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Methods

In This Apportionment Lottery, the House Always Wins

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Abstract. Apportionment is the problem of distributing h indivisible seats across states in proportion to the states' populations. In the context of the U.S. House of Representatives, this problem has a rich history and is a prime example of interactions between mathematical analysis and political practice. Grimmett suggests to apportion seats in a randomized way such that each state receives exactly its proportional share q_i of seats in expectation (ex ante proportionality) and receives either $\lfloor q_i \rfloor$ or $\lceil q_i \rceil$ many seats ex post (quota). However, there is a vast space of randomized apportionment methods satisfying these two axioms, and so we additionally consider prominent axioms from the apportionment literature. Our main result is a randomized method satisfying quota, ex ante proportionality, and house monotonicity—a property that prevents paradoxes when the number of seats changes and that we require to hold ex post. This result is based on a generalization of dependent rounding on bipartite graphs, which we call cumulative rounding and which might be of independent interest as we demonstrate via applications beyond apportionment.

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1. Introduction

The Constitution of the United States says, "Representatives [in the U.S. House of Representatives] shall be apportioned among the several States according to their respective numbers, counting the whole number of persons in each State … "

These "respective numbers," or populations, of the states are determined every decade through the census. For example, on April 1, 2020, the population of the United States was 331,108,434, and the state of New York had a population of 20,215,751. New York, therefore, deserves 6.105% of the 435 seats in the House, which is 26.56 seats, for the next 10 years. The puzzle of apportionment is what to do about New York's 0.56 seat—in this round of apportionment, it was rounded down to zero, and New York lost its 27th seat.

The mathematical question of how to allocate these fractional seats has riveted the American political establishment since the country's founding (Szpiro 2010). In 1792, Congress approved a bill that would enact an apportionment method proposed by Alexander Hamilton, the first secretary of the treasury and star of the eponymous musical. If we denote the standard quota of state i by q_i ($q_i = 26.56$ in the case of New York in 2020), Hamilton's method allocates to each state its lower quota $\lfloor q_i \rfloor$ (26 for New York). Then, Hamilton's method goes through the states in order of decreasing residue $q_i - \lfloor q_i \rfloor$ (0.56 for New York) and allocates an additional seat to each state until all house seats are allocated.

As sensible as Hamilton's method appears, it repeatedly led to bizarre results, which became known as apportionment paradoxes.

- The Alabama paradox: Using the 1880 census results, the chief clerk of the Census Office calculated the apportionment according to Hamilton's method for all House sizes between 275 and 350 and discovered that, as the size increased from 299 to 300, Alabama lost a seat. In 1900, the Alabama paradox reappeared, this time affecting Colorado and Maine.
- The population paradox: In 1900, the populations of Virginia and Maine were 1,854,184 and 694,466, respectively. Over the following year, the populations of the two states grew by 19,767 and 4,649, respectively. Even though Virginia's growth was larger even relative

to its population, Hamilton's method would have transferred a seat from Virginia to Maine.

Occurrences of these paradoxes caused partisan strife, which is only natural given that a state's representatives have a strong personal stake in their state not losing seats. In both Congress and the courts, this strife took the form of a tug-of-war over the choice of apportionment method, the size of the House, and the census numbers, which was driven by the states', parties', and individual representatives' self-interest rather than the public good.

This state of affairs improved in 1941 when Congress adopted an apportionment method that provably avoids the Alabama and population paradoxes, which had been developed by Edward Huntington, a Harvard mathematician, and Joseph Hill, the chief statistician of the Census Bureau. Even though the Huntington–Hill method is house monotone (i.e., it avoids the Alabama paradox) and population monotone (i.e., it avoids the population paradox), it has a different, equally bizarre weakness: it does not satisfy quota; that is, the allocation of some states may be different from $\lfloor q_i \rfloor$ or $\lceil q_i \rceil$. A striking impossibility result by Balinski and Young (1982) shows that this tension is inevitable: no apportionment method can simultaneously satisfy quota and be population monotone. (We revisit this result in Section 3 and show that, whereas the theorem by Balinski and Young (1982) makes additional implicit assumptions, the incompatibility between quota and population monotonicity continues to hold without these assumptions.)

Beyond the troubling Balinski–Young impossibility, there is, in our view, an even larger source of unfairness that plagues apportionment methods, which is rooted in their determinism. In addition to introducing bias (the Huntington–Hill method disadvantages larger states), deterministic methods often lead to situations in which small counting errors can change the outcome. For example, based on the 2020 census, New York lost its 27th House seat, but it would have kept it had its population count been higher by 89 residents! Indeed, current projections suggest that New York would have kept its seat were it not for distortions in census response rates (Elliott et al. 2021). After the 1990 and 2000 censuses, similar circumstances were the basis for lawsuits brought by Massachusetts and Utah. A second shortcoming of deterministic apportionment methods is a lack of fairness over time: for example, if the states' populations remain static, a state with a standard quota of, say, 1.5 might receive a single seat in every single apportionment and, therefore, only receive $2/3$ of its deserved representation.

To address these issues, an obvious solution is to use randomization in order to realize the standard quota of each state in expectation as Grimmett (2004) proposes. If such a randomized method was used, 89 additional residents would have shifted New York's expected number of seats by a negligible 0.0001, and the decision between 26 or 27 seats would have been made by an impartial random process, which is less accessible to political maneuvering than, say, the census (Stone 2011).

Grimmett's (2004) proposed apportionment method is easy to describe. First, it chooses a random permutation of the states; without loss of generality, that permutation is identity. Second, it draws U uniformly at random from [0, 1] and lets $Q_i := U + \sum_{j=1}^{i} q_j$. Finally, it allocates to each state i one seat for each integer contained in the interval $[Q_{i-1}, Q_i)$. In particular, this implies that the allocation satisfies quota.

Why this particular method? Grimmett (2004, p. 302) writes, "We offer no justification for this scheme apart from fairness and ease of implementation." Grimmett's (2004) method is easy to implement for sure, and what he refers to as "fairness"—realizing the fractional quotas in expectation—is arguably a minimal requirement for any randomized apportionment method. But his two axioms, "fairness" and quota, allow for a vast number of randomized methods: indeed, after allocating $\lfloor q_i \rfloor$ seats to each agent, the problem of determining which states to round up reduces to so-called πps sampling (which stands for inclusion probability proportional to size) without replacement, and dozens of such schemes have been proposed in the literature (Brewer and Hanif 1983). We believe, therefore, that additional criteria are needed to guide the design of randomized apportionment methods. To identify such criteria, we return to the classics: house and population monotonicity.

1.1. Our Approach and Results

In this paper, we seek randomized apportionment methods that satisfy natural extensions of house and population monotonicity to the randomized setting. We want these monotonicity axioms to hold even ex post, that is, after the randomization has been realized. We find such methods by taking a parameterized class of deterministic methods, all of which satisfy the desired ex post axioms (in our case, subsets of population monotonicity, house monotonicity, and quota), and to then randomize over the choice of parameters such that ex ante properties hold (here, ex ante proportionality). In mechanism design, a similar approach extends strategyproofness to universal strategyproofness (Nisan and Ronen 2001).

Guaranteeing monotonicity axioms ex post is helpful for preventing certain kinds of manipulation in the apportionment process. For instance, say that the census concludes and a randomized apportionment is determined, and only then does a state credibly contest that its population was undercounted (in the courts or in Congress with the support of a majority). Using an apportionment method without population monotonicity, states might strategically undercount their population in the census and only reveal the true count in case this turns out to be beneficial once the randomness is revealed. When using a population monotone method, by contrast, any revised apportionment would be made using the same deterministic and population monotone method, which implies that immediately revealing the full population count is a dominant strategy.

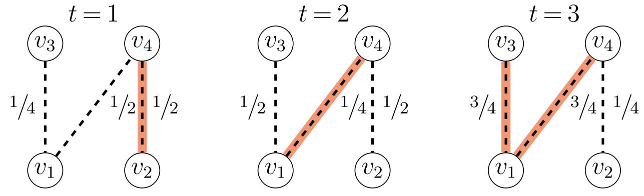
In Section 3, we first show that no such randomized methods exist for population monotonicity. This impossibility is not due to randomization or ex ante proportionality, but arises from the outright incompatibility of population monotonicity and quota. Thus, there do not exist suitable deterministic apportionment methods over which a randomized apportionment method could randomize. That population monotonicity and quota are incompatible is well-known from the Balinski–Young impossibility theorem (Balinski and Young 1982). But their proof uses seemingly mild background conditions that are not mild for our randomized purposes because, to provide ex ante proportionality, the randomized method must sometimes prioritize smaller states over larger states with positive probability, which is ruled out by those background conditions. We are able to prove a stronger version of their theorem, which derives the impossibility with no assumptions other than population monotonicity and quota. The deterministic apportionment methods that are most commonly used in practice (so called divisor methods, which include the Huntington–Hill method) satisfy population monotonicity but fail quota. So it makes sense to ask whether population monotonicity can be combined with ex ante proportionality (without requiring quota). We construct such a method, which is reminiscent of the family of divisor methods, except that the so-called division criterion (Balinski and Young 1982) is specific to each state and is given by a sequence of Poisson arrivals.

For house monotonicity, we provide in Section 4 a randomized apportionment method that satisfies house monotonicity, quota, and ex ante proportionality. To obtain this result, we generalize the classic result of Gandhi et al. (2006) on dependent rounding in a bipartite graph. We call this method cumulative dependent randomized rounding or cumulative rounding for short. Cumulative rounding allows to correlate dependent rounding processes in multiple copies of the same bipartite graph such that the result satisfies an additional guarantee across copies of the graph. This guarantee, which we describe in the next paragraph, generalizes the quota axiom of apportionment. As a side product, our existence proof for house monotonicity provides a new characterization of the deterministic apportionment methods satisfying house monotonicity and quota, which is based on the corner points of a bipartite matching polytope.

To describe cumulative rounding more precisely, we first sketch the result of Gandhi et al. (2006). For a bipartite graph (V, E) and edge weights $\{w_e\}_{e \in E}$ in $[0, 1]$, dependent rounding randomly generates a subgraph (V, E') with $E' \subseteq E$ providing three properties: marginal distribution (each edge $e \in E$ is contained in E' with probability w_e), degree preservation (in the rounded graph, the degree of a vertex v is the floor or the ceiling of v's fractional degree $\sum_{v \in e \in E} w_e$), and negative correlation. Cumulative rounding allows us to randomly round T many copies of (V, E), where each copy $1 \leq t \leq T$ has a set of weights $\{w_e^t\}_{e \in E}$. Each copy provides marginal distribution, degree preservation, and negative correlation. As we prove in Section 5, cumulative rounding additionally guarantees what we call cumulative degree preservation: for each vertex v and $1 \leq t \leq T$, the sum of degrees of v across copies 1 through t equals the sum of fractional degrees of v across copies 1 through t rounded either up or down. For example, node v_1 in Figure 1 is incident to edges with a total fractional weight of $2 \cdot 1/4 + 2 \cdot 1/2 = 1.5$ across copies $t = 1, 2$, and must, hence, be incident to one or two edges in total across the rounded versions of copies $t = 1, 2$. Because, across copies $t = 1, 2, 3$, v_1's total fractional degree is $1.5 + 2 \cdot 3/4 = 3$, v_1 must be incident to a total of exactly three rounded edges across the copies $t = 1, 2, 3$. By applying cumulative rounding to a star graph, we obtain a randomized apportionment method satisfying house monotonicity, quota, and ex ante proportionality.

We believe that cumulative rounding is of broader interest, and in Section 6, we present applications of cumulative rounding beyond apportionment. First, we consider a proposal of Buchstein and Hein (2009) for reforming the European Commission of the European Union: they propose a weighted lottery to determine which countries nominate commissioners. Using cumulative rounding to implement this lottery eliminates two key problems the authors identify in a simulation study, in particular, the possibility that some member states may not nominate any commissioners for a long time. We also describe how to apply cumulative rounding to round fractional allocations of goods or chores,

Figure 1. (Color online) Illustration of Cumulative Rounding



Notes. Dashed lines indicate edges $e \in E$ in the bipartite graph (V, E), which are labeled with weights w_e^t. The highlighted lines indicate a possible random outcome of cumulative rounding.

and we discuss a specific application of assigning faculty to teach courses.

1.2. Related Work

1.2.1. Apportionment Theory. We have already mentioned several seminal works of apportionment theory. Besides the axiomatic approach (e.g., Balinski and Young 1975, 1982; Balinski and Ramírez 1999, 2014; Palomares et al. 2024), which has arguably proved the most influential, deterministic apportionment has been extensively studied through the lens of constrained optimization (e.g., Huntington 1928, Burt and Harris 1963, Ernst 1994, Agnew 2008), with respect to bias against larger or smaller states (e.g., Pólya 1919, Balinski and Young 1982, Marshall et al. 2002, Lauwers and Van Puyenbroeck 2006, Janson 2014), and in multidimensional generalizations (e.g., Balinski and Demange 1989a, b; Maier et al. 2010; Cembrano et al. 2022; Mathieu and Verdugo 2022). For a comprehensive treatment of this theory, we refer the reader to Balinski and Young (1982) and Pukelsheim (2017).

1.2.2. Randomized Apportionment. A naïve form of randomized apportionment was suggested by Balinski (1993, p. 145), who immediately rejected it: "It is trivial to propose an unbiased method: assign the h seats at random with probabilities proportional to the fair shares. In this case none of the other desirable properties is guaranteed." The proposal by Grimmett (2004), which we discuss above, makes a much stronger case for randomized apportionment by showing one desirable property—quota—which can, in fact, be guaranteed ex post. Our work adds house and population monotonicity to the set of achievable properties.

Aziz et al. (2019) develop a random rounding scheme as part of a mechanism for strategyproof peer selection, which they simultaneously propose as a randomized apportionment method. As does Grimmett's (2004) method, their method satisfies ex ante proportionality and quota. The main advantage of their method is that its support consists of only linearly (not exponentially) many deterministic apportionments. This, Aziz et al. (2019) argue, is useful in repeated apportionment settings, in which one could repeat a periodic sequence of these deterministic apportionments and thereby limit the possibility of selecting the same state much too frequently or much too rarely because of random fluctuations. If this is the goal, cumulative rounding will arguably give better guarantees (see Section 6.1).

Hong et al. (2023) propose pipage rounding (Gandhi et al. 2006)—in this case, equivalent to pivotal sampling (Deville and Tille 1998)—as a randomized apportionment method without pursuing monotonicity.

Cembrano et al. (2024) propose a randomized apportionment scheme that circumvents the impossibility from Section 3.1 by allowing the house size to deviate ex post from its target. Their scheme satisfies ex ante proportionality, quota, and population monotonicity along with probabilistic bounds on how far the house size may deviate. Cembrano et al. (2024) also provide a conceptually simpler version of our characterization of house monotone and quota-compliant apportionment solutions in Theorem 6.

Evren and Khanna (2024) study a problem closely related to ours but inspired by affirmative action for faculty hiring in Indian universities. One can think of a house monotone and quota-compliant apportionment method as an iterative process that, in each time step $t = 1, 2, \ldots$, allocates the tth house seat to one of the states, ensuring that the total number of seats awarded to each state is proportional up to rounding. In the same way, Evren and Khanna (2024) successively allocate a university department's vacancies to demographic groups, and they also aim for quota and ex ante proportionality. Their algorithm is essentially equivalent to our randomized apportionment method; instead of randomly rounding a matching, they round a flow that appears similar to the one of Cembrano et al. (2024). By independently randomizing over hiring decisions in each of several departments, the authors immediately obtain concentration bounds, which imply that, in total across departments, demographic groups are likely almost proportionally represented.

Correa et al. (2024) study randomized apportionment to target—in addition to ex ante proportionality and quota—monotonicity axioms that are quite different from ours. The general flavor of these monotonicity axioms is to require that, if the standard quotas of some set S of states weakly increase and the standard quotas of all other states weakly decrease, the probability that the states in S simultaneously receive more seats should weakly increase; that is, they impose monotonicity on higher order correlations in the rounding. None of the apportionment methods they consider satisfy house monotonicity (Correa et al. 2024, appendix A).

1.2.3. Fair Division. Apportionment can be seen as a special case of the fair division of indivisible goods, which has recently received increased attention in operations research (e.g., Sandomirskiy and Segal-Halevi 2022, Aziz et al. 2023b, Benadè et al. 2023). The apportionment setting is characterized by the fact that the goods (i.e., the seats) are interchangeable and that the agents (i.e., the states) are weighted (in other words, have different entitlements) (e.g., Barbanel 1996, Aziz et al. 2020). Though the focus on interchangeable goods may appear very restrictive at first glance, Chakraborty et al. (2021) show that house monotone apportionment methods, when interpreted as picking sequences, induce allocation algorithms for the full setting of weighted fair division. In particular, an apportionment method satisfying house monotonicity and quota yields a fair division algorithm satisfying weighted proportionality up to one good (WPROP1) (Chakraborty et al. 2021, proposition 4.8). Hence, our randomized method in Section 4 can be seen as a randomized picking sequence algorithm that ensures WPROP1 ex post and ex ante satisfies that each agent i gets the tth pick (for each $t = 1, 2, \dots$) with probability proportional to i's weight. This latter property is not only an intuitively appealing fairness guarantee in its own right, but also immediately implies weighted proportionality (Barbanel 1996).

Our work is part of a larger thrust to develop allocation mechanisms that combine desirable ex ante and ex post guarantees, which have been termed best-of-both-worlds guarantees (Aziz et al. 2023b). The works of Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001) were early precursors to this idea in the setting of matchings in which the space of realizable ex ante probabilities has a very clean structure (Birkhoff 1946, von Neumann 1953). Budish et al. (2013) generalize this approach to more general combinatorial constraints. Aziz et al. (2023a) first study classic fair division axioms in this way, and Babaioff et al. (2022) and Feldman et al. (2023) extend this approach to additional fairness axioms and more general valuations. The corollary in the previous paragraph is a best-of-both-worlds fairness guarantee for weighted fair division; recently, Aziz et al. (2023a) and Hoefer et al. (2024) obtain such guarantees for this setting.

1.2.4. Randomly Rounding Bipartite Matchings. As a consequence of the Birkhoff–von Neumann theorem (Birkhoff 1946, von Neumann 1953), any fractional matching in a bipartite graph can be implemented as a lottery over integral matchings in the sense that each edge is present in the random matching with probability equal to its weight in the fractional matching. One algorithm for rounding a bipartite matching is pipage rounding (Ageev and Sviridenko 2004), which Gandhi et al. (2006) randomize in their dependent rounding technique. This rounding technique is powerful because it can directly accommodate fractional degrees larger than one and can provide negative correlation properties so that Chernoff concentration bounds apply (Panconesi and Srinivasan 1997). The technique of Gandhi et al. (2006) finds many applications in approximation algorithms (Kumar et al. 2009, Bansal et al. 2012) and in fair division (Saha and Srinivasan 2018, Akbarpour and Nikzad 2020, Cheng et al. 2020).

1.2.5. Just-in-Time Production. Steiner and Yeomans (1993) study a problem in just-in-time industrial manufacturing: how to alternate between the production of different types of goods in a way that produces each type in specified proportions. As pointed out by Bautista et al. (1996) and expanded upon by Balinski and Shahidi (1998), this problem is related to apportionment. In particular, a production schedule resembles a deterministic, house monotone apportionment method: as the available production time increases by one slot, the schedule needs to decide which type to produce in the next slot. Steiner and Yeomans (1993) end up with a property that nearly guarantees quota because they aim to minimize how far the prevalence of types among the goods produced so far deviates from the desired proportions. (Most of the literature on this just-in-time production problem minimizes other measures of deviation—see Kubiak (1993)—which are not connected to quota.) Now, Steiner and Yeomans (1993) only produce deterministic schedules, and the existence of deterministic house monotone and quota apportionment methods has long been known (Balinski and Young 1975, Still 1979). But we believe that the main construction in their proof could be randomized to obtain an alternative proof of Theorem 5 without, however, providing the generality of cumulative rounding. In fact, a similar graph construction to that by Steiner and Yeomans (1993) is randomly rounded within a proof by Gandhi et al. (2006) to obtain an approximation result about broadcast scheduling.

2. Model

Throughout this paper, fix a set of $n \geq 2$ states $N = \{1, 2, \dots, n\}$. For a given population profile $\vec{p} \in \mathbb{N}^n$, which assigns a population of $p_i \in \mathbb{N}$ to each state i, and for a house size $h \in \mathbb{N}$, an apportionment solution deterministically allocates to each state i a number $a_i \in \mathbb{Z}_{\geq 0}$ of house seats such that the total number of allocated seats is h. Formally, a solution is a function $f : \mathbb{N}^n \times \mathbb{N} \to \mathbb{Z}^n_{\geq 0}$ such that, for all $\vec{p}$ and h, $\sum_{i \in N} f_i(\vec{p}, h) = h$. For a population profile $\vec{p}$ and house size h, state i's standard quota is $q_i := (p_i / (\sum_{i \in N} p_i))\, h$.

Next, we define three axioms for solutions:

- Quota: A solution f satisfies quota if, for any $\vec{p}$ and h, it holds that $f_i(\vec{p}, h) \in \{\lfloor q_i \rfloor, \lceil q_i \rceil\}$ for all states i.
- House monotonicity: A solution f satisfies house monotonicity if, for any $\vec{p}$ and h, increasing the house size to $h+1$ does not reduce any state's seat number, that is, if $f_i(\vec{p}, h) \leq f_i(\vec{p}, h+1)$ for all $i \in N$.
- Population monotonicity: We say that a solution f, some $\vec{p}, \vec{p}\,' \in \mathbb{N}^n$, and some $h, h' \in \mathbb{N}$ exhibit a population paradox if there are two states $i \neq j$ such that $p'_i \geq p_i$, $p'_j \leq p_j$, $f_i(\vec{p}\,', h') < f_i(\vec{p}, h)$, and $f_j(\vec{p}\,', h') > f_j(\vec{p}, h)$ or, in words, if state i loses seats and j wins seats even though i's population weakly grew and j's population weakly shrunk. A solution f is population monotone if it exhibits no population paradoxes for any $\vec{p}, \vec{p}\,', h, h'$. By setting $\vec{p} = \vec{p}\,'$, one easily verifies that population monotonicity implies house monotonicity.

Note that the apportionment literature often considers two components of the quota axiom separately:

lower quota ("$f_i(\vec{\boldsymbol{p}}, h) \geq \lfloor q_i \rfloor$") and upper quota ("$f_i(\vec{\boldsymbol{p}}, h) \leq \lceil q_i \rceil$"). Also note that our definition of population monotonicity, taken from Robinson and Ullman (2010), is slightly weaker than the definition of other authors, whose violation we describe in the introduction. All results extend to this alternative notion of relative population monotonicity (Robinson and Ullman 2010): the proof of Theorem 1 immediately applies, and the proof of Theorem 2 is easy to adapt. Our results also continue to apply if one weakens population monotonicity by requiring that $h' = h$.

Finally, we define randomized apportionment methods. One potential definition, used by Grimmett (2004), is a function that, for each $\vec{\boldsymbol{p}}$ and h, specifies a probability distribution over seat allocations $(a_i)_{i \in N}$. For us, an apportionment method is instead a random process that determines an entire (deterministic) apportionment solution, that is, apportionments for all population profiles $\vec{\boldsymbol{p}}$ and house sizes h. The advantage of this definition is that it allows us to formulate axioms relating these different apportionments. We specify our apportionment methods by giving two components: first, a probability distribution for selecting some outcome ω from some suitable set Ω of possible outcomes and, second, the apportionment solution F^{ω} parameterized by ω. We refer to such an apportionment method as F, leaving the distribution over ω implicit. We treat an apportionment method F as a solution-valued random variable so that $F(\vec{\boldsymbol{p}}, h)$ refers to the method's random apportionment for $\vec{\boldsymbol{p}}, h$ and $F_i(\vec{\boldsymbol{p}}, h)$ refers to the random number of seats apportioned to state i. (We can ignore the measure theoretic complications of this statement as long as, for each $\vec{\boldsymbol{p}}, h$, the random apportionment $F^{\omega}(\vec{\boldsymbol{p}}, h)$ is a valid random variable, which is the case for all natural constructions following the two-component structure.) Our axioms, described in the next paragraph, constrain both the random behavior of F and the consistency of any solution F^{ω} in F's support across inputs.

A method F satisfies ex ante proportionality if, for any $\vec{\boldsymbol{p}}, h$ and for any state i, i's expected number of seats equals i's standard quota, that is, if $\mathbb{E}[F_i(\vec{\boldsymbol{p}}, h)] = q_i$, where the expected value is over the random choice of apportionment solution. A method F satisfies quota, house monotonicity, or population monotonicity if all solutions in the method's support satisfy the respective axiom. In this paper, we mainly search for apportionment methods that combine quota and ex ante proportionality—the two axioms obtained by Grimmett (2004)—with either population or house monotonicity.

3. Population Monotonicity

3.1. Population Monotonicity Is Incompatible with Quota

We begin by showing that no apportionment method satisfies population monotonicity, quota, and ex ante proportionality. In fact, quota and population monotonicity alone are incompatible: we show that no solution satisfies these two axioms. Because a method satisfying quota and population monotonicity would be a random choice over such solutions, no such method exists either.

At first glance, the incompatibility of quota and population monotonicity might seem to follow from existing results, but these results implicitly make assumptions that are not appropriate for randomized apportionment. Indeed, Balinski and Young (1982), who originally prove this incompatibility, as well as variations of their proof (Robinson and Ullman 2010, El-Helaly 2019) all assume what Robinson and Ullman (2010) call the order-preserving property; that is, if state i has a strictly larger population than state j, then i must receive at least as many seats as j. This property is usually proved as a consequence of neutrality together with population monotonicity.

The order-preserving property is reasonable for developing deterministic apportionment methods, but it is not desirable for the component solutions of a randomized apportionment method. This is clear for $h = 1$: the order-preserving property would mean that only the very largest state(s) can get a seat with positive probability; by contrast, the strength of randomization is that it allows us to allocate the seat to smaller states with some positive probability. To our knowledge, the existence of quota and population monotone solutions without the assumption of the order-preserving property was an open problem.

Theorem 1. *No (deterministic) apportionment solution satisfies population monotonicity and quota.*

Proof. Fix a set of five states, and let f be a solution satisfying quota.

We show that f must violate population monotonicity by analyzing three types of population profiles, which are given in Table 1, all for house size $h = 10$. The starting profile is $\vec{\boldsymbol{p}}^A$ in this table. By quota, state 1 must receive either eight or nine seats on this profile, but we show that either choice leads to a violation of population monotonicity: first, we show that allocating nine seats implies a violation of population monotonicity with respect to profile $\vec{\boldsymbol{p}}^B$; second, we show that allocating eight seats contradicts population monotonicity with respect to $\vec{\boldsymbol{p}}^C$.

Case 1: Allocating nine seats contradicts population monotonicity: Suppose that $f_1(\vec{\boldsymbol{p}}^A, 10) = 9$. Then, the remaining seat must be given to either state 2, 3, 4, or 5. Without loss of generality, we may assume that $f(\vec{\boldsymbol{p}}^A, 10) = (9, 0, 0, 0, 1)$.

Next, consider the profile $\vec{\boldsymbol{p}}^B$. Because quota prevents us from allocating more than seven seats to state 1 or more than two seats to state 5, at least one of the states 2, 3, and 4 must receive a seat on $\vec{\boldsymbol{p}}^B$. Thus, this

Table 1. Populations and Standard Quotas for Three Population Profiles with House Size $h = 10$ Used in Showing That Population Monotonicity and Quota Are Incompatible

	Profile $\vec{p}^A$		Profile $\vec{p}^B$		Profile $\vec{p}^C$	
State i	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

state's allocation strictly increases from its allocation of zero seats on $\vec{p}^A$ even though the state's population has not changed. Moreover, state 1 can receive at most seven seats on this profile by quota, which is strictly below the nine seats on $\vec{p}^A$, and state 1's population has also remained the same. But population monotonicity forbids there to be a pair of states with unchanged population such that one gains a seat and the other loses a seat. Hence, if state 1 receives nine seats on $\vec{p}^A$, then f violates population monotonicity.

Case 2: Allocating eight seats contradicts population monotonicity: Now, suppose that $f_1(\vec{p}^A, 10) = 8$. The remaining two seats must be given to two states out of 2, 3, 4, and 5; without loss of generality, we may assume that $f(\vec{p}^A, 10) = (8, 0, 0, 1, 1)$.

On profile $\vec{p}^C$, quota implies that state 1 receives at least nine seats—strictly more than the eight given on $\vec{p}^A$ even though the population has not changed. Given that there is at most one more seat to hand out, at least one state out of states 4 and 5 must receive zero seats on $\vec{p}^C$, which is a strict reduction with respect to $\vec{p}^A$ even though the state's population is the same. Thus, allocating eight seats to state 1 on $\vec{p}^A$ also leads to a violation of population monotonicity.

Because both possible choices for $f_1(\vec{p}^A, 10)$ imply a monotonicity violation, no solution can satisfy both quota and population monotonicity. □

The proof of Theorem 1 uses $n = 5$ states and can easily be generalized to any larger number of states. For $n = 3$ states, the Sainte-Laguë method (also known as Webster's method) satisfies both properties (Balinski and Young 1982, proposition 6.2). Whether impossibility holds for $n = 4$ remains an open question. The original impossibility result of Balinski and Young (1982) (which, in addition, assumes the order-preserving property) uses only $n = 4$ states.

3.2. A Population Monotone and Ex Ante Proportional (but Not Quota) Method

The incompatibility between population monotonicity and quota leaves open whether there are apportionment methods satisfying population monotonicity and ex ante proportionality. The answer is positive, and we now construct a method satisfying both axioms.

Because population monotonicity relates a solution's result for one input with those for infinitely many other inputs, it highly constrains the shape of population monotone solutions. In fact, under widely assumed regularity conditions, population monotone solutions are exactly characterized (Balinski and Young 1982) by the class of divisor methods (for consistency with our terminology, divisor solutions), which inspire our randomized method. A divisor solution is defined by a divisor criterion, which is a monotone increasing function $d : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$. (For instance, the Huntington–Hill solution is induced by $d(t) := \sqrt{t\,(t+1)}$.) For a population profile $\vec{p}$ and house size h, the divisor solution corresponding to d can be calculated by considering the sets $\{p_i/d(t) \mid t \in \mathbb{Z}_{\geq 0}\}$ for each state i, determining the h largest values across all sets, and allocating to each state i a number of seats equal to how many of the h largest values came from i's set.

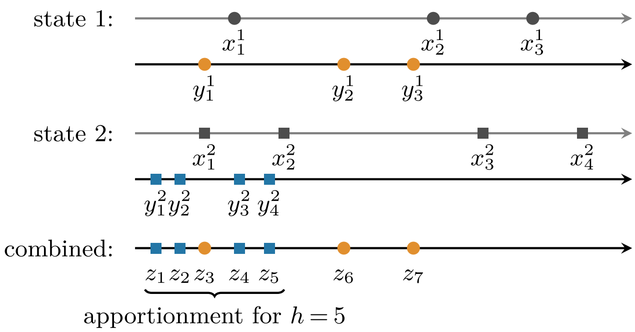
When state populations change, the values $p_i/d(t)$ evolve in a way that ensures population monotonicity: if state i grows and state j shrinks, a value belonging to state i might overtake a value belonging to j that was originally larger, but none of j's values can overtake any of i's values. As a result, i's number of seats cannot decrease if j's number of seats increases.

To avoid the order-preserving property (for reasons described in Section 3.1), we (randomly) choose a different divisor criterion for each state, which leaves the above argument for population monotonicity intact. The question is how to sample these divisor criteria such that ex ante proportionality holds. The answer lies in the properties of Poisson arrival processes: across independent, scaled Poisson processes, which process yields the first arrival is distributed with probabilities that are proportional to the reciprocals of the scaling factors, and because the interarrival times of the processes are memoryless, the same holds for each subsequent arrival. These properties allow us to randomly construct a generalized divisor solution such that the overall distribution satisfies population monotonicity and ex ante proportionality.

Theorem 2. *There exists an apportionment method F that satisfies population monotonicity and ex ante proportionality.*

Proof. Which solution is randomly chosen by the method depends on the values taken on by n independent Poisson arrival processes with rate one, which we define as our outcome ω. We now construct the solution F^ω corresponding to any given ω. For each state i, ω determines an infinite sequence $0 < x_1^i < x_2^i < \cdots$ of arrival times. We describe the apportionment given by F^ω on input $\vec{p}$ and h, which we illustrate in Figure 2: First, we divide each arrival time x_t^i by the corresponding state's population; that is, we

Figure 2. (Color online) Illustration of the Population Monotone Method in Theorem 2



set $y_t^i := x_t^i/p_i$. Second, we combine the y_t^i for all t and i in a single arrival sequence $(z_1, i_1), (z_2, i_2), \dots$ labeled with states; that is, each (z_j, i_j) corresponds to some arrival y_t^i for some i and t such that $z_j = y_t^i$ is the arrival time, $i_j = i$ is the agent label, and the z_j are sorted in increasing order. Third, we allocate $|\{1 \le j \le h | i_j = i\}|$ many seats to each state i, that is, a number of seats equal to how many among the h smallest scaled arrival times belonged to i's arrival process. This specifies the solution F^ω and, hence, the entire apportionment method F. Note that F^ω closely resembles a divisor solution, in which state i's set is $\{1/y_t^i | t \in \mathbb{Z}_{\ge 0}\} = \{p_i/x_t^i | t \in \mathbb{Z}_{\ge 0}\}$, that is, in which, for each state i, $t \longmapsto x_t^i$ plays the role of a state-specific divisor criterion.

First, we show that F satisfies ex ante proportionality. For this, fix some $\vec{\boldsymbol{p}}$ and h. Then, the $\{y_t^i\}_{t \ge 1}$ for each i are distributed as the arrival sequences of independent Poisson processes, in which i's arrival process has a rate of p_i. As stated by the coloring theorem for Poisson processes (Kingman 1993, p. 53), our labeled arrival sequence (z_j, i_j) has the same distribution as if we had sampled a Poisson arrival process $0 < z_1 < z_2 < \cdots$ with arrival rate $\sum_{i \in N} p_i$ and had drawn each i_j independently, choosing each $i \in N$ with probability proportional to p_i. Because the z_j and i_j are independent in this way, $F(\vec{\boldsymbol{p}}, h)$ is distributed as if sampling h states with probability proportional to the states' populations and with replacement. In particular, this implies ex ante proportionality.

It remains to show that F satisfies population monotonicity. Fix an ω, that is, the x_t^i. Consider two inputs $\vec{\boldsymbol{p}}, h$ and $\vec{\boldsymbol{p}}', h'$, for which we show that F^ω does not exhibit a population paradox. Denoting the inputs' respective variables by y_t^i, z_j and $y_t^{i\prime}, z_j'$, it is easy to see that, for all i for which $p_i' \ge p_i$, $y_t^{i\prime} \le y_t^i$ for all t, and that, for all i for which $p_i' \le p_i$, $y_t^{i\prime} \ge y_t^i$ for all t. Observe that each state i receives a number of seats equal to the number of its scaled arrival times y_t^i (respectively, $y_t^{i\prime}$) that are at most z_h (z_h').

Suppose that $z_h' \ge z_h$ (the reasoning for the case $z_h' \le z_h$ is symmetric). Then, whenever $y_t^i \le z_h$ for a state i for which $p_i' \ge p_i$, then $y_t^{i\prime} \le y_t^i \le z_h \le z_h'$, which shows that i's seat number must weakly increase. One verifies that this rules out a population paradox on $\vec{\boldsymbol{p}}, h$ and $\vec{\boldsymbol{p}}', h'$. Together with the symmetric argument for $z_h' \le z_h$, this establishes population monotonicity. □

Clearly, the solutions' resemblance to divisor solutions enables our proof of population monotonicity. At the same time, using different divisor criteria for different states allows us to avoid the order-preserving property, which would have prevented ex ante proportionality as described in Section 3.1. Less satisfying is that, whereas classic divisor criteria satisfy bounds such as $t \le d(t) \le t + 1$, the divisor criteria used in the last theorem do not satisfy any such bounds. As a result, solutions are likely to substantially deviate from proportionality ex post. An interesting question for future work is whether Theorem 2 can be strengthened to additionally satisfy lower quota or upper quota.

4. House Monotonicity

Whereas we cannot obtain population monotonicity without giving up on quota, we now propose an apportionment method that combines house monotonicity with quota and ex ante proportionality.

4.1. Examples of Pitfalls

An intuitive strategy for constructing a house monotone randomized apportionment method is to do it inductively, seat by seat. Thus, we would need a strategy for extending a method that works for all house sizes $h' \le h$ to a method that also works for house size $h + 1$. In this section, we give examples suggesting that this does not work by showing that some reasonable methods cannot be extended without violating quota or ex ante proportionality. This motivates a search for a more global strategy for constructing a house-monotone method.

Example 1. Our first example shows that there are apportionments for a given h that satisfy quota but that are "toxic" in that they can never be chosen by a house monotone solution that satisfies quota. Suppose that we have four states with populations $\vec{\boldsymbol{p}} = (1, 2, 1, 2)$. The distribution that we consider is the one given by Grimmett's (2004) method (as described in the introduction) for these inputs. Let $h = 2$. Observe that, if the random permutation chosen by Grimmett's (2004) method is identity and if, furthermore, $U > 2/3$, then Grimmett's (2004) method returns the allocation $(1, 0, 1, 0)$. But we show that no solution f such that $f(\vec{\boldsymbol{p}}, 2) = (1, 0, 1, 0)$ can satisfy house monotonicity and quota. Indeed, if f is house monotone, then at least one out of state 2 or state 4 must still be given zero seats by f when $h = 3$, but quota requires that both states receive exactly one seat when $h = 3$. It follows that Grimmett's (2004) method, the apportionment method of Aziz et al. (2019), or any other method satisfying quota and whose support contains solutions f with $f(\vec{\boldsymbol{p}},2)=(1,0,1,0)$ cannot be house monotone.

Thus, a first challenge that any quota and house monotone method must overcome is to never produce a toxic apportionment for a specific h that cannot be extended to all larger house sizes in a house monotone and quota-compliant way. Still (1979) and, later, Balinski and Young (1979) give a characterization of nontoxic apportionments, but we found no way of transforming this characterization into an apportionment method that would be ex ante proportional.

Example 2. Our second example shows that, even if there are no toxic apportionments in the support of a distribution, the wrong distribution over apportionments might still lead to violations of one of the axioms. Let there be four states with populations $\vec{\boldsymbol{p}}=(45,25,15,15)$ and let $h=3$; thus, the standard quotas are $(1.35,0.75,0.45,0.45)$. We consider the following distribution over allocations:

$$\vec{\boldsymbol{a}}=\begin{cases}(2,1,0,0) & \text{with probability 35\%,}\\(1,1,0,1) & \text{with probability 20\%,}\\(1,1,1,0) & \text{with probability 20\%,}\\(1,0,1,1) & \text{with probability 25\%.}\end{cases}$$

As we show in Appendix A, none of these allocations is toxic, and the distribution can be part of an apportionment method in which all three axioms hold for $\vec{\boldsymbol{p}}$ and all $h'\leq 3$. Nevertheless, we show in the following that any apportionment method F that satisfies house monotonicity and quota and that has the above distribution for $F(\vec{\boldsymbol{p}},3)$ must violate ex ante proportionality. Indeed, fix such an F. On the one hand, note that, for $h=4$, state 2's standard quota is $4\cdot 25/100=1$, so any quota apportionment must give the state 1 seat. Because any solution f in the support of F satisfies house monotonicity and quota by assumption, any f such that $f(\vec{\boldsymbol{p}},3)=(1,0,1,1)$ must satisfy $f(\vec{\boldsymbol{p}},4)=(1,1,1,1)$. Thus, with at least 25% probability, $F_1(\vec{\boldsymbol{p}},4)=1$. On the other hand, because state 1's standard quota for $h=4$ is $1.8\leq 2$, $F_1(\vec{\boldsymbol{p}},4)\leq 2$ holds deterministically by quota. It follows that $\mathbb{E}[F_1(\vec{\boldsymbol{p}},4)]\leq 25\%\cdot 1+75\%\cdot 2=1.75<1.8$, which means that F must violate ex ante proportionality as claimed. To avoid this kind of conflict between house monotonicity, ex ante proportionality, and quota, the distribution of $F(\vec{\boldsymbol{p}},3)$ must allocate at least 5% combined probability to the allocations $(2,0,1,0)$ and $(2,0,0,1)$, which to us is not obvious other than by considering the specific implications on $h=4$ as above.

4.2. Cumulative Rounding

The examples of the last section show that it is difficult to construct house monotone apportionment methods seat by seat. In this section, we develop an approach that is able to explicitly take into account how rounding decisions constrain each other across house sizes. Our approach is based on dependent randomized rounding in a bipartite graph that we construct. First, we state the main theorem by Gandhi et al. (2006).

Theorem 3 (Gandhi et al. 2006). *Let $(A\cup B,E)$ be an undirected bipartite graph with bipartition (A,B). Each edge $e\in E$ is labeled with a weight $w_e\in[0,1]$. For each $v\in A\cup B$, we denote the fractional degree of v by $d_v:=\sum_{v\in e\in E}w_e$.*

Then, there is a random process, running in $\mathcal{O}((|A|+|B|)\cdot|E|)$ time, that defines random variables $X_e\in\{0,1\}$ for all $e\in E$ such that the following properties hold:

- *Marginal distribution:*
  For all $e\in E$, $\mathbb{E}[X_e]=w_e$.
- *Degree preservation:*
  For all $v\in A\cup B$, $\sum_{v\in e\in E}X_e\in\{\lfloor d_v\rfloor,\lceil d_v\rceil\}$.
- *Negative correlation:*
  For all $v\in A\cup B$ and $S\subseteq\{e\in E\mid v\in e\}$, $\mathbb{P}[\bigwedge_{e\in S}X_e=1]\leq\prod_{e\in S}w_e$ and $\mathbb{P}[\bigwedge_{e\in S}X_e=0]\leq\prod_{e\in S}(1-w_e)$.

If $X_e=1$ for an edge e, we say that e gets rounded up, and if $X_e=0$, then e gets rounded down. We do not use negative correlation in our apportionment results, but it is crucial in many applications of dependent rounding because it implies that linear combinations of the shape $\sum_{e\in S}a_e X_e$ for some $a_e\in[0,1]$ obey Chernoff concentration bounds (Panconesi and Srinivasan 1997).

To see the connection to apportionment, let $\vec{\boldsymbol{p}}$ be a population profile. Then, to warm up, the problem of apportioning a single seat can be easily cast as dependent rounding in a bipartite graph: indeed, let A consist of a single special node a and let B contain a node b_i for each state i. We draw an edge $e=\{a,b_i\}$ with weight $w_e=p_i/\sum_{j\in N}p_j$ for each state i. Apply dependent rounding to this star graph. Then, a's fractional degree of exactly one means that, by degree preservation, exactly one edge $\{a,b_i\}$ gets rounded up, which we interpret as the seat being allocated to state i. Moreover, marginal distribution ensures that each state receives the seat with probability proportional to its population. This shows that randomized rounding can naturally express ex ante proportionality, which becomes a useful building block in the following.

Next, we expand our construction to multiple house seats and to satisfying house monotonicity across different house sizes. The most natural way is to duplicate the star graph from the last paragraph once per house size $h=1,2,\ldots$ with nodes a^h, $\{b_i^h\}_{i\in N}$ and edges $\{\{a^h,b_i^h\}\}_{i\in N}$. (In this intuitive exposition, we do not consider any explicit upper bound on the house sizes. Our formal result in Theorem 5 rounds a finite graph but this turns out to suffice for obtaining house monotonicity for all house sizes $h\in\mathbb{N}$.) If $\{a^h,b_i^h\}$ gets rounded up in the hth copy of the star graph, we interpret this as the hth seat going to state i. In other words, we determine how many seats get apportioned to state i for a house size h by counting how many edges $\{a^{h'}, b_i^{h'}\}$ got rounded up across all $h' \leq h$. This construction automatically satisfies house monotonicity and satisfies ex ante proportionality by the marginal distribution property but may violate quota by arbitrary amounts.

To explain how randomized rounding might be useful for guaranteeing quota, let us give a few details on how the pipage rounding procedure (Gandhi et al. 2006) randomly rounds a bipartite graph. In each step, pipage rounding selects either a cycle or a maximal path consisting of edges with fractional weights in $(0,1)$. The edges along this cycle or path are then alternatingly labeled even or odd, which is possible because each cycle in a bipartite graph has an even number of edges. Depending on a biased coinflip and appropriate numbers $\alpha, \beta > 0$, the algorithm either (1) increases all odd edge weights by α and decreases all even edge weights by α or (2) decreases all odd edge weights by β and increases all even edge weights by β. In this process, more and more edge weights become zero or one, which determines the X_e once no fractional edges remain.

The cycle/path rounding steps in pipage rounding represent an opportunity to couple the seat-allocation decisions across h in a way that ultimately allows us to guarantee quota. In our current graph consisting of disjoint stars, there are no cycles, and the maximal paths are always pairs of edges $\{a^h, b_i^h\}, \{a^h, b_j^h\}$ for two states i, j and some h. Thus, pipage rounding correctly anticorrelates the decision of giving the hth seat to state i and the decision of giving the hth seat to state j, but decisions for different seats remain independent. To guarantee quota, increasing (respectively, decreasing) the probability of the hth seat going to state i should also decrease (increase) the probability of some nearby seats h' going to state i and increase (decrease) the probability of seats h' going to some other state j. The difficulty is to choose these h' and j to provide quota, which is particular tricky because, in the course of running pipage rounding, some of the edge weights are rounded to zero and one and no longer available for paths or cycles.

Not only are we able to use pipage rounding to guarantee quota, but we do so through a general construction that adds quota-like guarantees to an arbitrary instance of repeated randomized rounding; we refer to this technique as cumulative rounding. In the following statement, the "time steps" t take the place of our possible house sizes h.

Theorem 4. *Let $(A \cup B, E)$ be an undirected bipartite graph. For each time step $t = 1, \ldots, T$, consider a set of edge weights $\{w_e^t\}_{e \in E}$ in [0, 1] for this bipartite graph. For each $v \in A \cup B$ and $1 \leq t \leq T$, we denote the fractional degree of v at time t by $d_v^t := \sum_{v \in e \in E} w_e^t$.*

Then, there is a random process, running in $\mathcal{O}(T^2 \cdot (|A| + |B|) \cdot |E|)$ time, that defines random variables $X_e^t \in \{0,1\}$ for all $e \in E$ and $1 \leq t \leq T$, such that the following properties hold for all $1 \leq t \leq T$. Let $D_v^t := \sum_{v \in e \in E} X_e^t$ denote the random degree of v at time t.

- *Marginal distribution:*
  For all $e \in E$, $\mathbb{E}[X_e^t] = w_e^t$.
- *Degree preservation:*
  For all $v \in A \cup B$, $D_v^t \in \{\lfloor d_v^t \rfloor, \lceil d_v^t \rceil\}$.
- *Negative correlation:*
  For all $v \in A \cup B$ and $S \subseteq \{e \in E \mid v \in e\}$, $\mathbb{P}[\bigwedge_{e \in S} X_e^t = 1] \leq \prod_{e \in S} w_e^t$ and $\mathbb{P}[\bigwedge_{e \in S} X_e^t = 0] \leq \prod_{e \in S}(1 - w_e^t)$.
- *Cumulative degree preservation:*
  For $v \in A \cup B$, $\sum_{t'=1}^{t} D_v^{t'} \in \{\lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor, \lceil \sum_{t'=1}^{t} d_v^{t'} \rceil\}$.

The first three properties can be achieved by simply applying Theorem 3 in each time step independently. Cumulative rounding correlates these rounding processes such that cumulative degree preservation (a generalization of quota) is additionally satisfied.

4.3. House Monotone, Quota-Compliant, and Ex Ante Proportional Apportionment

Before we prove Theorem 4, we explain how cumulative rounding can be used to construct an apportionment method that is house monotone and satisfies quota and ex ante proportionality.

None of these three axioms connects the outcomes at different population profiles $\vec{p}$, and so it suffices to consider them independently. Thus, let us fix a population profile $\vec{p}$. Denote the total population by $p := \sum_{i \in N} p_i$. The behavior of a house monotone solution on inputs with profile $\vec{p}$ and arbitrary house sizes can be expressed through what we call an infinite seat sequence, an infinite sequence $\alpha = \alpha_1, \alpha_2, \ldots$ over the states N. We also define finite seat sequences, which are sequences $\alpha = \alpha_1, \ldots, \alpha_p$ of length p over the states. Either sequence represents that, for any house size h (in the case of a finite seat sequence: $h \leq p$), the sequence apportions $a_i(h) := |\{1 \leq h' \leq h \mid \alpha_{h'} = i\}|$ seats to each state i. We can naturally express the quota axiom for seat sequences: α satisfies quota if, for all h ($h \leq p$ if α is finite) and all states i, we have $a_i(h) \in \{\lfloor h p_i / p \rfloor, \lceil h p_i / p \rceil\}$.

The main obstacle in obtaining a house monotone method via cumulative rounding is that we can only apply cumulative rounding to a finite number T of copies, whereas the quota axiom must hold for all house sizes $h \in \mathbb{N}$. However, it turns out that, for our purposes of satisfying quota, we can treat the allocation of seats $1, 2, \ldots, p$ separately from the allocation of seats $p+1, \ldots, 2p$, the allocation of seats $2p+1, \ldots 3p$, and so forth. The reason is that, when h is a multiple kp of p (for some $k \in \mathbb{N}$), each state i's standard quota is an integer kp_i. Thus, any solution that satisfies quota is forced to choose exactly the allocation $(kp_1, \ldots, kp_n)$ for house size h. At this point, the constraints for satisfying quota and house monotonicity reset to what they were at $h = 1$. We make this precise in the following lemma, proved in Appendix B.

Lemma 1. *An infinite seat sequence α satisfies quota iff it is the concatenation of infinitely many finite seat sequences $\beta^1, \beta^2, \beta^3, \ldots$ of length p each satisfying quota, that is,*

$$\alpha = \beta_1^1, \beta_2^1, \ldots, \beta_p^1, \beta_1^2, \beta_2^2, \ldots, \beta_p^2, \beta_1^3, \ldots.$$

This lemma allows us to apply cumulative rounding to only $T = p$ many copies of a star graph. Then, cumulative rounding produces a random matching that encodes a finite seat sequence satisfying quota, and Lemma 1 shows that the infinite repetition of this finite sequence describes an infinite seat sequence satisfying quota. This implies the existence of an apportionment method satisfying all three axioms for which we aimed. The formal proof is in Appendix B.

Theorem 5. *There exists an apportionment method F that satisfies house monotonicity, quota, and ex ante proportionality.*

4.3.1. Implications for Deterministic Methods. Our construction also increases our understanding of deterministic apportionment solutions satisfying house monotonicity and quota: indeed, the possible roundings of the bipartite graph constructed for cumulative rounding (plus some minor modifications described in the proof) turn out to correspond one to one to the finite seat sequences satisfying quota. Together with Lemma 1, this gives a characterization of all seat sequences that satisfy quota, providing a graph-theoretic alternative to the characterizations by Still (1979) and Balinski and Young (1979).

Theorem 6. *For any population profile $\vec{p}$, we can construct a bipartite graph whose perfect matchings are in one-to-one correspondence with the finite seat sequences satisfying quota.*

The proof is deferred to Appendix B. Combined with Lemma 1, this characterizes the set of infinite seat sequences satisfying quota and, thus, the apportionment solutions satisfying house monotonicity and quota.

A qualitative difference from the previous characterizations (Balinski and Young 1979, Still 1979) of apportionment solutions satisfying house monotonicity and quota is that the matchings allow for a *geometric* description as the corner points of the bipartite graph's matching polytope. Because a fractional matching assigning each edge $\{a, b_i\}$ a weight of $p_i/p > 0$ lies in the interior of this polytope of perfect fractional matchings, one immediate consequence of this characterization (equivalently, of ex ante proportionality in Theorem 5) is that, for each state i and $h \in \mathbb{N}$, there is a house monotone and quota-compliant solution that assigns the hth seat to i. To our knowledge, this result is not obvious based on the earlier characterizations. More generally, the polytope characterization might be useful in answering questions such as "For a set of pairs $(h_1, i_1), (h_2, i_2), \ldots, (h_t, i_t)$, is there a population-monotone and quota-compliant solution that assigns the h_jth seat to state i_j for all $1 \leq j \leq t$?" To answer this question, one can remove the nodes a^{h_j} and $b_{i_j}^{h_j}$ from the graph (simulating that they got matched) and check whether the remaining graph still permits a perfect matching, for example using Hall's (1935) marriage theorem. Finally, our formulation allows to optimize linear objectives over the space of solutions satisfying quota and house monotonicity. For example, such an optimization formulation might highlight natural quota-compliant and house monotone solutions other than the one by Balinski and Young (1975).

4.3.2. Computation. Before we prove the cumulative rounding result in Section 5, let us quickly discuss computational considerations of our house monotone apportionment method. Though it is possible to run dependent rounding on the constructed graph (for a given population profile $\vec{p}$), the running time would scale in $\mathcal{O}(p^2 n^2)$, and the quadratic dependence on the total population p might be prohibitive. In practice, we see two ways to avoid this computational cost.

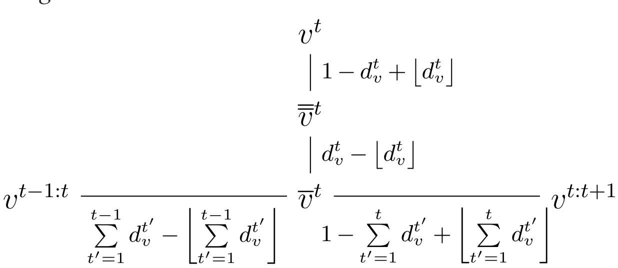
First, one might often not require a solution that is house monotone on all possible house sizes $h \in \mathbb{N}$; instead, it might suffice to rule out Alabama paradoxes for house sizes up to an upper bound h_{max}. In this case, it suffices to apply cumulative rounding on h_{max} many copies of the graph, leading to a much more manageable running time of $\mathcal{O}(h_{max}^2 n^2)$.

A second option would be to apply cumulative rounding on all p copies of the graph but to stop pipage rounding once all edge weights in the first h copies of the graph are integral even if edge weights for higher house sizes are still fractional. This allows to return an apportionment for inputs $\vec{p}, h$, without randomly determining a single house monotone solution. Instead, this process determines a conditioned distribution F^c over house monotone solutions, all of which agree on the apportionment for $\vec{p}$ and h. Because all solutions are house monotone, the expected number of seats for a party always monotonically increases in h across the conditioned distribution. Should it become necessary to determine apportionments for larger house sizes, one can simply continue the cumulative-rounding process where it left off. Because the pipage rounding used to prove Theorem 4 leaves open which cycles or maximal paths get rounded next, it seems likely that one can deliberately choose cycles/paths such that the apportionment for the first h seats is determined in few rounds.

5. Proof of Cumulative Rounding

We now prove Theorem 4 on cumulative rounding. Our proof constructs a weighted bipartite graph including T many copies of $(A \cup B, E)$, connected by appropriate additional edges and nodes and then applying dependent rounding to this constructed graph. The additional edges and vertices ensure that, if too many edges adjacent to some node v are rounded up in one copy of the graph, then this is counterbalanced by rounding down edges adjacent to v in another copy.

Construction 7. *Let $(A \cup B, E)$, T, and $\{w_e^t\}_{e,t}$ be given as in Theorem 4. We construct a new weighted, undirected, and bipartite graph as follows: For each node $v \in A \cup B$ and for each $t = 1, \dots, T$, create four nodes v^t, $\overline{v}^t$, $\overline{\overline{v}}^t$, and $v^{t:t+1}$; furthermore, create a node $v^{0:1}$ for each node v. For each $\{a, b\} \in E$ and $t = 1, \dots, T$, connect the nodes a^t and b^t with an edge of weight $w_{\{a,b\}}^t$. Additionally, for each node $v \in A \cup B$ and each $t = 1, \dots, T$, insert edges with the following weights.*



Before we go into the proof, we give in Figure 3 an interpretation for what it means for each edge in the constructed graph to be rounded up. One can easily verify that, under the (premature) assumption that cumulative rounding satisfies marginal distribution, degree preservation, and cumulative degree preservation, the edge weights coincide with the probabilities of each interpretation's event. We want to stress that it is not obvious that these descriptions are indeed consistent for any dependent rounding of the constructed graph, and we do not make use of these descriptions in the proof of Theorem 4. Instead, the characterizations follow from intermediate results in the proof. We give these interpretations here to make the construction seem less mysterious. We begin the formal analysis of the construction with a sequence of simple observations about the constructed graph (proofs are in Appendix C).

Lemma 2. *The graph of Construction 7 is bipartite.*

Lemma 3. *All edge weights lie between zero and one.*

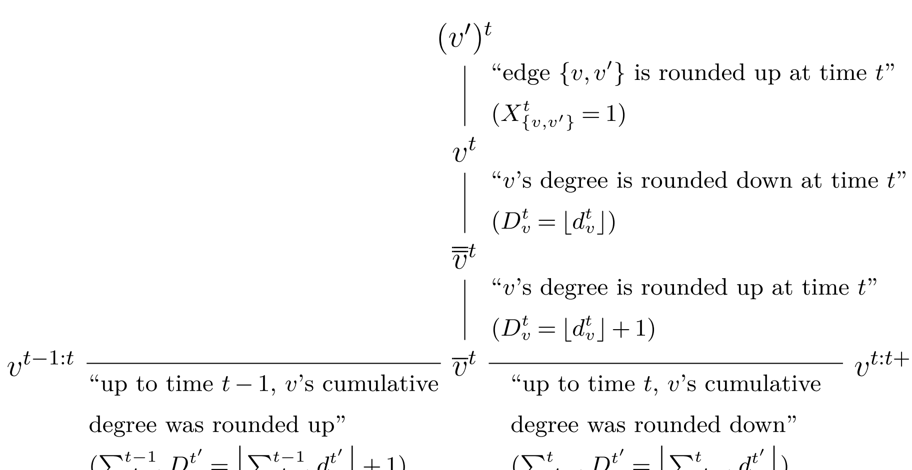
Lemma 4. *Let $1 \le t \le T$. The node v^t has fractional degree $\lfloor d_v^t \rfloor + 1$, the node $\overline{v}^t$ has fractional degree $\lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor - \lfloor \sum_{t'=1}^{t-1} d_v^{t'} \rfloor - \lfloor d_v^t \rfloor + 1$, and $\overline{\overline{v}}^t$ has fractional degree one.*

Further, the node $v^{0:1}$ has fractional degree zero, and the nodes $v^{1:2}, \dots, v^{t-1:t}$ have fractional degree one.

Proof of Theorem 4. We define cumulative rounding as the random process that follows construction 7 and then applies dependent rounding (Theorem 3) to the constructed graph, which is valid because the graph is bipartite and all edge weights lie in [0, 1] (Lemmas 2 and 3). For an edge e in the constructed graph, let $\hat{X}_e$ be the random variable indicating whether dependent rounding rounds it up or down. For any edge $\{a, b\} \in E$ in the underlying graph and some $1 \le t \le T$, we define the random variable $X_{\{a,b\}}^t$ to be equal to $\hat{X}_{\{a^t, b^t\}}$. Recall that we defined $D_v^t = \sum_{v \in e \in E} X_e^t$.

To prove the theorem, we have to bound the running time of this process and provide the four guaranteed

Figure 3. Interpretation of Each Edge Being Rounded up in the Constructed Graph for Arbitrary Nodes $v, v' \in A \cup B$ and $1 \le t \le T$



Note. The correctness of this characterization is shown along the proof of Theorem 4, specifically in the sections on degree preservation and cumulative degree preservation.

properties: marginal distribution, degree preservation, negative correlation, and cumulative degree preservation. The last property takes by far the most work.

Running Time. Without loss of generality, we may assume that each vertex $v \in A \cup B$ is incident to at least one edge because, otherwise, we could remove this vertex in a preprocessing step. From this, it follows that $|E| \in \Omega(|A| + |B|)$. Constructing the graph takes $\mathcal{O}(T|E|)$ time, which is dominated by the time required for running dependent rounding on the constructed graph. The constructed graph has $(1 + 4T)(|A| + |B|) \in \mathcal{O}(T(|A| + |B|))$ nodes and $T|E| + 4T(|A| + |B|) \in \mathcal{O}(T|E|)$ edges. Because the running time of dependent rounding scales in the product of the number of vertices and the number of edges, our procedure runs in $\mathcal{O}(T^2(|A| + |B|)|E|)$ time, as claimed.

Marginal Distribution. For an edge $\{a,b\} \in E$ and $1 \le t \le T$, $\mathbb{E}[X^t_{\{a,b\}}] = \mathbb{E}[\hat{X}_{\{a^t,b^t\}}] = w^t_{\{a,b\}}$, where the last equality follows from the marginal distribution property of dependent rounding.

Degree Preservation. Fix a node $v \in A \cup B$ and $1 \le t \le T$. By Lemma 4, the fractional degree of v^t is $\lfloor d^t_v \rfloor + 1$, and thus, by degree preservation of dependent rounding, exactly $\lfloor d^t_v \rfloor + 1$ edges adjacent to v^t must be rounded up. The only of these edges that does not count into D^t_v is $\{\overline{\overline{v}}^t, v^t\}$; depending on whether this edge is rounded up or down, D^t_v is either $\lfloor d^t_v \rfloor$ or $\lfloor d^t_v \rfloor + 1$. If d^t_v is not integer, the latter number equals $\lceil d^t_v \rceil$, which proves degree preservation. Else, if d^t_v is an integer, the edge weight of $\{\overline{\overline{v}}^t, v^t\}$ is one. Dependent rounding always rounds up edges with weight one, which means that D^t_v is definitely $\lfloor d^t_v \rfloor$ in this case. Thus, degree preservation holds in either case.

Negative Correlation. Negative correlation for $v \in A \cup B$, $S \subseteq \{e \in E | v \in e\}$, and $1 \le t \le T$ directly follows from the negative correlation property of dependent rounding for the node v^t and the edge set $S' := \{\{v^t, (v')^t\} | \{v, v'\} \in S\}$.

Cumulative Degree Preservation. Fix a node $v \in A \cup B$ and $1 \le t \le T$. We consider the rounded version of the constructed graph, that is, the unweighted bipartite graph over the nodes of the constructed graph in which exactly those edges are present that got rounded up by the randomized rounding process. We define five sets of nodes in the rounded graph (Figure 4):

$$V := \{v^{t'} | 1 \le t' \le t\}$$

$$\overline{V} := \{\overline{v}^{t'} | 1 \le t' \le t\}$$

$$\overline{\overline{V}} := \{\overline{\overline{v}}^{t'} | 1 \le t' \le t\}$$

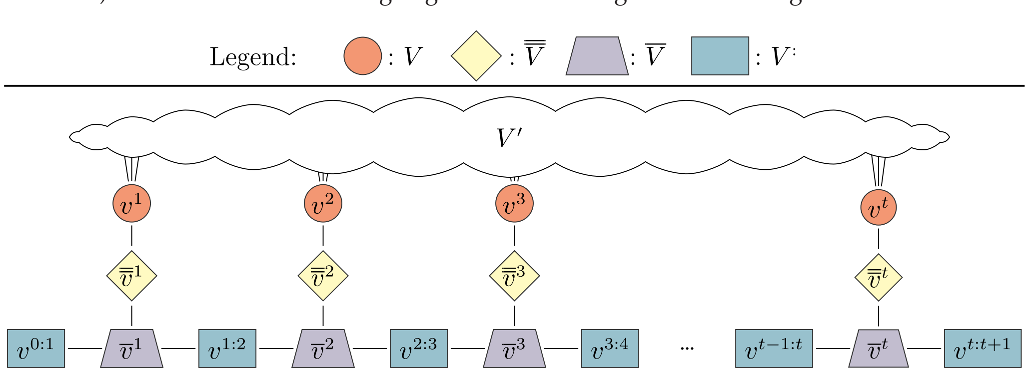
$$V^{:} := \{v^{t:t+1} | 0 \le t' \le t\}$$

$$V' := \{(v')^{t'} | v' \in (A \cup B) \setminus \{v\}, 1 \le t' \le t\}.$$

For any set of nodes V_1 in the rounded graph, we denote its neighborhood by $N(V_1)$, and we write $deg(V_1)$ for the sum of degrees of V_1 in the rounded graph. For any two sets of nodes V_1, V_2, we write $cut(V_1, V_2)$ to denote the number of edges between V_1 and V_2 in the rounded graph.

Note that $\sum_{t'=1}^{t} D^{t'}_v$, which we must bound, equals $cut(V, V')$. We bound this quantity by repeatedly using the following fact, which we refer to as pivoting: for pairwise disjoint sets of nodes V_0, V_1, V_2, if $N(V_0) \subseteq V_1 \cup V_2$, then $deg(V_0) = cut(V_0, V_1) + cut(V_0, V_2)$. Because Lemma 4 gives us a clear view of the fractional degrees of nodes in the constructed graph, and because, by degree preservation, a node's degree in the rounded graph must equal the fractional degree whenever the latter is an integer, this property allows us to express cuts in terms of other cuts. Figure 4 illustrates which of these sets border on each other and helps in following along with the derivation.

Figure 4. (Color online) Illustration of the Counting Argument for Proving Cumulative Degree Preservation



Notes. Edges in the figure are edges from the constructed graph, a superset of the edges in the rounded graph. Note shape (and, in the online version, node color) indicate the set to which a node belongs as indicated in the legend.

We begin by using a pivot with $V_0 = V, V_1 = V'$, $V_2 = \overline{\overline{V}}$, which gives

$$\begin{aligned}\sum_{t'=1}^{t} D_v^{t'} &= cut(V, V') \\ &= deg(V) - cut(V, \overline{\overline{V}}) \\ &= t + \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor - cut(V, \overline{\overline{V}}),\end{aligned}$$

noting that $deg(V) = t + \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor$ by Lemma 4. Using a pivot with $V_0 = \overline{\overline{V}}, V_1 = V, V_2 = \overline{V}$, we get

$$\begin{aligned}\sum_{t'=1}^{t} D_v^{t'} &= t + \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor - deg(\overline{\overline{V}}) + cut(\overline{\overline{V}}, \overline{V}) \\ &= \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor + cut(\overline{\overline{V}}, \overline{V}),\end{aligned}$$

because $deg(\overline{\overline{V}}) = t$ by Lemma 4. Using a pivot with $V_0 = \overline{V}, V_1 = \overline{\overline{V}}, V_2 = V^{:}$, we get

$$\begin{aligned}\sum_{t'=1}^{t} D_v^{t'} &= \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor + deg(\overline{V}) - cut(\overline{V}, V^{:}) \\ &= \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor - cut(\overline{V}, V^{:}) \\ &\quad + \sum_{t'=1}^{t} \left(\left\lfloor \sum_{t''=1}^{t'} d_v^{t''} \right\rfloor - \left\lfloor \sum_{t''=1}^{t'-1} d_v^{t''} \right\rfloor - \lfloor d_v^{t'} \rfloor + 1 \right) \\ &= \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor - cut(\overline{V}, V^{:}) \\ &\quad + \left\lfloor \sum_{t''=1}^{t} d_v^{t''} \right\rfloor - \sum_{t'=1}^{t} \lfloor d_v^{t'} \rfloor + t,\end{aligned}$$

using Lemma 4 and resolving the telescoping sum. Hence, we have that

$$\sum_{t'=1}^{t} D_v^{t'} = \left\lfloor \sum_{t'=1}^{t} d_v^{t'} \right\rfloor + t - cut(\overline{V}, V^{:}).$$

To bound $cut(\overline{V}, V^{:})$ in the last expression, observe that $N(V^{:} \setminus \{v^{t:t+1}\}) \subseteq \overline{V}$, from which it follows that $cut(\overline{V}, V^{:} \setminus \{v^{t:t+1}\}) = deg(V^{:} \setminus \{v^{t:t+1}\}) = t - 1$. Thus, $cut(\overline{V}, V^{:}) = t - 1 + \mathbb{1}\{\hat{X}_{\{\overline{v}^t, v^{t:t+1}\}}\}$, and hence,

$$\begin{aligned}\sum_{t'=1}^{t} D_v^{t'} &= \left\lfloor \sum_{t'=1}^{t} d_v^{t'} \right\rfloor + t - (t - 1 + \mathbb{1}\{\hat{X}_{\{\overline{v}^t, v^{t:t+1}\}}\}) \\ &= \left\lfloor \sum_{t'=1}^{t} d_v^{t'} \right\rfloor + 1 - \mathbb{1}\{\hat{X}_{\{\overline{v}^t, v^{t:t+1}\}}\}.\end{aligned}$$

If $\sum_{t'=1}^{t} d_v^{t'}$ is not an integer, the above shows that $\sum_{t'=1}^{t} D_v^{t'}$ is either the floor or ceiling of $\sum_{t'=1}^{t} d_v^{t'}$, establishing cumulative degree preservation. Else, if $\sum_{t'=1}^{t} d_v^{t'}$ is integer, note that the weight of the edge $\{\overline{v}^t, v^{t:t+1}\}$ in the constructed graph is one. Because dependent rounding always rounds such edges up, $\sum_{t'=1}^{t} D_v^{t'} = \lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor$. This establishes cumulative degree preservation, the last of the properties guaranteed by the theorem. □

6. Other Applications of Cumulative Rounding

Our exploration of house monotone randomized apportionment led us to the more general technique of cumulative rounding, which we believe to be of broader interest. We next illustrate this by discussing additional applications.

6.1. Sortition of the European Commission

The European Commission is one of the main institutions of the European Union, in which it plays a role comparable to that of a government. The commission consists of one commissioner from each of the 27 member states, and each commissioner is charged with a specific area of responsibility. Because the number of EU member states has nearly doubled in the past 20 years, so has the size of the commission. Besides making coordination inside the commission less efficient, the enlargement of the commission has led to the creation of areas of responsibility much less important than others. Because the important portfolios are typically reserved for the largest member states, smaller states have found themselves with limited influence on central topics being decided in the commission.

To remedy this imbalance, Buchstein and Hein (2009) propose to reduce the number of commissioners to 15, meaning that only a subset of the 27 member states would send a commissioner at any given time. Which states would receive a seat would be determined every five years by a weighted lottery ("sortition") in which states would be chosen with degressive proportional weights. Degressive means that smaller states get nonproportionately high weight; such weights are already used for apportioning the European parliament. The authors argue that, by the law of large numbers, political representation on the commission would be essentially proportional to these weights in politically relevant time spans.

However, a follow-up simulation study by Buchstein et al. (2013) challenges this assertion on two fronts: (1) First, the authors find that their implementation of a weighted lottery chooses states with probabilities that deviate from proportionality to the weights in a way that is not analytically tractable (see Brewer and Hanif 1983). (2) Second, and more gravely, their simulations undermine "a central argument in favor of legitimacy" in the original proposal, namely, that "in the long term, the seats on the commission would be distributed approximately like the share of lots" (Buchstein et al. 2013, own translation p. 222). From a mathematical point of view, the authors overestimated the rate of concentration across the independent lotteries. Instead, in the simulation, it takes 30 lotteries (150 years) until there is a probability of 99% that all member states have sent at least one commissioner.

These serious concerns could be resolved by using cumulative rounding to implement the weighted lotteries. Specifically, we would again construct a star graph with a special node a and one node b_i for each state i, setting T to the desired number of consecutive lotteries. For each $1 \le t \le T$, each edge $\{a, b_i\}$ would be weighted by $15\, w_i/\sum_{j\in N} w_j$, where w_j is state j's degressive weight. Degree preservation on a would ensure that, in each lottery t, exactly 15 distinct states are selected. By marginal distribution, the selection probabilities would be exactly proportional to the degressive weights, resolving issue 1. Furthermore, cumulative degree preservation on the state nodes would eliminate issue 2. If we take the effective selection probabilities of Buchstein et al. (2013) as the states' weights, even the smallest states i would have an edge weight $w^t_{\{a,b_i\}} \approx 0.187$. Then, cumulative quota prevents any state from getting rounded down in $11 = \lceil 2/0.187 \rceil$ consecutive lotteries: indeed, fixing any $0 \le t_0 \le T - 11$,

$$\begin{aligned}\sum_{t'=1}^{t_0+11} D^{t'}_{b_i} &\ge \lfloor (t_0+11)\,0.187 \rfloor \\ &\ge \lfloor t_0\, 0.187 \rfloor + 2 \\ &\ge \lceil t_0\, 0.187 \rceil + 1 \\ &\ge \sum_{t'=1}^{t_0} D^{t'}_{b_i} + 1,\end{aligned}$$

which means that state i must have been selected at least once between time $t_0 + 1$ and $t_0 + 11$. In political terms, this means that 55, not 150, years would be enough to deterministically ensure that each member state send a commissioner at least once in this period.

This cumulative rounding approach can accommodate weights that change across lotteries according to population projections (which Buchstein et al. (2013) do for some of their experiments) simply by choosing different weights across the copies of the star graph. It is necessary, however, that these population changes are known in advance because no algorithm can guarantee cumulative degree preservation in an iterated apportionment setting in which population changes are observed online (i.e., just in time for the next apportionment to be made). This is shown by the following example: Let there be four states and allocate a single seat per time step. At time $t = 1$, all four states have an equal population and, thus, an edge weight of $1/4$. Without loss of generality, the first seat goes to state 4. At time $t = 2$, state 4 disappears, while states 1 through 3 have equal population and, thus, each an edge weight of $1/3$. Without loss of generality, the second seat goes to state 3. At time $t = 3$, states 3 and 4 have zero population, while (note this is in a temporal context) states 1 and 2 have equal population (i.e., an edge weight of $1/2$). Because states 1 and 2 have a cumulative quota of $1/4 + 1/3 + 1/2 > 1$, cumulative degree preservation requires both states to have at least one among the first three seats, but this is clearly impossible. We have presented this argument for an adaptive adversary; for a nonadaptive adversary, essentially the same argument shows that any online apportionment mechanism violates cumulative degree preservation with probability at least $1/12$. Note that this impossibility holds even in the absence of marginal distribution and negative correlation. It is an intriguing question how one should design an online apportionment method that keeps violations from cumulative degree preservation at a minimum and ensures that the cost or benefit of such deviations is fairly spread across the remaining states.

6.2. Repeated Allocation of Courses to Faculty or Shifts to Workers

A common paradigm in fair division is to first create a fractional assignment between agents and resources, and to then implement this fractional assignment in expectation through randomized rounding. Below, we describe a setting of allocating courses to faculty members in a university department, in which implementing a fractional assignment using cumulative rounding is attractive.

For a university department, denote its set of faculty members by A and the set of possible courses to be taught by B. For each faculty member a and course b, let there be a weight $w_{\{a,b\}} \in [0,1]$ indicating how frequently course b should be taught by a on average. These numbers could be derived using a process such as probabilistic serial (Bogomolnaia and Moulin 2001), the Hylland–Zeckhauser mechanism (Hylland and Zeckhauser 1979), or the mechanisms by Budish et al. (2013), which would transform preferences of the faculty over which courses to teach into such weights. (Although these mechanisms are formulated for goods, they can be applied to bads when the number of bads allocated to each agent is fixed as it is when allocating courses to faculty or shifts to workers.) We allow arbitrary fractional degrees on the faculty side (so one person can teach multiple courses), and assume that the fractional degree of any course b is at most one.

Applying cumulative rounding to this graph (using the same edge weights in each period) for consecutive semesters $1 \leq t \leq T$, we get the following properties:

- Marginal distribution implies that, in each semester, faculty member a has a probability $w_{\{a,b\}}$ of teaching course b.
- Degree preservation on the course side means that a course is never taught by two different faculty members in the same semester.
- Degree preservation on the faculty side implies that a faculty member a's teaching load does not vary by more than one between semesters; it is either the floor or the ceiling of a's expected teaching load.
- Cumulative degree preservation on the course side ensures that courses are offered with some regularity. For example, if a course's fractional degree is 1/2, it is taught exactly once in every academic year (either in fall or in spring).
- Cumulative degree preservation on the faculty side allows for a noninteger teaching load. For example, a faculty member with fractional degree 1.5 has a "2-1" teaching load; that is, the faculty member will teach three courses per year: either two in the fall and one in the spring or vice versa.

The same approach is applicable for matching workers to shifts.

One could also use cumulative rounding to repeatedly round a fractional assignment of general chores, such as the ones computed by the online platform spliddit.org (Goldman and Procaccia 2014). In this case, a caveat is that (cumulative) degree preservation only ensures that the number of assigned chores is close to its expected number per time period, not necessarily the cost of the assigned chores. However, if many chores are allocated per time step and if costs are additive, then an agent's per-timestep cost is well-concentrated, which follows from the negative correlation property that permits the application of Chernoff concentration bounds (Panconesi and Srinivasan 1997).

7. Discussion

Though our work is motivated by the application of apportioning seats at random, the technical questions we posed and addressed are fundamental to the theoretical study of apportionment. In a sense, any deterministic apportionment solution is unproportional; after all, its role is to decide which agents receive more or fewer seats than their standard quota. By searching for randomized methods satisfying ex ante proportionality, we ask whether these unproportional solutions can be combined (through random choice) such that these deviations from proportionality cancel out to achieve perfect proportionality and whether this remains possible when we restrict the solutions to those satisfying subsets of the axioms population monotonicity, house monotonicity, and quota. Naturally, this objective pushes us to better understand the whole space of solutions satisfying these subsets of axioms, including the space's more extreme elements. Therefore, it is in hindsight not surprising that our work led to new insights for deterministic apportionment: a more robust impossibility between population monotonicity and quota (Theorem 1), an exploration of solutions generalizing the divisor solutions (Theorem 2), and a geometric characterization of house monotone and quota compliant solutions (Theorem 6).

Concerning the cumulative rounding technique introduced in this paper, we have only scratched the surface in exploring its applications. In particular, we hope to investigate whether cumulative rounding can extend existing algorithmic results that use dependent rounding and whether it can be used to construct new approximation algorithms. For both of these purposes, the negative correlation property, which we have not used much so far, will, we hope, turn out to be valuable.

Despite their advantageous properties, randomized mechanisms have in the past often met with resistance by practitioners and the public (Kurokawa et al. 2018), but we see signs of a shift in attitudes. Citizens' assemblies, deliberative forums composed of a random sample of citizens, are quickly gaining usage around the world (Organisation for Economic Co-operation and Development 2020) and proudly point to their random selection—often carried out using complex algorithms from computer science (Flanigan et al. 2021) as a source of legitimacy. If this trend continues, randomness will be associated by the public with neutrality and fairness, not with haphazardness. Hence, randomized apportionment methods (though, perhaps, simpler ones than the ones we develop here) might yet receive serious consideration.

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