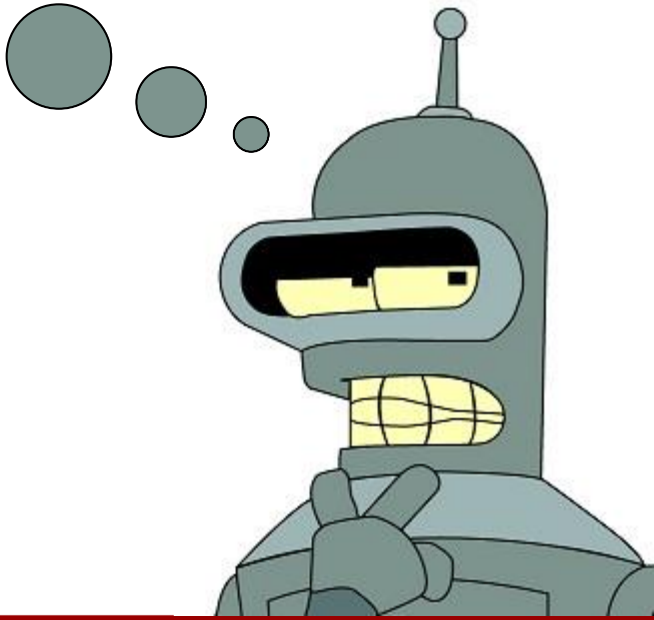
	Dare	Chicken
Dare	0,0	4,1
Chicken	1,4	3,3

- Social welfare of CE $= \frac{16}{3}$

Implementation of CE

- Instead of a mediator, use a hat!
- Balls in hat are labeled with "chicken" or "dare", each blindfolded player takes a ball
- Poll 3: Which balls implement the distribution of slide 19?
    1. 1 chicken, 1 dare
    2. 2 chicken, 1 dare
    3. 2 chicken, 2 dare
    4. 3 chicken, 2 dare

What is the relation between CE and NE?



CE As LP

- Can compute CE via linear programming in polynomial time!

$$\textbf{find } \forall s_1, s_2 \in S, p(s_1, s_2)$$

$$\textbf{s.t. } \forall s_1, s_1' \in S, \sum_{s_2 \in S} p(s_1, s_2) u_1(s_1, s_2) \geq \sum_{s_2 \in S} p(s_1, s_2) u_1(s_1', s_2)$$

$$\forall s_2, s_2' \in S, \sum_{s_1 \in S} p(s_1, s_2) u_2(s_1, s_2) \geq \sum_{s_1 \in S} p(s_1, s_2) u_2(s_1, s_2')$$

$$\sum_{s_1, s_2 \in S} p(s_1, s_2) = 1$$

$$\forall s_1, s_2 \in S, p(s_1, s_2) \in [0,1]$$

Summary

- Terminology:
  - Normal-form game
  - Nash equilibrium
  - Mixed strategies
  - Correlated equilibrium
- Algorithms:
  - LP for CE