15-251 : Great Theoretical Ideas In Computer Science

Fall 2013

Assignment 10 (With the Beatles)

Due: Tuesday, Nov. 26, 2013 11:59 PM

Name: ______________________________

Andrew ID: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	20	15	20	25	20	10	110
Score:							

1. Strawberry Fields Forever

One ordinary Day in the Life, John was frolicking Here, There, and Everywhere along the countryside, just as he did as a kid on Penny Lane. After talking to The Fool on the Hill and passing by tangerine trees and marmalade skies, he came across a great strawberry field. The field consisted of countably many sets of strawberries, each of which contained a countable number of strawberries. Good Day Sunshine!

(10) (a) First, John wondered how many total strawberries this field contained - if it was countable, he figured he could eat one every minute for the rest of time and eat all of them. However, he worried that if it was uncountable, there would be no way to eat all the strawberries.

Prove or disprove the following:
The union of countably many countable sets is countable.

Solution:

(10) (b) Next, John wondered how many ways he could select a single strawberry from each set so that he could bring them home to his mother Julia.

Again, prove or disprove the following:
The product of countably many countable sets is countable.

Solution:

2. Being for the Benefit of Mr. Kite

"I want to create the greatest performance I've ever done In My Life," Mr. Kite thought. He'd been working Eight Days a Week and had at his disposal an act i for each $i \in \mathbb{N}$ (that's a lot of acts!), but wasn't sure the absolute best way to arrange them, so asked for a little help from his friend Ringo. They wanted to arrange the acts so that no act was repeated and so that every act eventually appeared. Ringo worried that it would be quite A Hard Day's Night to find the absolute best ordering, as he thought the total number of orderings is uncountable. Prove Ringo's theory.

(15) (a) Formally, define $\mathbb{N}!$ to be the set of permutations of $\mathbb{N}$. In particular, the elements of $\mathbb{N}!$ are the infinite sequences that contain every natural number exactly once. Prove that $\mathbb{N}!$ is uncountable.

(Hint: Note that a permutation requires two things: no numbers should repeat, and every number should be included. For the latter part, use lazy diagonalization (recall this technique from recitation).)

Solution:

3. Sun King

Paul, John, and George each came up with some great parts of songs (melodies, chord progressions, and lyrics) that could be described using regular languages. However, they wanted to make them even more awesome by adding some extra effects before putting them together to make songs such as reversing and repeating the sounds. These techniques allowed them to create such songs as "I'm Only Sleeping" and "Revolution 9". Before they added these effects, they wanted to make sure that it would still sound good by proving that the results could still be described using regular languages.

Let L be a regular language. Prove that each of the following languages are also regular. Given a DFA for L, provide a formal construction (explicitly define the 5-tuple) for a DFA or NFA that accepts the new language, and give a clear and concise argument of why it works. A super-formal argument of correctness would involve induction on the length of input strings, but we only want an intuitive explanation.

(10) (a) Reverse$(L) = \{x_1 x_2 ... x_n | x_n ... x_2 x_1 \in L\}$

Solution:

(10) (b) Kleene$(L) = \{w_1 ... w_k | k \geq 0$ and each $w_i \in L\}$.

Solution:

4. **Help!**

Yesterday, Eleanor Rigby was feeling lonely Because it was raining, so she decided to work on some fun puzzles. After struggling for a while, she started to feel so tired that her mind was on the blink. Eleanor decided to sleep on it before starting again the next day (which happened to be her Birthday), because Tomorrow Never Knows. She had very strange dreams that night involving a Glass Onion, human walruses, and Bungalow Bill. When she woke up from her Golden Slumbers, these dreams gave her the idea to call up her friends from the Beatles.
"Hey Jude, can you help me out with some problems?" Eleanor asked.
"Sure, I think We Can Work It Out," Jude replied.
"Prudence, won't you come out to help me figure out some puzzles?"
"Yeah, I'll be right over! Will it be just the Two of Us?"
"Nope, Jude is coming and I'm also going to ask a few more friends from all Across the Universe!"
"Hello Paul, would you like to come over and work on some puzzles with me? Do You Want To Know a Secret? It's my Birthday!"
"You say it's your birthday, well happy birthday to you! I'll Drive My Car over but I might be a little late. They're Fixing a Hole on Abbey Road so I'll have to take Blue Jay Way instead."
"Okay, Don't Let Me Down!"
Eleanor next called Lady Madonna: "Hello?"
"Goodbye. Don't Bother Me!"
"Oh well, I'll try Sgt. Pepper even though he's been feeling sick," Eleanor thought.
"Hi Sgt. Pepper, are you Getting Better? Will you come to my birthday party?"
"I Feel Fine, and I Will. I've Got a Feeling this will be really fun!"
When everyone arrived at Eleanor's, they started working on the problems together.

Are the following languages regular? Prove your result.

(15) (a) $P = \{a^p | p \text{ is prime}\}$
(Hint: Recall the pigeonhole principle and that there are infinitely many primes.)

Solution:

(10) (b) $D_k = \{n | k \text{ divides } n\}$ where the input n is in binary and k is a fixed natural number.

Solution:

5. Within You Without You

After a long day of working with regular languages to create new music, George was relaxing at home when he noticed the floor needed sweeping. The DFAs and NFAs they had been using were starting to get dusty, so he started thinking of a cleaner way to represent regular languages. He came up with Something, and everything seemed to get a little better (just as it does all the time). "Finally, Here Comes the Sun," he thought. George came up with the idea to use simple patterns to express languages without automata, and realized that his scheme still fit within the definition they had been using before.

A regular expression is a way of representing a set of strings. We define a regular expression inductively as follows:
- Any single string A is a regular expression representing the singleton set containing A.
- If A and B are regular expressions, then (A + B) is a regular expression representing the union of A and B.
- If A and B are regular expressions, then (AB) is a regular expression representing the concatenation of A and B - that is, a string in (AB) can be written as w_1w_2 such that w_1 is in A and w_2 is in B.
- If A is a regular expression, then (A)* is a regular expression that gives the Kleene Closure of the set described by A.

As the name implies, a language is regular if and only if it can be represented by a regular expression. Draw DFAs that accept the regular languages represented by the following regular expressions, over the alphabet $\{0, 1\}$. You do not need to prove correctness of your answer.

(10) (a) ((01) + (10))*

Solution:

(10) (b) ((01)* + (10)*) + ((1)* + (0)*)

Solution:

6. Bonus Question (Ob-La-Di Ob-La-Da)

It's been quite a Long and Winding Road, and the Beatles were starting to drift apart. However, they decided to Get Back and Come Together for one last concert on the roof before they would Let It Be and finally break up. They all took a Day Tripper on the Magical Mystery Tour (which was a Flying Yellow Submarine made of pure Norwegian Wood) and ended up Back In the USSR, where they'd play their last concert. Anyone who wanted to join them was given a Ticket To Ride, including the Taxman, Polythene Pam, and Michelle, For No One could be left out of this adventure. They came up with an interesting idea for their last concert, but weren't quite sure that it would work.

(10) (a) Let L be a regular language and define $\text{HelterSkelter}(L) = \{0^n | 0^n 1^n \in L\}$.

Prove that HelterSkelter(L) is regular. As in Q3, you should provide a formal construction supplemented with an intuitive argument rather than a formal proof for correctness.

(Hint: Let M be the DFA of L. To construct a DFA M' for HelterSkelter(L), note that if you could "guess" which state of M you'll end up in after reading the input 0^n, then you could re-read the input, run from that state, simulating 1^n (instead of 0^n) from there to see if it leads to an accept state. Now, instead of guessing, keep multiple copies simulated in parallel, so that in the end you can verify that "the correct copy" led to an accept state.)

Solution: