A Computational Approach to Organizational Structure

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Abstract

An organizational structure determines how an organization arranges and manages its individuals. Sociologists have studied organizational structure from an empirical perspective for decades, but their work has been limited by small sample sizes. By contrast, we study organizational structure from a computational and theoretical perspective.

Specifically, we introduce a model of organizations that involves both computation and communication, and captures the spirit of past sociology experiments. In our model, each node in a graph starts with a token, and at any time can either perform computation to merge two tokens in $t_c$ time, or perform communication by sending a token to a neighbor in $t_m$ time. We study how to schedule computation and communication so as to merge all tokens as quickly as possible.

As our first result, we give a polynomial-time algorithm that optimally solves this problem on a complete graph. This result characterizes the optimal graph structure—the edges used for communication in the optimal schedule—and therefore informs the optimal design of large-scale organizations. Moreover, since pre-existing organizations may want to optimize their workflow, we also study this problem on arbitrary graphs. We demonstrate that our problem on arbitrary graphs is not only NP-complete but also hard to approximate within a multiplicative 1.5 factor. Finally, we give an $O(\log n \cdot \log \frac{OPT}{t_m})$-approximation algorithm for our problem on arbitrary graphs, where $n$ is the number of nodes and $OPT$ is the length of the optimal schedule.

1 Introduction

Large-scale organizations accomplish otherwise prohibitively difficult tasks by distributing work. For example, companies distribute tasks among employees. Moreover, not only do organizations delegate work, but they also determine how members communicate with one another—both in how they choose to structure themselves and in how they instruct employees to utilize this structure [14, 32]. Considering how these two facets of organizations—computation and communication—interact is paramount to ensuring organizational efficiency. For instance, taking both into account enables powerful optimizations such as pipelining, a technique that parallelizes computation and communication [37].
Sociologists have studied how organizational structure—both how an organization is arranged and how an organization is managed—impacts the performance of organizations for decades \[19, 56, 58\]. An especially influential experiment was conducted by sociologists Alex Bavelas and Harold Leavitt \[6, 7, 43\]. In this experiment, groups of five participants were placed in one of four communication patterns, depicted in Figure 1. Each participant was given a set of five distinct items chosen from a set of six total items. The goal of all participants was to determine which item all five participants had in common by communicating via written notes over the specified communication pattern and repeatedly taking set intersections. Each group was evaluated based on the total time it took for the participants to complete the task. Notably, both the time it took participants to communicate messages and the time it took participants to perform computation on the messages they received counted towards the total time taken by the solution.

Many other sociologists have further studied organizational structure through empirical experimentation \[12, 15, 16, 28, 29, 33, 41, 42, 47, 49, 50, 62, 63, 64, 65, 66, 67, 68\]. Many of these studies observe a strong pattern in which “more peripheral members of a network [channeled] information to the most central node ... who then decided what the correct answer was” \[10\].

However, sociology experiments on organizational structure suffer from two serious shortcomings. First, these empirical studies are limited in size by physical and logistical constraints. For example, the Bavelas-Leavitt experiment studied groups of only five participants, and most subsequent experiments did not consider groups of more than five people \[12, 15, 16, 28, 29, 33, 41, 42, 47, 49, 50, 62, 63, 64, 65, 66, 67, 68\]. These small sample sizes have led to reproducibility issues and generally inconclusive experiments as observed by Burgess \[12\]: “Unfortunately, [this research] has not produced consistent and cumulative results. Indeed, the results are contradictory as well as inconclusive”. Second, although “these works initiated a plethora of empirical research... [they] were not accompanied by much theoretical development” \[13\]. Moreover, the few theoretical results that have been given \[6\] are not only rudimentary but also old and so fail to capitalize on modern advances in theoretical computer science.

In this paper, we provide a modern algorithmic perspective on organizational structure. Our main research question is twofold:

1. **How should one construct a large-scale organization?**

2. **How should an organization coordinate communication and computation?**

We present a model of organizations, directly inspired by the Bavelas-Leavitt experiment, which takes both communication and computation into account. With this model we give a formal treatment of the tendency of members of networks to channel information to centrally located nodes. Examining organizational structure through a computational lens—with tools from approximation
algorithms—allows us to develop a formal understanding of organizational structure. Moreover, our results apply to organizations of any size.

1.1 Our Model and Problem

The Token Network Model. We now introduce our model, the Token Network model. A Token Network is given by an undirected graph $G = (V, E)$ with parameters $t_c, t_m \in \mathbb{N}^1$ which describe the time it takes nodes to do computation and communication, respectively.

Time proceeds in synchronous rounds during which nodes can compute on or communicate atomic tokens. Specifically, in any given round a node is busy or not busy. If a node is not busy and has at least one token it can communicate: any node that does so is busy for the next $t_m$ rounds, at the end of which it passes one of its tokens to a neighbor in $G$. If a node is not busy and has two or more tokens, it can compute: any node that does so is busy for the next $t_c$ rounds, at the end of which it combines (a.k.a. aggregates) two of its tokens into a single new token.$^2$ At a high level, this means that communication takes $t_m$ rounds and computation takes $t_c$ rounds.

The Token Computation Problem. We use our Token Network model to give a formal treatment of the tendency of members of networks to channel information to centrally located nodes. In particular, we study the Token Computation problem. Given an input Token Network, an algorithm for the Token Computation problem must output a schedule $S$ which directs each node when to compute and when and with whom to communicate. A schedule is valid if after the schedule is run on the input Token Network where every node begins with a single token, there is a single token in the entire network; i.e., there is one node that has aggregated all the information in the network. We use $|S|$ to notate the length of $S$—i.e., the number of rounds $S$ takes—and measure the quality of an algorithm by the length of the schedules that it outputs. So as to leave no ambiguity as to the precise problem that we are solving, we give a more formal definition of the Token Computation problem in Appendix A, though we note that this formalization is exactly what one would expect.

Discussion of Modeling Choices. Our Token Network model and the Token Computation problem are designed to formally capture the challenges articulated by Bavelas and Leavitt of an organization collectively accomplishing a task. In particular, combining tokens can be understood as applying some commutative, associative function to the private input of all organization members. For instance, summing up private inputs, taking a minimum of private inputs, or computing the intersection of input sets in an organization can all be cast as instances of the Token Computation problem. Computing the intersection of input sets is, in fact, exactly the setting studied by the Bavelas-Leavitt experiment. Additionally, unlike most models of distributed computing, we assume that both the computation and the communication by members take time, as time to compute over received messages was factored into that experiment. We consider how a centralized agent can coordinate an organization rather than a distributed model both because it reflects how organizations operate in practice—managers coordinate employees—and because it

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$^1$We assume $t_c, t_m = \text{poly}(n)$ throughout this paper.

$^2$Throughout this paper, we assume for ease of exposition that the smaller of $t_c$ and $t_m$ evenly divides the larger of $t_c$ and $t_m$. 

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was an intended object of study of the Bavelas-Leavitt experiment—“there is an understandable urge to proceed with the business of coordination” [6]. We assume that the computation time is the same for every operation and that the output of a computation is roughly the size of each of the inputs as a simplifying assumption; however, we do note that the sets computed over in the Bavelas-Leavitt experiment stayed roughly of the same size, so this assumption, too, is motivated by the Bavelas-Leavitt experiment. Note that unlike the Bavelas-Leavitt experiment, in the Token Computation problem, we only require a single arbitrary node in the network to compute the final answer. We purposefully make this choice as even if we were to require that all nodes in a network know the final answer, once one node has aggregated all the information in the network, it is well-understood how to spread said value in the network [35, 40, 60]. Moreover, we feel that this sort of workflow, in which only a single node must determine the answer, captures the structure of companies—often, lower-level employees perform tasks and propagate information up a network, but they may not always be told the final result after higher-level management integrates their input. We allow nodes to receive information from multiple neighbors in each round to reflect the structure of the Bavelas-Leavitt experiment. Lastly, it is not hard to see that our assumption that a node can send its token to only a single neighbor rather than multiple copies of its token to multiple neighbors is without loss of generality: One can easily modify a schedule in which nodes send multiple copies to neighbors to one of equal length in which a node only ever sends one token per round.

1.2 Results, Discussion and Techniques

We now give a high-level description of our results for solving the Token Computation problem.

Optimal Schedule on Complete Graphs (Section 4). We begin by considering how to construct the optimal schedule for the complete graph for given values of $t_c$ and $t_m$. The principal challenge in constructing such a schedule is formalizing how to optimally pipeline computation and communication and showing that any valid schedule needs at least as many rounds as one’s constructed schedule. We overcome this challenge by showing how to modify a given optimal schedule into an efficiently computable one in a way that preserves its pipelining structure. Specifically, we show that one can always modify a valid optimal schedule into another valid optimal schedule with a well-behaved recursive form. We show that this well-behaved schedule can be computed in polynomial time. Even stronger, we show that the edges over which communication takes place in this schedule induce a tree which generalizes Fibonacci trees [34]. It is important to emphasize, then, that this result has implications beyond producing the optimal schedule for a complete graph; it shows how one ought to construct a network over which both computation and communication occur (if one had the freedom to include any edge), thereby answering the first of our two main research questions.

Hardness Results for Arbitrary Graphs (Section 5). We next consider the hardness of producing good schedules efficiently for arbitrary graphs for given values of $t_c$ and $t_m$. The challenge in demonstrating the hardness of our problem is that the optimal schedule for an arbitrary graph does not have well-behaved structure. Our insight here is that by forcing a single node to do a great deal of computation we can impose structure on the optimal schedule in a way that makes the optimal schedule reflect the minimum dominating set of the graph. Using this insight, we show that an approximation algorithm is the best one can hope for in our problem, by demonstrating
that the problem is NP-complete. We then show that no polynomial-time algorithm can produce
a schedule of length within a multiplicative 1.5 factor of the optimal schedule unless P = NP.
This result shows that one can only coordinate a pre-existing network over which computation and
communication occur so well.

**Approximately-Optimal Schedule for Arbitrary Graphs (Section 6).** Given that an ap-
proximation algorithm is the best one can hope for, we next give an algorithm which in polynomial
time produces an approximately-optimal Token Computation schedule. Our algorithm is based
on the simple observation that after $O(\log n)$ repetitions of pairing off nodes with tokens, having
one node in each pair route a token to the other node in the pair and then having every node com-
pute, there will be a single token in the network. The difficulty in this approach lies in showing that
one can route pairs of tokens in a way that is competitive with the length of the optimal schedule.
We show that by considering the paths in $G$ traced out by tokens sent by the optimal schedule,
we can get a concrete hold on the optimal schedule. Specifically, we show that a polynomial-time
algorithm based on our observation produces a valid schedule of length $O(\text{OPT} \cdot \log n \cdot \log \frac{\text{OPT}}{t_m})$
with high probability,\(^3\) where $\text{OPT}$ is the length of the optimal schedule. Using an easy bound on
$\text{OPT}$, this can be roughly interpreted as an $O(\log^2 n)$-approximation algorithm. This result shows
how one can coordinate a pre-existing network over which both computation and communication
occur fairly well.

We believe that these results motivate interesting empirical directions. Previous sociology exper-
iments have been performed in networks with decentralized coordination. Running an experiment
similar to that of Bavelas and Leavitt using our optimal networks with a centralized authority
would measure to what degree optimal network structure and a centralized authority expedite the
time groups take to solve tasks.

Lastly, while we have thus far framed the applications of our work to organizations of people,
they just as well apply to distributed systems of computers. In particular, our results give algorithms
for scheduling computation and communication on networks of computers. Furthermore, it is not
hard to see that when $t_c = 0$ and $t_m > 0$ or $t_c > 0$ and $t_m = 0$, our problem is trivially solvable in
polynomial time. However, we show hardness for the case where $t_c, t_m > 0$, which gives a formal
sense in which computation and communication cannot be considered in isolation, as assumed in
previous models of distributed computation.

# 2 Related Work

Though we have reviewed the primary related work in sociology in Section 1, problems related
to ours have been studied in both theoretical and applied computer science and economics. We
describe some of this related work here and discuss additional related work in Appendix B.

## 2.1 Related Theoretical Work

The previous theoretical work which most closely resembles our own is a line of work in centralized
algorithms for scheduling information dissemination [35, 40, 60]. In this problem, an algorithm is
given a graph and a model of distributed communication, and must output a schedule that instructs
\(^3\)Meaning at least $1 - \frac{1}{\text{poly}(n)}$ henceforth.
nodes how to communicate in order to spread some information. For instance, in one setting an
algorithm must produce a schedule which, when run, broadcasts a message from one node to all
other nodes in the graph. The fact that these problems consider spreading information is com-
plementary to the way in which we consider consolidating it. However, we note that computation
plays no role in these problems, in contrast to our Token Computation problem.

Of these prior models of communication, the model which is most similar to our own is the
telephone broadcast model. In this model in each round a node can “call” another node to transmit
information or receive a call from a single neighbor. Previous results have given a hardness of
approximation of 3 [21] for broadcasting in this model and logarithmic as well as sublogarithmic
approximation algorithms for broadcasting [22]. The two notable differences between this model
and our own are (1) in our model nodes can receive information from multiple neighbors in a single
round\(^4\) and (2) again, in our model computation takes a non-negligible amount of time. Note, then,
that even in the special case when \(t_c = 0\), our model does not generalize the telephone broadcast
model; as such we do not immediately inherit prior hardness results from the telephone broadcast
problem. Relatedly, (1) and (2) preclude the possibility of an easy reduction from our problem to
the telephone broadcast problem.

In the setting of radio networks, researchers have examined the Convergecast scheduling prob-
lem, which resembles the aggregation aspect of the Token Computation problem [23, 24]. How-
ever, this work does not consider the cost of computation. Additionally, researchers have studied
algorithms for the general problem of multi-level aggregation, which shares many of the same ap-
plications as our work [9, 11, 52]. Again, these works do not consider the cost of computation,
and, moreover, they only examine the online setting in which requests for information aggregation
arrive over time.

2.2 Related Applied Work

There is a significant body of related work in resource-aware scheduling, sensor networks, and high-
performance computing that considers both the relative costs of communication and computation,
often bundled together in an energy cost. However, these studies have been largely empirical rather
than theoretical, and much of the work considers distributed algorithms.

The most closely related applied work is in the high-performance computing space, where many
researchers have studied the problem of AllReduce. In this problem one must compute a function
of the input of all nodes in a network and then distributes the result to all other nodes in the network
[27, 59]. However, while there has been significant research on communication-efficient AllReduce
algorithms, there has been relatively little work that explicitly considers the cost of computation,
and even less work that considers the construction of optimal topologies for efficient distributed
computation. Researchers have empirically evaluated the performance of different models of com-
munication [39, 51] and have proven (trivial) lower bounds for communication without considering
computation [53, 54]. Indeed, to the best of our knowledge, the extent to which they consider
computation is through an additive penalty that consists of a multiplicative factor times the size
of all inputs at all nodes, as in [36]; crucially, this penalty is the same for any schedule and cannot
be reduced via intelligent scheduling. Therefore, there do not seem to exist theoretical results for
efficient algorithms that consider both the cost of communication and computation.

\(^4\)See above for the justification of this assumption.
3 Definitions and Notation

Before moving on to our formal results, we introduce the definitions we use in the remainder of the paper.

**Definition 1 (Idle).** A node is *idle* in round $t$ if it is not computing or communicating in round $t$.

**Definition 2 (Contains).** A token $a$ contains token $a'$ if $a = a'$ or $a$ was created by combining two tokens, one of which contains $a'$. For shorthand we write $a' \in a$ to mean that $a$ contains $a'$.

**Definition 3 (Singleton token).** A singleton token is a token that only contains itself; i.e., it is a token with which a node started. We let $a_v$ be the singleton token with which vertex $v$ starts and refer to $a_v$ as $v$’s singleton token.

**Definition 4 (Size of a token).** The size of a token is the number of singleton tokens it contains.

**Definition 5 (Terminus).** Let $a_f$ be the last token of a valid schedule $S$; i.e., $a_f$ contains all singleton tokens. The *terminus* of $S$ is the node at which $a_f$ is formed by a computation.

4 Optimal Algorithm for Complete Graphs

In this section we provide an optimal polynomial-time algorithm for the Token Computation problem on a complete graph. The schedule output by our algorithm ultimately only uses the edges of a particular tree, and so, although we reason about our algorithm in a fully connected graph, in reality our algorithm works equally well on said tree. This result, then, informs the design of an optimal organization structure.

4.1 Binary Trees (Warmup)

We build intuition by considering a natural solution to Token Computation on the complete graph: naive aggregation on a rooted binary tree. In this schedule, nodes do computations and communications in lock-step. In particular, consider the schedule $S$ which alternates the following two operations until only a single node with tokens remains on a fixed binary tree: (1) every non-root node that has a token sends its token to its parent in the binary tree; (2) every $v$ with at least two tokens performs one computation. Once only one node has any tokens, that node performs computation until only a single token remains. After $\log n$ iterations of this schedule, the root of the binary tree is the only node with any tokens, and thereafter only performs computation for the remainder of $S$. However, $S$ does not efficiently pipeline communication and computation: after each iteration of (1) and (2), the root of the tree gains an extra token. Therefore, after $\log n$ repetitions of this schedule, the root has $\log n$ tokens. In total, then, this schedule aggregates all tokens after essentially $\log n(t_c + t_m) + \log n(t_c)$ rounds. See Figure 2.

For certain values of $t_c$ and $t_m$, we can speed up naive aggregation on the binary tree by pipelining the computations of the root with the communications of other nodes in the network. In particular, consider the schedule $S'$ for a fixed binary tree for the case when $t_c = t_m$ in which every non-root node does exactly what it does in $S$ but the root always computes. Since the root is always computing in $S'$, even as other nodes are sending, it does not build up a surplus as in $S$. Thus, this schedule aggregates all tokens after essentially $\log n(t_c + t_m)$ rounds when $t_c = t_m$. 


(a) Round 1  (b) Round 2  (c) Round 3  (d) Round 4

Figure 2: The naive aggregation schedule on a binary tree for $t_c = t_m = 1$ and $n = 7$ after 4 rounds. Tokens are represented by blue diamonds; a red arrow from node $u$ to node $v$ means that $u$ sends to $v$; and a double-ended blue arrow between two tokens $a$ and $b$ means that $a$ and $b$ are combined at the node. Notice that the root gains an extra token every 2 rounds.

(a) Round 1  (b) Round 2  (c) Round 3  (d) Round 4

Figure 3: The aggregation schedule on a binary tree for $t_c = t_m = 1$ and $n = 7$ after 4 rounds where the root pipelines its computations. Again, tokens are represented by blue diamonds; a red arrow from node $u$ to node $v$ means that $u$ sends to $v$; and a double-ended blue arrow between two tokens $a$ and $b$ means that $a$ and $b$ are combined at the node. Notice that the root will never have more than 3 tokens when this schedule is run.

However, as we will show, binary trees are not optimal even when they pipeline computation at the root and $t_c = t_m$. In the remainder of this section, we generalize this pipelining intuition to arbitrary values of $t_c$ and $t_m$ and formalize how to show a schedule is optimal.

4.2 Complete Graphs

We now describe our optimal polynomial-time algorithm for complete graphs. This algorithm produces a schedule which greedily aggregates on a particular tree, $T^*_n$. In order to describe this tree, we first introduce the tree $T(R, t_c, t_m)$. This tree can be thought of as the largest tree for which greedy aggregation aggregates all tokens in $R$ rounds given computation cost $t_c$ and communication cost $t_m$. We will overload notation and let $T(R)$ denote $T(R, t_c, t_m)$ for some fixed values of $t_c$ and $t_m$. Let the root of a tree be the node in that tree with no parents. Also, given a tree $T_1$ with root $r$ we define $T_1$ JOIN $T_2$ as $T_1$ but where $r$ also has $T_2$ as an additional subtree. We define $T(R)$ as follows.
Figure 4: An example of $T(16)$ where $t_c = 2$, $t_m = 1$.

$T(R) := \begin{cases} 
\text{A single leaf} & \text{if } R < t_m + t_c \\
T(R - t_c) \text{ JOIN } T(R - t_c - t_m) & \text{otherwise}
\end{cases}$

We give an example of $T(16)$ for $t_c = 2$ and $t_m = 1$ in Figure 4. Notice that for $t_c = t_m = 1$ the tree $T(R)$ just is a Fibonacci tree [34] of recursion depth $R$; $T(R)$, then, can be thought of as a generalization of Fibonacci trees.\(^5\)

Since an input to the TOKEN COMPUTATION problem consists of $n$ nodes, and not a desired number of rounds, we define $R^*(n, t_c, t_m)$ to be the minimum value such that $T(R^*(n, t_c, t_m)) \geq n$. We again overload notation and let $R^*(n)$ denote $R^*(n, t_c, t_m)$. Formally,

$$R^*(n) := \arg \min_{R:|T(R)| \geq n} R.$$ 

We let $T^*_n$ denote $T(R^*(n))$. For ease of presentation throughout this section we assume that $|T^*_n| = n$.\(^6\)

The schedule produced by our algorithm will simply perform greedy aggregation on $T^*_n$. We now formally define greedy aggregation and establish its runtime on the tree $T(R)$.

**Definition 6** (Greedy Aggregation). Given an $r$-rooted tree, let the greedy aggregation schedule be defined as follows. In the first round, every node except for $r$ sends its token to its parent. In subsequent rounds we do the following. If a node is not busy and has at least two tokens, it performs a computation. If a non-root node is not busy, has exactly one token, and has received a token from every child in previous rounds, it forwards its token to its parent.

**Lemma 7.** Greedy aggregation on $T(R)$ terminates in $R$ rounds.

\(^5\)Echoing the fact that only graphs containing binomial heaps have log $n$-length telephone broadcast schedules [21].

\(^6\)If $|T^*_n| > n$, then we could always “hallucinate” extra nodes where appropriate.
Figure 5: An illustration of the optimal schedule length for different sized trees. The solid red line is the number of rounds taken by greedy aggregation with pipelining on a binary tree (i.e., $\lceil 2 \cdot t_c \cdot \log n + t_m \cdot \log n \rceil$); the dashed blue line is the number of rounds taken by greedy aggregation on $T^*_n$; and the dotted green line is the trivial lower bound of $\lceil t_c \cdot \log n \rceil$ rounds. Note that though we illustrate the trivial lower bound of $\lceil t_c \cdot \log n \rceil$ rounds, as we prove the true lower bound is given by the number of rounds taken by greedy aggregation on $T^*_n$.

Proof. We will show by induction on $k \geq 0$ that greedy aggregation results in the root of $T(k)$ having a token of size $|T(k)|$ after $k$ rounds. The base cases of $k \in [0, t_m + t_c]$ are trivial, as nothing needs to be combined. For the inductive step, applying the inductive hypothesis and using the recursive structure of our graph tells us that the root of $T(k + t_c)$ has a token of size $|T(k)|$ at its root in $k$ rounds, and the root of the child $T(k - t_m)$ has a token of size $|T(k - t_m)|$ at its root in $k - t_m$ rounds. Therefore, by the definition of greedy aggregation, the root of $T(k - t_m)$ sends its token of size $|T(k - t_m)|$ to the root of $T(k + t_c)$ at time $k - t_m$, which means the root of $T(k + t_c)$ can compute a token of size $|T(k - t_m)| + |T(k)| = |T(k + t_c)|$ by round $k + t_c$, as desired.

See Figure 5 for an illustration of the $|T^*_n|$ as a function of $n$ for specific values of $t_c$ and $t_m$. Furthermore, notice that that $T(R)$ and $T^*_n$ are constructed in such a way that greedy aggregation pipelines computation and communication. We now state our optimal algorithm, which simply outputs the greedy aggregation schedule on $T^*_n$, and give the main theorem of this section.

Algorithm 1 OptComplete($t_c$, $t_m$, $n$)

Input: $t_c$, $t_m$, $n$
Output: A schedule for Token Computation on $K_n$ with parameters $t_c$ and $t_m$
Arbitrarily embed $T^*_n$ into $K_n$
return Greedy aggregation schedule on $T^*_n$ embedded in $K_n$

Theorem 8. Given a complete graph $K_n$ on $n$ vertices and any $t_m, t_c \in \mathbb{Z}^+$, OptComplete optimally solves Token Computation on the Token Network $(K_n, t_c, t_m)$ in polynomial time.
To show that Theorem 8 holds, we first note that OptComplete trivially runs in polynomial time (proof given in Appendix C). Therefore, we focus on showing that greedy aggregation on \( T_n^* \) optimally solves the Token Computation problem on \( K_n \). We start by defining \( N^*(R) \) as follows.

**Definition 9.** Let \( N^*(R) \in \mathbb{Z}^+ \) be the number of nodes in the maximum size complete graph on which one can solve the Token Computation problem in at most \( R \) total rounds. Equivalently, \( N^*(R) \) is the size of the largest token produced by a schedule of length \( R \) in a sufficiently large complete graph.

We show that \( |T(R)| = N^*(R) \), thereby showing that greedy aggregation on \( T_n^* \) is optimal. Roughly, we do this by showing that \( |T(R)| \) and \( N^*(R) \) satisfy the same recurrence.

First notice that the base case of \( N^*(R) \) is trivially 1.

**Lemma 10.** For \( R \in \mathbb{Z}_0^+ \) we have that \( N^*(R) = 1 \) for \( R < t_c + t_m \).

**Proof.** That \( N^*(R) = 1 \) for \( R < t_c + t_m \) follows immediately from the fact that if \( R < t_c + t_m \) there are not enough rounds to send and combine a token and so the Token Computation problem can only be solved on a graph with one node.

We now show that for the recursive case \( N^*(R) \) is always at least as large as \( N^*(R - t_c) + N^*(R - t_c - t_m) \), which is the recurrence that defines \( |T(R)| \).

**Lemma 11.** For \( R \in \mathbb{Z}_0^+ \) we have that \( N^*(R) \geq N^*(R - t_c) + N^*(R - t_c - t_m) \) for \( R \geq t_c + t_m \).

**Proof.** Suppose \( R \geq t_c + t_m \). Let \( S_1 \) be the optimal schedule on the complete graph of \( N^*(R - t_c) \) nodes with terminus \( v_{t1} \) and let \( S_2 \) be the optimal schedule on the complete graph of size \( N^*(R - t_c - t_m) \) with corresponding terminus \( v_{t2} \). Now consider the following solution on the complete graph of \( N^*(R - t_c) + N^*(R - t_c - t_m) \) nodes. Run \( S_1 \) and \( S_2 \) in parallel on \( N^*(R - t_c) \) and \( N^*(R - t_c - t_m) \) nodes respectively, and once \( S_2 \) has completed, forward the token at \( v_{t2} \) to \( v_{t1} \) and, once it arrives, have \( v_{t1} \) perform one computation. This is a valid schedule which takes \( R \) rounds to solve Token Computation on \( N^*(R - t_c) + N^*(R - t_c - t_m) \) nodes. Thus, we have that \( N^*(R) \geq N^*(R - t_c) + N^*(R - t_c - t_m) \) for \( R \geq t_c + t_m \).

It remains to show that this bound on the recursion is tight. To do so, we case on whether \( t_c \geq t_m \) or \( t_c < t_m \).

### 4.2.1 \( t_c \geq t_m \)

In this case, where computation takes at least as much time as communication, it is straightforward to show that \( N^*(R) \) follows the same recurrence as \( |T(R)| \).

**Lemma 12.** When \( t_c \geq t_m \) for \( R \in \mathbb{Z}_0^+ \) it holds that \( N^*(R) = N^*(R - t_c) + N^*(R - t_c - t_m) \) for \( R \geq t_c + t_m \).

**Proof.** Suppose that \( R \geq t_c + t_m \). By Lemma 11, it is sufficient to show that \( N^*(R) \leq N^*(R - t_c) + N^*(R - t_c - t_m) \). Consider the optimal solution given \( R \) rounds. The last action performed by any node must have been a computation that combines two tokens, \( a \) and \( b \), at the terminus \( v_t \) because, in an optimal schedule, any further communication of the last token increases the length of the schedule. We now consider three cases.
• In the first case, $a$ and $b$ were both created at $v_t$. Because both of $a$ or $b$ could not have been created at time $R - t_c$, one of them must have been created at time $R - 2t_c$ at the latest. This means that $N^*(R) \leq N^*(R - t_c) + N^*(R - 2t_c) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$.

• In the second case, exactly one of $a$ or $b$ (without loss of generality, $a$) was created at $v_t$. This means that $b$ must have been sent to $v_t$ at latest at time $R - t_c - t_m$. This means that $N^*(R) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$.

• In the last case, neither $a$ nor $b$ was created at $v_t$. This means that both must have been sent to $v_t$ at the latest at time $R - t_c - t_m$. This means that $N^*(R) \leq N^*(R - t_c - t_m) + N^*(R - t_c - t_m) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$.

Thus, in all cases we have $N^*(R) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$.

\hfill $\Box$

4.2.2 $t_c < t_m$

We now consider the case in which communication is more expensive than computation, $t_c < t_m$. One might hope that the same sort of simple case-wise analysis as used when $t_c \geq t_m$ would prove the desired result for when $t_c < t_m$. However, it is not difficult to see that that the preceding analysis breaks down for this case. Thus, we have to do significantly more work to show that $N^*(R) = N^*(R - t_c) + N^*(R - t_c - t_m)$. In particular, we have to establish some structure on the schedule which solves Token Computation on $K_{N^*(R)}$ in $R$ rounds that will allow us to show that $N^*(R) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$ when $t_c < t_m$. We do this by considering an arbitrary schedule and then performing successive modifications to this schedule that do not affect the validity or length of the schedule. In particular, we leverage the following insights—illustrated in Figure 6—to perform these successive modifications.

• Combining insight: Suppose node $v$ has two tokens in round $t$, $a$ and $b$, and $v$ sends $a$ to node $u$ in round $t$. Node $v$ can just as well aggregate $a$ and $b$, treat this aggregation as it treats $b$ in the original schedule and $u$ can just pretend that it receives $a$ in round $t + t_m$. That is, $u$ can “hallucinate” that it has token $a$. Note that this insight crucially leverages the fact that $t_c < t_m$, since otherwise the performed computation would not finish before round $t + t_m$.

• Shortcutting insight: Suppose node $v$ sends a token to node $u$ in round $t$ and node $u$ sends a token to node $w$ in a round in $[t, t + t_m]$. Node $v$ can “shortcut” node $u$ and send to $w$ directly and $u$ can just not send.

Using these insights, we establish the following structure on a schedule which solves Token Computation on $K_{N^*(R)}$ in $R$ rounds when $t_c < t_m$.

Lemma 13. When $t_c < t_m$, for all $R \in \mathbb{Z}_0^+$, there exists a schedule, $\tilde{S}^*$, of length $R$ that solves Token Computation on $K_{N^*(R)}$ such that the terminus of $\tilde{S}^*$, $v_t$, never communicates and every computation performed by $v_t$ involves a token that contains $v_t$’s singleton token, $a_{v_t}$.

Proof. Let $S^*$ be some arbitrary schedule of length $R$ which solves Token Computation on $K_{N^*(R)}$; we know that such a schedule exists by definition of $N^*(R)$. We first show how to modify $S^*$ into another schedule, $S^*_{1-4}$, which not only also solves Token Computation on $K_{N^*(R)}$ in $R$ rounds, but which also satisfies the following four properties.
Figure 6: An illustration of the shortcutting and combining insights. Here, tokens are denoted by blue diamonds, and hallucinated tokens are denoted by striped red diamonds. As before, a red arrow from node \( u \) to node \( v \) means that \( u \) sends to \( v \), and a double-ended blue arrow between two tokens \( a \) and \( b \) means that \( a \) and \( b \) are combined at the node. Notice that which nodes have tokens and when nodes have tokens are the same under both modifications (though in the combining insight, a node is only hallucinating that it has a token).

(1) \( v \) only sends at time \( t \) if \( v \) at time \( t \) has exactly one token for \( t \in [R] \);
(2) if \( v \) sends in round \( t \) then \( v \) does not receive any tokens in rounds \( [t, t+tm] \) for \( t \in [R] \);
(3) if \( v \) sends in round \( t \) then \( v \) is idle during rounds \( t'>t \) for \( t \in [R] \);
(4) the terminus never communicates.

Achieving property (1). Consider an optimal schedule \( S^* \). We first show how to modify \( S^* \) to an \( R \)-round schedule \( S_1^* \) that solves Token Computation on \( K_{N^*(R)} \) and satisfies property (1). We use our combining insight here. Suppose that (1) does not hold for \( S^* \); i.e., a node \( v \) sends a token \( a_1 \) to node \( u \) at time \( t \) and \( v \) has at least one other token, say \( a_2 \), at time \( t \). We modify \( S^* \) as follows. At time \( t \), node \( v \) combines \( a_1 \) and \( a_2 \) into a token which it then performs operations on (i.e., computes and sends) as it does to \( a_2 \) in the original schedule. Moreover, node \( u \) pretends that it receives token \( a_1 \) at time \( t + tm \); any round in which \( S^* \) has \( u \) compute on or communicate \( a_1 \), \( u \) now simply does nothing; nodes that were meant to receive \( a_1 \) do the same. It is easy to see that by repeatedly applying the above procedure to every node when it sends when it has more than one token, we can reduce the number of tokens every node has whenever it sends to at most one. The total runtime of this schedule is no greater than that of \( S^* \), namely \( R \), because \( t_c < t_m \). Moreover, it clearly still solves Token Computation on \( K_{N^*(R)} \). Call the schedule \( S_{1-2}^* \).

Achieving properties (1) and (2). Now, we show how to modify \( S_1^* \) into \( S_{1-2}^* \) such that properties (1) and (2) both hold. Again, \( S_{1-2}^* \) is of length \( R \) and solves Token Computation on \( K_{N^*(R)} \). We use our shortcutting insight here. Suppose that (2) does not hold for \( S_1^* \); i.e., there
exists a $v$ that receives a token $a_1$ from node $u$ while sending another token $a_2$ to node $u'$. We say that node $u$ is bothering node $v$ in round $t$ if node $u$ communicates a token $a_1$ to $v$ in round $t$, and node $v$ communicates a token $a_2$ to node $u' \in V \setminus \{v, u\}$ in round $[t, t + t_m]$. Say any such pair is a bothersome pair. Furthermore, given a pair of nodes $(u, v)$ and round $t$ such that node $u$ is bothering node $v$ in round $t$, let the resolution of $(u, v)$ in round $t$ be the modification in which $u$ sends its token directly to the node $u'$ to which $v$ sends its token. Note that each resolution does not increase the length of the optimal schedule because, by the definition of bothering, this will only serve as a shortcut; $u'$ will receive a token from $u$ at the latest in the same round it would have received a token from $v$ in the original schedule, and nodes $u'$ and $v$ can pretend that they received tokens from $v$ and $u$, respectively. However, it may now be the case that node $u$ ends up bothering node $u'$. We now show how to repeatedly apply resolutions to modify $S^*_1$ into a schedule $S^*_{1-2}$ in which no node bothers another in any round $t$.

Consider the graph $B_t(S^*_1)$ where the vertices are the nodes in $G$ and there exists a directed edge $(u, v)$ if node $u$ is bothering node $v$ in round $t$ in schedule $S^*_1$. First, consider cycles in $B_t(S^*_1)$. Note that, for any time $t$ in which $B_t(S^*_1)$ has a cycle, we can create a schedule $\tilde{S}^*_1$ in which no nodes in any cycle in $B_t(S^*_1)$ send their tokens in round $t$; rather, they remain idle this round and pretend they received the token they would have received under $S^*_1$. Clearly, this does not increase the length of the optimal schedule and removes all cycles in round $t$. Furthermore, this does not violate property (1) because fewer nodes send tokens in round $t$, and no new nodes send tokens in round $t$.

Therefore, it suffices to consider an acyclic, directed graph $B_t(\tilde{S}_1)$. Now, for each round $t$, we repeatedly apply resolutions until no node bothers any other node during that round. Note that for every $t$, each node can only be bothering at most one other node because nodes can only send one message at a time. This fact, coupled with the fact that $B_t(\tilde{S}_1)$ is acyclic, means that $B_t(\tilde{S}_1)$ is a DAG where nodes have out-degree 1. It is not hard to see that repeatedly applying resolutions to a node $v$ which bothers another node will decrease the number of edges in $B_t(\tilde{S}_1)$ by 1. Furthermore, because there are $n$ total nodes in the network, the number of resolutions needed for any node $v$ at time $t$ is at most $n$.

Furthermore, repeatedly applying resolutions to $B_t(\tilde{S}_1)$ for times $t = 1, \ldots, R$ in order results in a schedule $S^*_{1-2}$ with no bothersome pairs at any time $t$ and that still satisfies property (1), and so schedule $S^*_{1-2}$ satisfies properties (1) and (2). Since each resolution did not increase the length of the schedule we also have that $S^*_{1-2}$ is of length $R$. Lastly, $S^*_{1-2}$ clearly still solves Token Computation on $K_{N^*(R)}$.

**Achieving properties (1) - (3).** Now, we show how to modify $S^*_{1-2}$ into $S^*_{1-3}$ which satisfies properties (1), (2), and (3). We use our shortcutting insight here as well as some new ideas. Given $S^*_{1-2}$, we show by induction over $k$ from 0 to $R - t_m$, where $R$ is the length of an optimal schedule, that we can modify $S^*_{1-2}$ such that if a node finishes communicating in round $R - k$ (i.e., begins communicating in round $R - k - t_m$), it remains idle in rounds $t' \in (R - k, R]$ in the modified optimal schedule. The base case of $k = 0$ is trivial: If a node communicates in round $R - t_m$, it must remain idle in round $R$ because the entire schedule is of length $R$.

Suppose there exists a node $v$ that finishes communicating in round $t = R - k$ but is not idle in some round $t' > R - k$ in $S^*_{1-2}$; furthermore, let round $t'$ be the first round after $t$ in which node $v$ is not idle. By property (1), node $v$ must have sent its only token away in round $t$, and therefore node $v$ must have received at least one other token after round $t$ but before round $t'$. We now case
on the type of action node $v$ performs in round $t'$.

- If node $v$ communicates in round $t'$, it must send a token it received after time $t$ but before round $t'$. Furthermore, as this is the first round after $t$ in which $v$ is not idle, $v$ cannot have performed any computation on this token, and by the inductive hypothesis, $v$ must remain idle from round $t' + t_m$ on. Therefore, $v$ receives a token $a_u$ from some node $u$ and then forwards this token to node $u'$ at time $t'$. One can modify this schedule such that $u$ sends $a_u$ directly to $u'$ instead of sending to $v$.

- If node $v$ computes in round $t'$, consider the actions of node $v$ after round $t' + t_c$. Either $v$ eventually performs a communication after some number of computations, after which point it is idle by the inductive hypothesis, or $v$ only ever performs computations from time $t'$ on.

In round $t'$, $v$ must combine two tokens it received after time $t + t_m$ by property (1). Note that two distinct nodes must have sent the two tokens to $v$ because, by the inductive hypothesis, each node that sends after round $t$ remains idle for the remainder of the schedule. Therefore, the nodes $u_1$ and $u_2$ that sent the two tokens to $v$ must have been active at times $t'_1, t'_2 > t$, where $t_1 \leq t_2$, after which they remain idle for the rest of the schedule. Call the tuple $(v, u_1, u_2)$ a switchable triple. We can modify the schedule to make $v$ idle at round $t'$ by picking the node that first sent to $v$ and treating it as $v$ while the original $v$ stays idle for the remainder of the schedule. In particular, we can modify $S^*_{1-2}$ such that, without loss of generality, $u_2$ sends its token to $u_1$ and $u_1$ performs the computation that $v$ originally performed in $S^*_{1-2}$. Note that this now ensures that $v$ will be idle in round $t'$ and does not increase the length of the schedule, as $u_1$ takes on the role of $v$. Furthermore, node $u_1$'s new actions do not violate the inductive hypothesis: Either $u_1$ only ever performs computations after time $t'$, or it eventually communicates and thereafter remains idle.

We can repeat this process for all nodes that are not idle after performing a communication in order to produce a schedule $S^*_{1-3}$ in which property (3) is satisfied.

First, notice that these modifications do not change the length of $S^*_{1-2}$: in the first case $u'$ can still pretend that it receives $a_u$ at time $t' + t_m$ even though it now receives it in an earlier round and in the second case $u_2$ takes on the role of $v$ at the expense of no additional round overhead. Also, it is easy to see that $S^*_{1-3}$ still solves Token Computation on $K_{N^*}(R)$.

We now argue that the above modifications preserve (1) and (2). First, notice that the modifications we do for the first case do not change when any nodes send and so (1) is satisfied. In the second case, because we switch the roles of nodes, we may potentially add a send for a node. However, note that we only require a node $u_1$ to perform an additional send when it is part of a switchable triple $(v, u_1, u_2)$, and $u_1$ takes on the role of $v$ in the original schedule from time $t'$ on. However, because $S^*_{1-2}$ satisfies (1), $u$ was about to send its only token away and therefore only had one token upon receipt of the token from $u_2$. Therefore, because $u_1$ performs the actions that $v$ performs in $S^*_{1-2}$ from time $t'$ on, and because at time $t'$, both $u_1$ and $v$ have exactly two tokens, (1) is still satisfied by $S^*_{1-3}$. Next, we argue that (3) is a strictly stronger condition than (2). In particular, we show that since $S^*_{1-3}$ satisfies (3) it also satisfies (2). Suppose for the sake of contradiction that $S^*_{1-3}$ satisfies (3) but not (2). Since (2) is not satisfied there must exist some node $v$ that sends in some round $t$ to, say node $u$, but receives a token in some round in $[t, t + t_m]$. By (3) it then follows that $v$ is idle in all rounds after $t$. However, $u$ also receives a token in round $t + t_m$. Therefore, in round $t + t_m$, two distinct nodes have tokens, one of which is idle in all rounds.
after $t + t_m$; this contradicts the fact that $S_{1-3}^*$ solves Token Computation. Thus, $S_{1-3}^*$ must also satisfy (2)

**Achieving properties (1) - (4).** It is straightforward to see that $S_{1-3}^*$ also satisfies property (4). Indeed, by property (3), if the terminus ever sends in round $t < R - t_c$, then the terminus must remain idle during rounds $t' > t$, meaning it must be idle in round $R - t_c$ which contradicts the fact that in this round the terminus performs a computation. Therefore, $S_{1-4}^* = S_{1-3}^*$ satisfies properties (1) - (4), and we know that there exists an optimal schedule in which $v_t$ is always either computing or idle.

**Achieving the final property.** We now argue that we can modify $S_{1-4}^*$ into another optimal schedule $\tilde{S}^*$ such that every computation done at the terminus $v_t$ involves a token that contains the original singleton token that started at the terminus. Suppose that in $S_{1-4}^*$, $v_t$ performs computation that does not involve $a_{v_t}$. Take the first instance in which $v_t$ combines tokens $a_1$ and $a_2$, neither of which contains $a_{v_t}$, in round $t$. Because this is the first computation that does not involve a token containing $a_{v_t}$, both $a_1$ and $a_2$ must have been communicated to the terminus in round $t - t_m$ at the latest.

Consider the earliest time $t' > t$ in which $v_t$ computes a token $a_{comb}$ that contains all of $a_1$, $a_2$, and $a_{v_t}$. We now show how to modify $S_{1-4}^*$ into $\tilde{S}'$ such that $v_t$ computes a token $a_{comb}$ at time $t'$ that contains all of $a_1$, $a_2$, and $a_{v_t}$ and is at least the size of $a_{comb}$ by having nodes swap roles in the schedule between times $t$ and $t'$. Furthermore, because the rest of the schedule remains the same after time $t'$, this implies that $\tilde{S}'$ uses at most as many rounds as $S_{1-4}^*$, and therefore that $\tilde{S}'$ uses at most $R$ rounds.

The modification is as follows. At time $t$, instead of having $v_t$ combine tokens $a_1$ and $a_2$, have $v_t$ combine one of them (without loss of generality, $a_1$) with the token containing $a_{v_t}$. Now, continue executing $S_{1-4}^*$ but substitute $a_2$ for the token containing $a_{v_t}$ from round $t$ on; this is a valid substitution because $v_t$ possesses $a_2$ at time $t$. In round $t'$, $v_t$ computes a token $a'_{comb} = a_{comb}$; the difference from the previous schedule is that the new schedule has one fewer violation of property (4), i.e., one fewer round in which it computes on two tokens, neither of which contains $a_{v_t}$.

We repeat this process for every step in which the terminus does not compute on the token containing $a_{v_t}$, resulting in a schedule $\tilde{S}^*$ in which the terminus is always combining a communicated token with a token containing its own singleton token. Note that these modifications do not affect properties (1) - (4) because this does not affect the sending actions of any node, and therefore $\tilde{S}^*$ still satisfies properties (1) - (4). It easily follows, then, that $\tilde{S}^*$ solves Token Computation on $K_{N^*}(R)$ in $R$ rounds. Thus, $\tilde{S}^*$ is a schedule of length $R$ that solves Token Computation on $K_{N^*}(R)$ in which every computation the terminus $v_t$ does is on two tokens, one of which contains $a_{v_t}$, and, by (4), the terminus $v_t$ never communicates.

Having shown that the schedule corresponding to $N^*(R)$ can be modified to satisfy a nice structure when $t_c < t_m$, we can conclude our recursive bound on $N^*(R)$.

**Lemma 14.** When $t_c < t_m$, for $R \in \mathbb{Z}_0^+$ it holds that $N^*(R) = N^*(R - t_c) + N^*(R - t_c - t_m)$ for $R \geq t_c + t_m$.

**Proof.** Suppose $R \geq t_c + t_m$. We begin by applying Lemma 13 to show that $N^*(R) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$. Let $v_t$ be the terminus of the schedule $\tilde{S}^*$ using $R$ rounds as given in Lemma 13.
By Lemma 13, in all rounds after round $t_m$ of $\tilde{S}^*$ it holds that $v_i$ is either computing on a token that contains $a_{v_i}$ or busy because it did such a computation. Notice that it follows that every token produced by a computation at $v_i$ contains $a_{v_i}$.

Now consider the last token produced by our schedule. Call this token $a$. By definition of the terminus, $a$ must be produced by a computation performed by $v_i$, combining two tokens, say $a_1$ and $a_2$, in round $R - t_c$ at the latest. Since every computation that $v_i$ does combines two tokens, one of which contains $a_{v_i}$, without loss of generality let $a_1$ contain $a_{v_i}$.

We now bound the size of $a_1$ and $a_2$. Since $a_1$ exists in round $R - t_c$ we know that it is of size at most $N^*(R - t_c)$. Now consider $a_2$. Since every token produced by a computation at $v_i$ contains $a_{v_i}$ and $a_2$ does not contain $a_{v_i}$ it follows that $a_2$ must either be a singleton token that originates at a node other than $v$, or $a_2$ was produced by a computation at another node. Either way, $a_2$ must have been sent to $v$, who then performed a computation on $a_2$ in round $R - t_c$ at the latest. It follows that $a_2$ exists in round $R - t_c - t_m$, and so $a_2$ is of size no more than $N^*(R - t_c - t_m)$.

Since the size of $a$ just is the size of $a_1$ plus the size of $a_2$, we conclude that $a$ is of size no more than $N^*(R - t_c) + N^*(R - t_c - t_m)$. Since, $\tilde{S}^*$ solves Token Computation on a complete graph of size $N^*(R)$, we have that $a$ is of size $N^*(R)$ and so we conclude that $N^*(R) \leq N^*(R - t_c) + N^*(R - t_c - t_m)$ for $R \geq t_c + t_m$ when $t_c < t_m$.

Lastly, since $N^*(R) \geq N^*(R - t_c) + N^*(R - t_c - t_m)$ for $R \geq t_c + t_m$ by Lemma 11, we conclude that $N^*(R) = N^*(R - t_c) + N^*(R - t_c - t_m)$ for $R \geq t_c + t_m$ when $t_c < t_m$.

4.2.3 Putting Both Cases Together

The preceding lemmas have established that the recursive structure of $N^*(R)$ is the same as that of $|T(R)|$ for both cases. We now prove Theorem 8.

Proof of Theorem 8. On a high level, we argue that the greedy aggregation schedule on $T(R)$ combines $N^*(R)$ nodes in $R$ rounds and is therefore optimal. Combining Lemma 10, Lemma 12, and Lemma 14 we have the following recurrence on $N^*(R)$ for $R \in \mathbb{Z}_0^+$.

$$N^*(R) = \begin{cases} 1 & \text{if } R < t_c + t_m \\ N^*(R - t_c) + N^*(R - t_c - t_m) & \text{if } R \geq t_c + t_m \end{cases}$$

However, notice that this is just the recurrence which defines $|T(R)|$. Thus, for $R \in \mathbb{Z}_0^+$ we have that $N^*(R) = |T(R)|$, and by Lemma 7, the greedy aggregation schedule on $T(R)$ terminates in $R$ rounds.

Thus, the greedy aggregation schedule on $T(R)$ solves Token Computation on $K_{|T(R)|} = K_{N^*(R)}$ in $R$ rounds, and therefore is an optimal solution for $K_{N^*(R)}$. Since $T_n^*$ is the smallest $T(R)$ with at most $n$ nodes, greedy aggregation on $T_n^*$ is optimal for $K_n$ and so $\text{OptComplete}$ optimally solves Token Computation on $K_n$. Finally, as stated above, a polynomial runtime is trivial. 

5 Hardness of the Token Computation Problem

We now consider the Token Computation problem on arbitrary graphs. In this section we demonstrate that not only is Token Computation NP-complete, but no polynomial time algorithm approximates Token Computation to a factor better than 1.5.
5.1 NP-Completeness

As a warmup for our hardness of approximation result, and to introduce some of the techniques we use for our hardness of approximation, we begin with our proof that the decision version of Token Computation is NP-complete. An instance of the decision version of Token Computation is given by an instance of Token Computation and a candidate $\ell$. An algorithm must decide if there exists a schedule that solves Token Computation in at most $\ell$ rounds.

We reduce from $k$-dominating set.

**Definition 15** ($k$-dominating set). An instance of $k$-dominating set consists of a graph $G = (V,E)$; the decision problem is to decide whether there exists $\kappa \subseteq V$ where $|\kappa| = k$ such that for all $v \in V \setminus \kappa$ there exists $v' \in \kappa$ such that $(v,v') \in E$.

Recall that $k$-dominating set is NP-complete.

**Lemma 16** (Garey and Johnson [25]). $k$-dominating set is NP-complete.

Given an instance of $k$-dominating set, we would like to transform $G$ into another graph $G'$ in polynomial time such that $G'$ has a $k$-dominating set iff there exists a Token Computation schedule of some particular length for $G'$ for some values of $t_c$ and $t_m$.

We begin by describing the intuition behind the transformation we use, R. Any schedule on graph $G$ in which every node only performs a single communication and which aggregates all tokens down to at most $k$ tokens corresponds to a $k$-dominating set of $G$; in particular, those nodes that do computation form a $k$-dominating set of $G$. If we had a schedule of length $< 2t_m$ which aggregated all tokens down to $k$ tokens, then we could recover a $k$-dominating set from our schedule. However, our problem aggregates down to only a single token, not $k$ tokens. Our crucial insight, here, is that by structuring our graph such that a single node, $a$, must perform a great deal of computation, $a$ must be the terminus of any short schedule. The fact that $a$ must be the terminus and do a great deal of computation, in turn, forces any short schedule to aggregate all tokens in $G$ down to at most $k$ tokens at some point, giving us a $k$-dominating set.

Formally, R is as follows. R takes as input a graph $G$ and a value for $t_m$ and outputs $G' = (V',E')$. $G'$ has $G$ as a sub-graph and in addition has auxiliary node $a$ where $a$ is connected to all $v \in V$; $a$ is also connected to dangling nodes $d \in D$, where $|D| = \Delta + t_m$, along with a special dangling node $d^*$. Thus, $G' = (V \cup \{a,d^*\} \cup D, E \cup \{(a,v'): v' \in V' \setminus \{a\}\})$. See Figure 7.

We now prove that the optimal Token Computation schedule on $G' = R(G,t_m)$ can be upper bounded as a function of the size of the minimum dominating set of $G$.

**Lemma 17.** The optimal Token Computation schedule on $G' = R(G,t_m)$ is of length at most $2t_m + \Delta + k^*$ for $t_c = 1$, where $k^*$ is the minimum dominating set of $G$.

*Proof.* We know by definition of $k^*$ that there is a dominating set of size $k^*$ on $G$. Call this set $\kappa$ and let $\sigma : V \rightarrow \kappa$ map any given $v \in V$ to a unique node in $\kappa$ that dominates it. We argue that it must be the case that Token Computation requires at most $2t_m + t_c(\Delta + k^*)$ rounds on $G'$ for $t_c = 1$. Roughly, we solve Token Computation by first aggregating at $\kappa$ and then aggregating at $a$.

In more detail, in stage 1 of the schedule, every $d \in D$ sends to $a$, every node $v \in V$ sends to $\sigma(v)$ and $a$ sends to $d^*$. This takes $t_m$ rounds. In stage 2, each node does the following in parallel.

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$\Delta$ is the max degree of $G$.  

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Figure 7: An example of \( R \) for a given graph \( G \) and \( t_m = 1 \). Nodes and edges added by \( R \) are dashed and in blue. Notice that \( |D| = \Delta + t_m = 3 + 1 = 4 \).

Node \( d^* \) computes and sends its single token to \( a \). Each \( \nu \in \kappa \) computes until it has a single token and sends the result to \( a \). Node \( a \) combines all tokens from \( D \cup \{d^*\} \). Node \( d^* \) takes \( 1 + t_m \) rounds to do this. Each \( \nu \in \kappa \) takes at most \( \Delta + t_m \) rounds to do this since \( a \) will receive \( d^* \)'s token after \( t_m + 1 \) rounds (and \( \Delta \geq 1 \) without loss of generality). Thus, stage 2, when done in parallel, takes \( \Delta + t_m \) rounds. At this point \( a \) has \( k^* + 1 \) tokens and no other node in \( G' \) has a token. In stage 3, \( a \) computes until it has only a single token, which takes \( k^* \) rounds. In total the number of rounds used by this schedule is \( t_m + \Delta + t_m + k^* = 2t_m + \Delta + k^* \). Thus, the total number of rounds used by the optimal Token Computation schedule on \( G' \) is at most \( 2t_m + \Delta + k^* \).

Next, we show that any valid Token Computation schedule on \( G' = R(G,t_m) \) that has at most two serialized sends corresponds to a dominating set of size bounded by the length of the schedule.

**Lemma 18.** Given \( G' = R(G,t_m) \) and a Token Computation schedule \( S \) for \( G' \) where \( |S| < 3t_m \), \( t_c = 1 \), \( \kappa = \{v : v \in G, \ v \text{ sends to } a \text{ in } S\} \) is a dominating set of \( G \) of size \( |S| - 2t_m - \Delta \).

**Proof.** Roughly, we argue that \( a \) must be the terminus of \( S \) and must perform at most \( |S| - 2t_m - \Delta \) computations on tokens from \( G \), each of which is the aggregation of a node’s token and some of its neighbors’ tokens. We begin by arguing that \( a \) must be the terminus.

First, we prove that no \( d \in D \cup \{d^*\} \) is the terminus of \( S \). Suppose for the sake of contradiction that some \( d \in D \cup \{d^*\} \) is the terminus. Since our schedule takes fewer than \( 3t_m \) rounds, we know that every node sends a token that is not just the singleton token with which it starts at most once. Thus, \( a \) sends tokens that are not just the singleton token that it starts with at most once. Since \( |D \cup \{d^*\} \setminus \{d\}| = \Delta + t_m \) and \( a \) is the only node connected to these nodes, we know that every singleton token that originates in \( D \cup \{d^*\} \setminus \{d\} \) must travel through \( a \). Moreover, since \( a \) sends tokens that are not just the singleton token that it starts with at most once, \( a \) must send all such tokens as a single token. It follows that \( a \) must perform at least \( \Delta + t_m \) computations, but then our entire schedule takes at least \( t_m + \Delta + t_m + t_m = 3t_m + \Delta > 3t_m \) rounds—a contradiction to our assumption that our schedule takes less than \( 3t_m \) rounds.

We now argue that no \( v \in G \) is the terminus. Suppose for the sake of contradiction that some \( \bar{v} \in V \) is the terminus. Again, we know that \( a \) sends tokens that are not just the singleton token that
The problem is clearly in NP. To show hardness, we reduce from Theorem 19.

Assume for the sake of contradiction that \( a \) sends a token other than that with which it starts, a contradiction to the fact that our schedule takes \( < 3t_m \) rounds.

Thus, since \( a \) is the terminus, our schedule solves Token Computation in fewer than \( 3t_m \) rounds, and no computations occur in the first \( t_m \) rounds, \( a \) does at most \( |S| - t_m \) computations. Since \( a \) never sends any token aside from its singleton token, and \( a \) is the only node to which \( D \cup \{d^*\} \) are connected, we know that \( a \) must combine all tokens of nodes in \( D \cup \{d^*\} \), where \( a \) must take \( \Delta + t_m \) rounds to do so. Thus, since \( a \) takes \( \Delta + t_m \) rounds to aggregate tokens from \( D \cup \{d^*\} \) and it performs at most \( |S| - t_m \) computations in total, \( a \) must receive at most \( |S| - 2t_m - \Delta \) tokens from \( G \). It follows that \(|\kappa| \leq |S| - 2t_m - \Delta\).

Since each token sent by a node in \( \kappa \) to \( a \) must be sent at the latest in round \( |S| - t_m \) and since \(|S| < 3t_m\), we have that every token sent by a node in \( \kappa \) is formed in fewer than \( 2t_m \) rounds. It follows that each such token is formed by tokens that travel at most 1 hop in \( G \). Since every node in \( G \) must eventually aggregate its tokens at \( a \), it follows that every node in \( G \) is adjacent to a node in \( \kappa \). Thus \( \kappa \) is a dominating set of \( G \), and as shown before \(|\kappa| \leq |S| - 2t_m - \Delta\).

Having shown that the optimal Token Computation schedule of \( G' = R(G, t_m) \) is closely related to the size of the minimum dominating set, we prove that Token Computation is NP-complete.

**Theorem 19.** The decision version of Token Computation is NP-complete.

**Proof.** The problem is clearly in NP. To show hardness, we reduce from \( k \)-dominating set. Specifically, we give a polynomial-time Karp reduction from \( k \)-dominating set to the decision version of Token Computation.

Our reduction is as follows. First, run \( R(G, t_m) \) for \( t_m = \Delta + k + 1 \) to get back \( G' \). Next, return a decision version instance of Token Computation given by graph \( G' \) with \( t_m = \Delta + k + 1 \), \( t_c = 1 \) and \( \ell = 2t_m + \Delta + k \). We now argue that \( G' \) has a schedule of length \( \ell \) iff \( G \) has a \( k \)-dominating set.

- Suppose that \( G \) has a \( k \)-dominating set. We know that \( k \geq k^* \), where \( k^* \) is the minimum dominating of \( G \), and so by Lemma 17 we know that \( G' \) has a schedule of length at most \( 2t_m + \Delta + k^* \leq 2t_m + \Delta + k \).

- Suppose that \( G' \) has a Token Computation schedule \( S \) of length at most \( 2t_m + \Delta + k \). Notice that by our choice of \( t_m \), we have that \(|S| = 2t_m + \Delta + k < 3t_m \) and so by Lemma 18 we know that \( \kappa = \{v : v \in G, v \text{ sends to } a \text{ in } S\} \) is a dominating set of \( G \) of size \(|S| - 2t_m - \Delta \). Since \(|S| \leq 2t_m + \Delta + k \) we conclude that \(|\kappa| = |S| - 2t_m - \Delta \leq k\).
Lastly, notice that our reduction, R, runs in polynomial time since it adds at most a polynomial number of vertices and edges to G. Thus, we conclude that k-dominating is polynomial-time reducible to the decision version of Token Computation, and therefore the decision version of Token Computation is NP-complete.

### 5.2 Hardness of Approximation

In this section, we show that unless P = NP there exists no polynomial-time algorithm that approximates Token Computation multiplicatively better than 1.5. Specifically, unless P = NP, no polynomial-time algorithm can produce a schedule that uses at most 1.5 − ϵ times more rounds than the optimal schedule for ϵ \( \geq \frac{1}{o(\log n)} \).

Recall that k-dominating set is \( \Omega(\log n) \) hard to approximate.

**Lemma 20** (Dinur and Steurer [20]). *Unless P = NP every polynomial-time algorithm approximates minimum dominating set at best within a \((1 - o(1))(\log n)\) multiplicative factor.*

We prove hardness of approximation by using a \((1.5 - \epsilon)\) algorithm for Token Computation to approximate minimum dominating set with a polynomial-time algorithm better than \(O(\log n)\). Similar to our proof of NP-completeness, given input graph \(G\) whose minimum dominating set we would like to approximate, we would like to transform \(G\) into another graph \(G'\) such that a \((1.5 - \epsilon)\)-approximate Token Computation schedule for \(G'\) allows us to recover an approximately minimum dominating set. We denote copy \(i\) as \(G_i\).

**Lemma 21.** *Given \(G'_D = R(G_D, t_m)\) and a valid Token Computation schedule \(S\) for \(G'_D\) where \(|S| < 3t_m\), \(t_c = 1\) and \(\epsilon \in (0, 1]\), DSFROMSCHEDULE outputs in polynomial time a dominating set of \(G\) of size \(\frac{1}{2}(|S| - 2t_m - \Delta)\).*

---

8 Since the max degree of \(G\) and \(G_D\) are the same, throughout this section \(\Delta\) will be used to refer to both the max degree of \(G\) and the max degree of \(G_D\).

9 Since this lemma allows for \(\epsilon \in (0, 1]\), it may appear that we will be able to achieve an arbitrarily good approximation for minimum dominating set. In fact, it might even seems as though we can produce a dominating set of size smaller than the minimum dominating set by simply letting \(\epsilon\) be arbitrarily small. However, this is not the case. Intuitively, the smaller \(\epsilon\) is, the larger \(G_D\) is and so the longer any feasible schedule \(S\) must be. Thus, decreases in \(\epsilon\) are balanced out by increases in \(|S|\) with respect to the size of our dominating set, \(\frac{1}{2}(|S| - 2t_m - \Delta)\).
Algorithm 2 DSFromSchedule

Input: $G_D' = R(G_D, t_m)$; a valid Token Computation schedule for $G_D'$, $S$, of length $< 3t_m$; $\epsilon$

Output: A dominating set for $G$ of size $|S| - 2t_m - \Delta$

$K \leftarrow \emptyset$

for $i \in \left[\frac{\Delta}{\epsilon}\right]$ do

$\kappa_i \leftarrow \{v \in V_i : v \in G_D$ sends to $a$ in $S\}$

$K \leftarrow K \cup \{\kappa_i\}$

return $\arg\min_{\kappa_i \in K} |\kappa_i|$

Proof. Polynomial runtime is trivial, so we focus on the size guarantee. By Lemma 18 we know that $\kappa = \{v : v \in G_D, v$ sends to $a in S\}$ is a dominating set of $G_D$ of size $|S| - 2t_m - \Delta$. Moreover, notice that $\kappa_i = \kappa \cap G_i$, and so it follows that $\kappa_i$ is a dominating set of $G_i$, or equivalently $G$ because $G_i$ is just a copy of $G$. Thus we have that $\arg\min_{\kappa_i \in K} |\kappa_i|$ will return a dominating set of $G$.

We now prove that $\arg\min_{\kappa_i \in K} |\kappa_i|$ is small. Since each $\kappa_i$ is disjoint we have $\sum_{i=1}^{\Delta/\epsilon} |\kappa_i| = |\kappa| \leq |S| - 2t_m - \Delta$. Thus, by an averaging argument we have that there must be some $\kappa_i$ such that $|\kappa_i| \leq \frac{\Delta}{\epsilon} (|S| - 2t_m - \Delta)$. It follows that $\min_{\kappa_i \in K} |\kappa_i| \leq \frac{\Delta}{\epsilon} (|S| - 2t_m - \Delta)$, meaning the $\kappa_i$ that our algorithm returns is not only a dominating set of $G$ but of size at most $\frac{\Delta}{\epsilon} (|S| - 2t_m - \Delta)$.

Lastly, we combine $R$ with DSFromSchedule to get MDSAPx, our algorithm for approximating minimum dominating set. Roughly, MDSAPx constructs $G_D'$ by applying $R$ to $G_D$, uses a $(1.5 - \epsilon)$ approximation to Token Computation to get a schedule to $G_D'$ and then uses DSFromSchedule to extract a minimum dominating set for $G$ from this schedule. MDSAPx will carefully choose a $t_m$ that is large enough so that the schedule produced by the $(1.5 - \epsilon)$ approximation for Token Computation is of length $< 3t_m$ but also small enough so that the produced schedule can be used to recover a small dominating set.

Algorithm 3 MDSAPx

Input: Graph $G$; $(1.5 - \epsilon)$ Token Computation approximation algorithm $A$

Output: An $O(1/\epsilon)$-approximation for the minimum dominating set of $G$

$D \leftarrow \emptyset$

for $\hat{k} \in [n]$ do

$G_D \leftarrow \bigcup_{i=1}^{\Delta/\epsilon} G_i$

$t_m \leftarrow \frac{1}{\epsilon} \left(\Delta + \frac{\hat{k} \Delta}{\epsilon}\right) + 1; t_c \leftarrow 1$

$G_D' \leftarrow R(G_D, t_m)$

$S_{\hat{k}} \leftarrow A(G_D', \frac{\hat{k}}{\epsilon}, t_m, t_c)$

if $|S_{\hat{k}}| < 3t_m$ then

$\kappa_{\hat{k}} \leftarrow$ DSFromSchedule$(G_D, S, \epsilon)$

$D \leftarrow D \cup \{\kappa_{\hat{k}}\}$

return $\arg\min_{\kappa \in D} |\kappa|$.  

Lemma 22. Given graph $G$ and a $(1.5 - \epsilon)$-approximation algorithm for Token Computation, $A$, MDSAPx outputs in poly $\left(n, \frac{1}{\epsilon}\right)$ time a dominating set of $G$ of size $O\left(\frac{k^*}{\epsilon}\right)$, where $k^*$ is the
Thus, since \(|S_k| - 2t_m - \Delta| = \frac{\Delta}{\epsilon}\). Thus, it suffices to show that \(D\) contains at least one dominating set of \(G\), \(k\) such that \(S_k\) is small. We do so now.

Let \(k\) be the size of the minimum dominating set of \(G\). We know that \(k \leq n\) and so in some iteration of MDSAPX we will have \(k = k^*\). Moreover, the minimum dominating set of \(G\) in this iteration just is \(\frac{\Delta}{\epsilon}\) since \(G_D\) is just \(\frac{\Delta}{\epsilon}\) copies of \(G\). Consider this iteration. Let \(S^*\) be the optimal schedule for \(G_D\) when \(k = k^*\). By Lemma 17 we know that \(|S^*| \leq 2t_m + \Delta + \frac{k^*\Delta}{\epsilon}\). We now leverage the fact that we chose \(t_m\) to be large enough so that \(|S^*| < 3t_m\). In particular, combining the fact that \(|S^*| \leq 2t_m + \Delta + \frac{k^*\Delta}{\epsilon}\) with the fact that \(A\) is a \((1.5 - \epsilon)\) approximation we have that

\[
|S_k| \leq (1.5 - \epsilon)|S^*| \\
\leq (1.5 - \epsilon) \left(2t_m + \Delta + \frac{k^*\Delta}{\epsilon}\right) \\
= (1.5 - \epsilon) \left(2t_m - 2\epsilon t_m + (1.5 - \epsilon)\left(\Delta + \frac{k^*\Delta}{\epsilon}\right)\right) \\
= (1.5 - \epsilon) \left(2t_m - 2\epsilon t_m + (1.5 - \epsilon)(t_m - 1)\right) \\
= 3t_m - \epsilon(0.5 + \epsilon)t_m - \epsilon(1.5 - \epsilon) \\
< 3t_m. \\
\tag{1}
\]

Thus, since \(|S_k| < 3t_m\) we know that \(k \leq n\). Lastly, we argue that \(|k^*| = O\left(\frac{k^*}{\epsilon}\right)\), thereby showing that \(\arg\min_{k \in D} |k|\), the returned dominating set of our algorithm, is \(O\left(\frac{k^*}{\epsilon}\right)\).

We now leverage the fact that we chose \(t_m\) to be small enough to give us a small dominating set. Applying Lemma 21 we have that

\[
|k^*| \leq \frac{\epsilon}{\Delta} \left(|S_k| - 2t_m - \Delta\right) \tag{By Lemma 21} \\
< \frac{\epsilon}{\Delta} (t_m - \Delta) \tag{By Equation (1)} \\
= \frac{\epsilon}{\Delta} \left(\frac{1}{\epsilon}\left(\Delta + \frac{k^*\Delta}{\epsilon}\right) + 1 - \Delta\right) \tag{By t_m dfn.} \\
= \left(1 + \frac{k^*}{\epsilon}\right) \frac{\epsilon}{\Delta} - \epsilon \\
= O\left(\frac{k^*}{\epsilon}\right)
\]

Thus, we conclude that MDSAPX produces an \(O\left(\frac{k^*}{\epsilon}\right)\) minimum dominating set of \(G\).

Lastly, we argue a polynomial in \(n\) and \(1/\epsilon\) runtime of MDSAPX. First we argue that each iteration requires polynomial time. Constructing \(G_D\) takes polynomial time since the algorithm need only create \(\frac{\Delta}{\epsilon} = \text{poly}\left(n, \frac{1}{\epsilon}\right)\) copies of \(G\). Running \(R\) also requires polynomial time since it simple adds polynomially many nodes to \(G_D\). \(A\) is polynomial by assumption and DSFROMSCHEDULE is polynomial by Lemma 21. Thus, each iteration takes polynomial time and since MDSAPX has \(n\) iterations, MDSAPX takes polynomial time in \(n\) and \(1/\epsilon\).
Given that MDSAPx demonstrates an efficient approximation for minimum dominating set given a polynomial-time \((1.5 - \epsilon)\) approximation for Token Computation, we conclude our hardness of approximation.

**Theorem 23.** Token Computation cannot be approximated by a polynomial-time algorithm within \((1.5 - \epsilon)\) for \(\epsilon \geq \frac{1}{o(\log n)}\) unless \(P = NP\).

**Proof.** Assume for the sake of contradiction that \(P \neq NP\) and there existed a polynomial-time algorithm \(A\) that approximated Token Computation within \((1.5 - \epsilon)\) for \(\epsilon = \frac{1}{o(\log n)}\). It follows by Lemma 22 that MDSAPx when run with \(A\) is a \(o(\log n)\)-approximation for minimum dominating set. However, this contradicts Lemma 20, and so we conclude that Token Computation cannot be approximated within \((1.5 - \epsilon)\) for \(\epsilon \geq \frac{1}{o(\log n)}\). \(\square\)

6 Approximation Algorithm for the Token Computation Problem

In this section we give a polynomial-time algorithm that produces a schedule that solves Token Computation on arbitrary graphs using at most \(O(\log n \cdot \log \frac{OPT}{t_m})\) multiplicatively more rounds than the optimal schedule, where \(OPT\) is the length of the optimal schedule. Notice that \(\frac{OPT}{t_m}\) is at most \((n-1)t_c/t_m + D\) since \(OPT \leq (n-1)(t_c + D \cdot t_m)\): the schedule that picks a pair of nodes, routes one to the other then aggregates and repeats \(n-1\) times is valid and takes \((n-1)(t_c + D \cdot t_m)\) rounds. Thus, roughly our algorithm can be understood as an \(O(\log^2 n)\) approximation algorithm.

6.1 Token Computation Extremes (Warmup)

We build intuition by considering two extremes of Token Computation in the Token Network model.

\(t_m \ll t_c\). First, consider the case where \(t_m \ll t_c\); that is, communication is very cheap compared to computation. As computation is the bottleneck in this setting, we can achieve an essentially optimal schedule by parallelizing computation as much as possible. In particular, consider a schedule consisting of \(O(\log n)\) repetitions of: (1) each node with a token uniquely pairs off with another node with a token; (2) one node in each pair routes its token to the other node in its pair; (3) every node that received a token performs one computation. In total this schedule will take \(O(t_c \cdot \log n)\) rounds to perform its computations along with some amount of time to perform communications. However, any schedule takes at least \(\Omega(t_c \cdot \log n)\) rounds, even if communication were free and computation were perfectly parallelized. Therefore, because the time to perform communications is negligible, this schedule is essentially optimal.

\(t_c \ll t_m\). Now consider the case where \(t_c \ll t_m\); that is, computation is very cheap compared to communication. In this setting, we can provide an essentially optimal schedule by minimizing the amount of communication that occurs. In particular, we pick a center \(c\) of the graph\(^{10}\) and have all nodes send their tokens along the shortest path towards \(c\). At any point during this schedule, it is always more time efficient a node with multiple tokens to combine its tokens together before forwarding them since \(t_c \ll t_m\). Thus, if at any point a node has multiple tokens, it combines these

\(^{10}\)The center of graph \(G\) is \(\text{arg min}_v \max_u d(v, u)\) where \(d(v, u)\) is the shortest path distance between \(u\) and \(v\) in \(G\).
Figure 8: An illustration of essentially-optimal schedules for the extremes of the **Token Computation** problem. Dotted red arrows give the node towards which each node routes. In the case where $t_m \ll t_c$ one would repeat this sort of routing $O(\log n)$ times.

Into one token and forwards the result towards $c$. Lastly, $c$ aggregates all tokens it receives. This schedule takes $t_m \cdot R$ time to perform its communications, where $R$ is the radius of the graph,\(^{11}\) and some amount of time to perform its computations. However, because for every schedule there exists a token that must travel at least $R$ hops, any schedule takes at least $\Omega(R \cdot t_m)$ rounds. Furthermore, computations take a negligible amount of time since $t_c \ll t_m$, which means that this schedule is essentially optimal.

See Figure 8 for an illustration of these two schedules. Thus, in the case when $t_m \ll t_c$, we have that routing between pairs of nodes and delaying computations is essentially optimal, and in the case when $t_c \ll t_m$, we have that it is essentially optimal for nodes to greedily aggregate tokens before sending. These two observations will form the foundation of our approximation algorithm.

### 6.2 Approximation Algorithm

Our approximation algorithm, **SolveTC**, performs $O(\log n)$ repetitions of: designate some subset of nodes with tokens sinks and the rest of the nodes with tokens sources; route tokens at sources to sinks. If $t_c > t_m$, our algorithm will delay computations until tokens from sources arrive at sinks, and if $t_m \geq t_c$, our algorithm will immediately aggregate tokens that arrive at the same node. More formally, we define the problem which our algorithm solves $O(\log n)$ times as follows.

**Definition 24 (Route and Compute Problem).** The input to the Route and Compute Problem consists of a set $U \subseteq V$ and a set of directed paths $\mathcal{P}_U = \{ \vec{P}_u : u \in U \}$ where: (1) $u \in U$ has a token and is the source of $\vec{P}_u$; (2) every sink of every path $\vec{P}_u$ has a token; (3) if $u$ and $t_u$ are the sources and sinks of $\vec{P}_u \in \mathcal{P}_U$ respectively then neither $u$ nor $t_u$ are endpoints of any $\vec{P}_{u'} \in \mathcal{P}_U$ for $u' \neq u$. A solution of cost $C$ is a schedule of length $C$ which, when run, performs computations on a constant fraction of tokens belonging to nodes in $U$.

**SolveTC** repeatedly calls a subroutine, **GetDirectedPaths**, to get a set of paths for which it would like to solve the Route and Compute Problem. It then solves the Route and Compute Problem for these paths, using **ROUTEPATHS$_m$** if $t_c \leq t_m$ or **ROUTEPATHS$_c$** if $t_c > t_m$.

\(^{11}\)The radius of graph $G$ is $\min_u \max_u d(v, u)$.
Algorithm 4 \texttt{SolveTC}

\textbf{Input:} Token Computation instance given by graph $G = (V,E), t_c, t_m$

\textbf{Output:} A schedule for the input Token Computation problem

$W \leftarrow V$

\textbf{for} iteration $i \in O(\log n)$ \textbf{do}

$P_U \leftarrow \text{GetDirectedPaths}(W,G)$

\textbf{if} $t_c > t_m$ \textbf{then} \text{RoutePaths}_m(P_U)$

\textbf{if} $t_c \leq t_m$ \textbf{then} \text{RoutePaths}_c(P_U)$

$W \leftarrow \{v : v \text{ has 1 token}\}$

6.2.1 Producing Paths on Which to Route

We begin by describing how we produce the paths along which we route. In particular, we describe \texttt{GetDirectedPaths}. Roughly, given that nodes in $W \subseteq V$ have tokens, \texttt{GetDirectedPaths} works as follows. \texttt{GetDirectedPaths} solves a flow LP which has a flow for each $w \in W$ whose sinks are $w' \in W$ such that $w' \neq w$. The flow for each $w \in W$ defines a probability distribution over (undirected) paths with endpoints $w$ and $w'$ where $w' \neq w$ and $w' \in W$. Given these probability distributions, we repeatedly sample paths for every $w \in W$ until we produce a set of paths with low vertex congestion. Lastly, given our undirected paths, we apply another subroutine, \texttt{AssignPaths}, to fix some subset of nodes $U \subset W$ as sources such that $\lvert U \rvert$ is within a constant fraction of $\lvert W \rvert$. Throughout this section, given a set of paths $P$, we define the vertex congestion of $P$ as $\text{con}(P) = \max_v \sum_{P \in P} (\# \text{ occurrences of } v \in P)$ and the dilation of $P$ as $\max_{P \in P} \lvert P \rvert$.

We now describe the flow LP we use given that $W = \{v : v \text{ has at least 1 token}\}$. The flow LP we use for \texttt{GetDirectedPaths} can be thought of as flow on a $G$ “time-expanded” by the maximum length that a token in the optimal schedule travels. Given any schedule we define the distance that singleton token $a$ travels as the number of times any token containing $a$ is sent in said schedule. Let $L^*$ be the furthest distance a singleton token travels in the optimal schedule. Given a guess for $L^*$, namely $\hat{L}$, we define a graph $G_{\hat{L}}$ with vertices $\{v_r : v \in V, r \in [\hat{L}]\}$ and edges $\{e = (u_r, v_{r+1}) : (u,v) \in E, r \in [\hat{L} - 1]\}$. We have a flow type for each $w \in W$ which uses $\{w' : w' \in W \wedge w' \neq w\}$ as sinks. Correspondingly, we have a flow variable, $f_w(x_r,y_{r+1})$ for every $r \in [\hat{L} - 1], w \in W$ and $(x,y) \in E$. The objective function of the LP is to minimize the maximum vertex congestion, given by variable $z$. See Figure 9 for an illustration of a feasible solution to our LP. Formally, our LP is as follows, where $\Gamma(v)$ gives the neighbors of $v$ in $G$.  

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Consider a \( W \) as above and the optimal schedule that solves \textsc{Token Computation} in time \( \text{OPT} \). The key property of our LP that we use is that it has value commensurate with \( \text{OPT} \). In particular, we show that by sending flow along paths taken by certain tokens in the optimal schedule, we can provide a feasible solution to \textsc{PathsFlowLP(\( \hat{L} \))} with value commensurate with \( \text{OPT} \). For this reason we now formally define these paths, \textsc{OptPaths(}\( W \)). Roughly, these are the paths taken by tokens containing singleton tokens that originate in \( W \). Formally, these paths are as follows. Recall that \( a_w \) is the singleton token with which \( w \) starts in the optimal schedule. Notice that in any given round of the optimal schedule exactly one token contains \( a_w \). As such, order every round in which a token containing \( a_w \) is received by a node in ascending order as \( r_0(w), r_1(w) \ldots \) where we think of \( w \) as receiving \( a_w \) in the first round. Correspondingly, let \( v_i(w) \) be the vertex that receives a token containing \( a_{u_i} \) in round \( r_i(w) \); that is \( (v_1(w), v_2(w), \ldots) \) is the path “traced out” by \( a_w \) in the optimal schedule. For token \( a \), let \( C(a) := \{ a_{w'} : w' \in W \land a_{w'} \in a \} \) stand for all singleton tokens contained by token \( a \) that originated at a \( w' \in W \). Say that token \( a \) is \textit{active} if
$|C(a)|$ is odd. Let $v_{L_u}(w)$ be the first vertex in $(v_1(w), v_2(w), \ldots)$ where an active token containing $a_w$ is combined with another active token. Correspondingly, let $c(w)$ be the first round in which an active token containing $a_w$ is combined with another active token. Say that a singleton token $a_w$ is pending in round $r$ if $r < c(w)$. We note the following behavior of pending singleton tokens.

**Lemma 25.** In every round of the optimal schedule, if a token is active then it contains exactly one pending singleton token and if a token is inactive then it contains no pending singleton tokens.

**Proof.** We prove this by induction over the rounds of the optimal schedule. As a base case, we note that in the first round of the optimal schedule a token is active if it is a singleton node and every singleton node is pending. Now consider an arbitrary round $i$ and assume that our claim holds in previous rounds. Consider an arbitrary token $a$. If $a$ is not computed on by a node in this round then by our inductive hypothesis we have that it contains exactly one pending singleton token if it is active and no pending singleton tokens if it is not active. If $a$ is active and combined with an inactive token, by our inductive hypothesis, the resulting token contains exactly one pending singleton token. Lastly, if $a$ is active and combined with another active token by our inductive hypothesis these contain pending singletons $a_w$ and $a_u$ respectively such that $c(w) = c(u) = i$; it follows that the resulting token is inactive and contains no pending singleton tokens. This completes our induction. \hfill \Box

This behavior allows us to pair off vertices in $W$ based on how their singleton tokens are combined.\footnote{Without loss of generality we assume that $|W|$ is even here; if not, we can simply drop one element from $W$ each time we construct $\text{OPTPaths}(W)$.}

**Lemma 26.** For each $w \in W$ there exists a unique $u \in W$ such that $u \neq w$ and $v_{L_u}(w) = v_{L_u}(u)$ and $c(w) = c(u)$.

**Proof.** Consider the round in which a token containing $a_w$, say $a$, is combined with an active token, say $b$, at vertex $v_{L_w}(w)$. Recall that this round is notated $c(w)$. By Lemma 25 we know that $a$ and $b$ contain exactly one pending singleton token, say $a_w$ and $a_u$ respectively. Since both $a$ and $b$ are active in this round and $b$ contains $a_u$ we have $c(u) = c(w)$. Moreover, since both $a$ and $b$ are combined at the same vertex we have $v_{L_u}(u) = v_{L_u}(w)$. Lastly, notice that this $u$ is unique since by Lemma 25 there is exactly one singleton token, $a_u$, contained by $b$ such that $c(u) \leq c(w)$. \hfill \Box

Having paired off vertices in $W$, we can now define $\text{OPTPaths}(W)$. Fix a $w$ and let $u$ be the vertex it is paired off with as in Lemma 26. We define $\text{OptPath}(w) := (v_1(w), v_2(w), \ldots, v_{L_w}(w) = v_{L_u}(w), v_{L_u-1}(w), \ldots, v_1(u))$. Lastly, define $\text{OPTPaths}(W) = \bigcup_{w \in W} \text{OptPath}(w)$. See Figure 10 for an illustration of how $\text{OPTPaths}(W)$ is constructed from the optimal schedule.

The critical property of $\text{OPTPaths}(W)$ is that it has vertex congestion commensurate with OPT as follows.

**Lemma 27.** $\text{con}(\text{OPTPaths}(W)) \leq \frac{2 \cdot \text{OPT}}{\min(t_c, t_m)}$.

**Proof.** Call a pair of directed paths in $\text{OPTPaths}(W)$ complementary if one path is $\text{OptPath}(w)$ and the other $\text{OptPath}(u)$ where $u$ is to $w$ as in Lemma 26. We argue that each pair of complementary paths passing through a given vertex $v$ uniquely account for either $t_c$ or $t_m$ rounds of $v$’s OPT rounds in the optimal schedule. Consider a pair of complementary paths, $P =$
Figure 10: An illustration of the optimal schedule and how $\text{OptPaths}(W)$ is constructed from it for a particular $G$. Active tokens are denoted by blue diamonds; inactive tokens are denoted by white diamonds; a dotted red arrow from node $u$ to node $v$ means that $u$ sends to $v$; a double-ended blue arrow between two tokens $a$ and $b$ means that $a$ and $b$ are combined at the node; thick, dashed green lines give a path and its reversal in $\text{OptPaths}(W)$ (for a total of 4 paths across all rounds) where $(v_1(w), v_2(w), \ldots v_{L_w}(w) = v_{L_u}(u), v_{L_u-1}(u), \ldots, v_1(u)) = P \in \text{OptPaths}(W)$ drawn only in round $c(w)$. Furthermore, token $a$ labeled with $\{v : a$ contains $a_v\}$ and $W = \{1, 3, 4, 6\}$. 

(OptPath(w), OptPath(u)), passing through a given vertex v. This pair of paths pass through v because in some round, say r_P, v sends a token containing a_u or a_w or v combines together tokens a and a' containing a_u and a_w respectively. Say that whichever of these operations accounts for P is responsible for P. Now suppose for the sake of contradiction that this operation of v in round r_p is responsible for another distinct pair P' of complementary paths, OptPath(w') and OptPath(u'). Notice that a_w, a_w', a_u and a_w' are all pending in round r_P. We case on whether v's action is a communication or a computation and show that v's operation cannot be responsible for P' in either case.

- Suppose that v is responsible for P and P' because it performs a computation in r_P. It follows that v combines an active token a and another active token a' where without loss of generality a_w, a_w' ∈ a and a_u, a_u' ∈ a'. However, it then follows that a is active and contains two pending singleton tokens, which contradicts Lemma 25.

- Suppose that v is responsible for P and P' because it performs a communication in r_P by sending token a. It follows that without loss of generality a_w, a_w' ∈ a. However, either a is active or it is not. But by Lemma 25 if a is active it contains 1 pending singleton token and if a is not active then it contains 0 pending singleton tokens. Thus, the fact that v sends a token containing two pending singleton tokens contradicts Lemma 25.

Thus, it must be the case that v's action in r_P is uniquely responsible for P.

It follows that each computation and communication performed by v uniquely corresponds to a pair of complementary paths (consisting of a pair of paths in OptPaths(W)) that passes through v. Since v performs at most OPT/\min(t_c, t_m) operations in the optimal schedule, it follows that there are at most OPT/\min(t_c, t_m) pairs of complementary paths in OptPaths(W) incident to v. Since each pair consists of two paths, there are at most 2·OPT/\min(t_c, t_m) paths in OptPaths(W) incident to v and so v has vertex congestion at most 2·OPT/\min(t_c, t_m) in OptPaths(W). Since v was arbitrary, this bound on congestion holds for every vertex.

We now use OptPaths(W) to construct a feasible solution for PathsFlowLP(2L^*). We let \tilde{f} be this feasible solution. Intuitively, \tilde{f} simply sends flow along the paths of OptPaths(W). More formally define \tilde{f} as follows. For w ∈ W and its corresponding path OptPath(w) = (v_1(w), v_2(w), ...) we set \tilde{f}_w(v_i, v_{i+1}) = 1. We set all other variables of \tilde{f} to 0 and let \tilde{z} be the vertex congestion of OptPaths(W).

Lemma 28. (\tilde{f}, \tilde{z}) is a feasible solution for PathsFlowLP(2L^*) where \tilde{z} ≤ 2OPT/\min(t_c, t_m).

Proof. We begin by noting that every path in OptPaths(W) is of length at most 2L^*: for each w ∈ W, OptPath(w) is the concatenation of two paths, each of which is of length no more than L^*. Moreover, notice that for each w ∈ W, the sink of OptPath(w) is a w' ∈ W such that w' ≠ w.

We now argue that (\tilde{f}, \tilde{z}) is a feasible solution for PathsFlowLP(2L^*): each vertex v with incoming w-flow that is not in W \ w sends out this unit of flow and so Equation (2) is satisfied; since each OptPath(w) is of length at most 2L^* and ends at a w' ∈ W we have that every w ∈ W is a source for f_w and not a sink for f_w, satisfying Equation (3); for the same reason, Equation (4) is satisfied; letting \tilde{z} be the vertex congestion of OptPaths(W) clearly satisfies Equation (5); and flow is trivially non-zero.

Lastly, since \tilde{f} simply sends one unit of flow along each path in OptPaths(W), our bound of \tilde{z} ≤ 2OPT/\min(t_c, t_m) follows immediately from Lemma 27.
We conclude that \( \tilde{f} \) demonstrates that our LP has value commensurate with OPT.

**Lemma 29.** \( \min(t_c, t_m) \cdot z(2L^*) \leq 2\text{OPT} \).

**Proof.** Since Lemma 28 shows that \((\tilde{f}, \tilde{z})\) is a feasible solution for \(\text{PathsFlowLP}(2L^*)\) with cost at most \(2\text{OPT}/\min(t_c, t_m)\), our claim immediately follows. \(\square\)

Having shown that our LP has value commensurate with OPT, we now provide the algorithm which we use to sample paths from our LP solution, \texttt{SampleLPPaths}. This algorithm produces a single sample by taking a random walk from each \(w \in W\) where edges are taken with probability corresponding to their LP value. It repeats this \(O(\log n)\) times to produce \(O(\log n)\) samples. It then takes the sample with the most low congestion paths, discarding any high congestion paths in said sample. In particular, \texttt{SampleLPPaths} takes the sample \(P = \{P_w : P_w \in P_W, \text{con}(P_w) \leq 10 \cdot z(\hat{L}) \log \hat{L}\}\) for an input \(\hat{L}\).

**Algorithm 5** \texttt{SampleLPPaths}(\(f_w^*\))

**Input:** \(f_w^*, \text{solution to PathsFlowLP}(\hat{L}); \hat{L}, \text{guess of } L^*; W \subseteq V\)

**Output:** Undirected paths between nodes in \(W\)

\[
\begin{align*}
C &\leftarrow \emptyset \\
\text{for } &\text{ sample } i \in O(\log n) \text{ do} \\
&\text{ } \text{ } P^i_W \leftarrow \emptyset \\
&\text{ } \text{ } \text{for } w \in W \text{ do} \\
&\text{ } \text{ } \text{ } v \sim f_w^*(w_1, v_2) \\
&\text{ } \text{ } \text{ } P_w \leftarrow (w, v) \\
&\text{ } \text{ } \text{while } v \not\in W \text{ do} \\
&\text{ } \text{ } \text{ } \text{ } v' \sim f_w^*(v|P_w|, v'|P_w|+1) \\
&\text{ } \text{ } \text{ } \text{ } v \leftarrow v' \\
&\text{ } \text{ } \text{ } \text{ } P_w^{+} = v \\
&\text{ } \text{ } \text{ } \text{ } P^i_W \leftarrow P^i_W \cup \{P_w\} \\
&\text{ } \text{ } C \leftarrow C \cup P^i_W \\
&\text{ } \text{ } P_S \leftarrow Q(\arg \max_{P_w \in C} |Q(P_w)|) \\
&\text{ } \text{return } P_S \\
\end{align*}
\]

The properties of \texttt{SampleLPPaths} are as follows.

**Lemma 30.** For any fixed \(W \subseteq V\), \(\hat{L}\) and an optimal solution \(f_w^*\) to \(\text{PathsFlowLP}(\hat{L})\), \texttt{SampleLPPaths} is a polynomial-time randomized algorithm that outputs a set of undirected paths \(P_W\) such that \(P_w \in P_W\) is an undirected path with endpoints \(w, w' \in S\) where \(w \neq w'\). Also \(|P_W| \geq \frac{1}{3}|W|\) w.h.p., \(\text{con}(P_W) \leq z(\hat{L}) \cdot O(\log \hat{L})\), and \(\text{dil}(P_W) \leq \hat{L}\).

**Proof.** Our proof consists of a series of union and Chernoff bounds over our samples. Consider an arbitrary \(W \subseteq V\). Define \(T_w\) for \(w \in W\) as the (directed) subgraph of \(G_{\hat{L}}\) containing arc \((v, u) \in E_{\hat{L}}\) if for some \(r\) we have \(f_w^*(x, y) > 0\). Notice that \(T_w\) is a weakly connected DAG where \(w\) has no edges into it: \(T_w\) does not contain any cycles since flow only moves from \(x_r\) to \(y_{r+1}\) for \(x, y \in V\); by our flow constraints \(T_w\) must be weakly connected and \(w_r\) must have no edges into it for any \(r\). Moreover, notice that \(P_w\) is generated by a random walk on \(T_w\) starting at \(w_1\), where if the last
vertex added to $P_w$ was $v$, then we add $u$ to $P_w$ in step $r$ of the random walk with probability $f^*_w(v_r, u_{r+1})$.

We first argue that every $P_w \in P_W$ has endpoints $w, w' \in W$ for $w \neq w'$ and $\text{dil}(P_W) \leq \hat{L}$. By construction, one endpoint of $P_w$ is $w$. Moreover, the other endpoint of $P_w$ will necessarily be a $w' \in W$ such that $w' \neq w$: by Equation (2) flow is conserved and by Equation (4) all flow from $w$ must end at a point $w' \in W$ such that $w' \neq w$; thus our random walk will always eventually find such an $w'$. Moreover, notice that our random walk is of length at most $\hat{L}$ since $T_w$ is of depth at most $\hat{L}$. Thus, every $P_w$ is of length at most $\hat{L}$, meaning $\text{dil}(P_W) \leq \hat{L}$.

Next, notice that, by the definition of $Q$, $\text{con}(P_W) \leq z(\hat{L}) \cdot O(\log \hat{L})$ by construction since every element in $Q(\arg \max_{P_w \in C} |Q(P_W)|)$ has $O(z(\hat{L}) \cdot O(\log \hat{L}))$ congestion.

Thus, it remains only to prove that $|P_W| \geq \frac{1}{3}|W|$. We begin by arguing that for a fixed path $P_w$ in a fixed set of sampled paths, $P_W$ we have $\text{con}(P_w) \geq z(\hat{L}) \cdot O(\log \hat{L})$ with probability at most $\frac{1}{3}$. Consider a fixed path $P_w \in P_W$ and fix an arbitrary $v \in P_w$. Now let $X_{uw}$ stand for the random variable indicating the number of times that path $P_w$ visits vertex $w$. Without loss of generality we know that $P_w$ contains no cycles (since if it did we could just remove said cycles) and so $X_{uw}$ is either 0 or 1. By a union bound over rounds, then, we have $\mathbb{E}[X_{uw}] \leq \sum_{r} \sum_{u \in \Gamma(v)} f^*_w(u_r, v_{r+1}) \cdot \text{Pr}(u \text{ taken in }(r-1)\text{th step}) \leq \sum_{u \in \Gamma(v)} \sum_{r} f^*_w(u_r, v_{r+1})$.

Now note that the congestion of a single vertex under our solution is just $\text{con}(v) = \sum_{w \in W} X_{uw}$. It follows that $\mathbb{E}[\text{con}(v)] = \sum_{w \in W} \mathbb{E}[X_{uw}] \leq \max_{v} \sum_{w} \sum_{u \in \Gamma(v)} \sum_{r} f^*_w(u_r, v_{r+1}) \leq z(\hat{L})$. Also notice that for a fixed $v$ every $X_{uw}$ is independent. Thus, we have by a Chernoff bound that that

\[
\Pr(\text{con}(v) \geq z(\hat{L}) \cdot O(\log \hat{L})) \leq \Pr\left(\sum_{w \in W} X_{uw} \geq \mathbb{E}\left[\sum_{w \in W} X_{uw}\right] \cdot O(\log \hat{L})\right) \leq \frac{1}{(L)c} \tag{7}
\]

for $c$ given by constants of our choosing. $P_w$ is of length at most $\hat{L}$ by construction. Thus, by a union over $v \in P_w$ and Equation (7) we have that

\[
\Pr\left(\text{con}(P_w) \geq z(\hat{L}) \cdot O(\log \hat{L})\right) \leq \frac{1}{Lc-1} \leq \frac{1}{3}
\]

Thus, for a fixed path $P_w \in P^i_W$ we know that this path has congestion at least $z(\hat{L}) \cdot O(\log \hat{L})$ with probability at most $\frac{1}{3}$.

We now argue at least one of our $O(\log n)$ samples is such that at least $\frac{1}{3}$ of the paths in the sample have congestion at most $z(\hat{L}) \cdot O(\log \hat{L})$. Let $Y_{uw}$ be the random variable that is 1 if $P_w \in P^i_W$ is such that $\text{con}(P_w) \geq z(\hat{L}) \cdot O(\log \hat{L})$ and 0 otherwise. Notice that $\mathbb{E}[Y_{uw}] \leq \frac{1}{3}$ by the fact that a path has congestion at least $z(\hat{L}) \cdot O(\log \hat{L})$ with probability at most $\frac{1}{3}$. Now let $Z_i = \sum_{w \in W} Y_{uw}$ stand for the number of paths in sample $i$ with high congestion. By linearity of expectation we have $\mathbb{E}[Z_i] \leq |W| \frac{1}{3}$. By Markov’s inequality we have for a fixed $i$ that $\Pr(Z_i \geq \frac{2}{3}|W|) \leq \Pr(Z_i \geq 2 \mathbb{E}[Z_i]|W|) \leq \frac{1}{3}$. Now consider the probability that every sample $i$ is such that more than $\frac{2}{3}$ of the paths have congestion more than $z(\hat{L}) \cdot O(\log \hat{L})$, i.e. consider the probability that for all $i$ we have $Z_i \geq \frac{2}{3}|W|$. We have
from $G_P$ in-degree of at least two we do the following: if $\Gamma$ consists only of nodes with in-degree at most 1 and out-degree at most 1. The remaining graph, with in-degree of two or more and every vertex has out-degree consisting of nodes $W_P$ returns a set of directed paths, \[ \text{con} \]

Lemma 32. Given $W \subseteq V$, \text{GetDirectedPaths} is a randomized polynomial-time algorithm that returns a set of directed paths, $\bar{P}_U = \{P_u : u \in U\}$ for $U \subseteq W$, such that with high probability at least 1/12 of nodes in $W$ are sources of paths in $\bar{P}_U$ each with a unique sink in $W$. Moreover, $\text{con}(\bar{P}_U) \leq O\left(\frac{\text{OPT}}{\text{min}(t_c, t_m)} \log \frac{\text{OPT}}{t_m}\right)$ and $\text{dil}(\bar{P}_U) \leq \frac{8 \text{OPT}}{t_m}$.

Thus, with high probability there will be some sample, $i$, such that $Z_i \leq |W|^2/3$. It follows that with high probability $\max_{P_w \in C} |Q(P_w)| \geq 1/4 |W|$ and since $P_W = Q(\arg \max_{P_w \in C} |Q(P_w)|)$, we conclude that with high probability $P_W \geq |W|^{1/2}$.
Lemma 31.

Proof. The fact that $\text{GetDirectedPaths}$ returns a set of directed paths, $\mathcal{P}_U$, such that at least $1/12$ of nodes in $W$ are sources in a path with a sink in $W$ follows directly from Lemma 30 and Lemma 31.

We now give the stated bounds on congestion and dilation. First notice that $2L^* \in [R]$. Moreover, $2\text{OPT} \leq 2(n-1)(t_c + D \cdot t_m)$: the schedule that picks a pair of nodes, routes one to the other then aggregates and repeats $n-1$ times is always feasible and takes $(n-1)(t_c + D \cdot t_m)$ rounds. Thus, $2L^* \leq 2\text{OPT} \leq R$.

Thus, by definition of $L$ we know that

$$t_m \cdot L + \min(t_c, t_m) \cdot t(L) \leq 2t_m \cdot L^* + \min(t_c, t_m) \cdot z(2L^*)$$

(By Lemma 29)

$$\leq 2L^* + 2\text{OPT}$$

(By dfn. of $L^*$)

It follows, then, that $t_m \cdot L \leq 4\text{OPT}$ and so $L \leq \frac{4\text{OPT}}{t_m}$. Similarly, we know that $\min(t_c, t_m) \cdot z(L) \leq 4\text{OPT}$ and so $z(L) \leq \frac{4\text{OPT}}{\min(t_c, t_m)}$.

Lastly, by Lemma 30 we know that $\text{dil} (\mathcal{P}_W) \leq L \leq \frac{4\text{OPT}}{t_m}$ and $\text{con} (\mathcal{P}_W) \leq t(L) \cdot O(\log L) \leq O \left( \frac{\text{OPT}}{\min(t_c, t_m)} \cdot \log \frac{\text{OPT}}{t_m} \right)$. By Lemma 31 we get that the same congestion bound holds for $\mathcal{P}_U$ and $\text{dil} (\mathcal{P}_U) \leq \frac{\text{OPT}}{\min(t_c, t_m)}$.

A polynomial runtime comes from the fact that we solve at most $(n-1)(t_c + D \cdot t_m) = \text{poly}(n)$ LPs and then sample at most $(n-1)(t_c + D \cdot t_m)$ edges $O(\log n)$ times to round the chosen LP. □

6.2.2 Routing Along Produced Paths

We now specify how we route along the paths produced by $\text{GetDirectedPaths}$. In particular, if $t_c \leq t_m$ we use one subroutine, $\text{RoutePaths}_c$, and if $t_c > t_m$ we use another subroutine, $\text{RoutePaths}_m$.

$t_c > t_m$ In this section we provide a polytime algorithm, $\text{RoutePaths}_m$, that we use to solve the Route and Compute Problem when $t_c > t_m$. $\text{RoutePaths}_m$ adapts the routing algorithm of Leighton et al. [44] (simplified by Rothvoß [61]) to efficiently route from sources to sinks.\footnote{Our approach for the case when $t_c > t_m$ can be simplified using techniques from Srinivasan and Teo [69]. In fact, using their techniques we can even shave the $\frac{\text{OPT}}{t_c}$ factor in our approximation. However, because these techniques do not take computation into account, they do not readily extend to the case when $t_c \leq t_m$. Thus, for the sake of a unified exposition, we omit the adaptation of their results.} We

\begin{algorithm}
\begin{algorithmic}
\State \textbf{Input:} $W \subseteq V$ where $w \in W$ has a token
\State \textbf{Output:} Paths between nodes in $W$
\State $L \leftarrow \arg\min_{L \in [R]} \left[ t_m \cdot L + \min(t_c, t_m) \cdot t(L) \right]$
\State $f_w^* \leftarrow \text{PATHSFLOWLP}(L)$
\State $\mathcal{P}_W \leftarrow \text{SAMPLELPPaths}(f_w^*, L, W)$
\State $\mathcal{P}_U \leftarrow \text{ASSIGNPATHS}(\mathcal{P}_W, W)$
\State return $\mathcal{P}_U$
\end{algorithmic}
\caption{$\text{GetDirectedPaths}(G, W)$}
\end{algorithm}
let $\text{OPTROUTE}$ be this adaptation of the algorithm of Leighton et al. [44] and describe it and its properties now.

**Lemma 33** (Adapted from Leighton et al. [44] and Rothvoß [61]). Given a set of directed paths $\mathcal{P}_U$ with some subset of endpoints of paths in $\mathcal{P}_U$ designated sources and the rest of the endpoints designated sources, $\text{OPTROUTE}$ is a randomized polynomial-time algorithm that produces a Token Network schedule that sends from all sources to sinks in $O(\text{con}(\mathcal{P}_W) + \text{dil}(\mathcal{P}_W))$.

**Proof.** Given a set of paths $\mathcal{P}_U$, Rothvoß [61] provide a polynomial-time algorithm that produces a schedule that routes along all paths in $O(\text{con}_E(\mathcal{P}_U) + \text{dil}(\mathcal{P}_U))$ where $\text{con}_E(\mathcal{P}) = \max_e \sum_{P \in \mathcal{P}} \mathbb{1}(e \in P)$ is the edge congestion. However, the algorithm of Rothvoß [61] assumes that in each round a vertex can send a token along each of its incident edges whereas we assume that in each round a vertex can only forward a single token.

However, it is easy to use the algorithm of Rothvoß [61] to produce an algorithm that produces a Token Network routing schedule using $O(\text{con}(\mathcal{P}_U) + \text{dil}(\mathcal{P}_U))$ rounds which assumes that vertices only send one token per round as we assume in the Token Network model as follows. Let $G$ be our input network with paths $\mathcal{P}_U$ along which we would like to route where we assume that vertices can only send one token per round. We will produce another graph $G'$ on which to run the algorithm of Rothvoß [61]. For each node $v \in G$ add nodes $v_i$ and $v_o$ to $G'$. Project each path $P \in \mathcal{P}_U$ into $G'$ to get $P' \in \mathcal{P}_U'$ as follows: if edge $(u, v)$ is in path $P \in \mathcal{P}_S$ then add edge $(v_o, v_i)$ and edge $(v_i, v_o)$ to path $P'$ in $G'$. Notice that $\text{con}(\mathcal{P}_U) = \text{con}_E(\mathcal{P}_U')$ and $\text{dil}(\mathcal{P}_U) = 2\text{dil}(\mathcal{P}_U')$.

Now run Rothvoß [61] on $G'$ with paths $\mathcal{P}_U'$ to get back some routing schedule $S'$.

Without loss of generality we can assume that $S'$ only has nodes in $G'$ send along a single edge in each round: every $v_i$ is incident to a single outbound edge across all paths (namely $(v_i, v_o)$) and so cannot send more than one token per round; every $v_o$ has a single incoming edge and so receives at most one token per round which, without loss of generality, we can assume $v_o$ sends as soon as it receives (it might be the case that $v_o$ collects some number of tokens over several rounds and then sends them all out at once but we can always just have $v_o$ forward these tokens as soon as they are received and have the recipients “pretend” that they do not receive them until $v_o$ would have sent out many tokens at once).

Now generate a routing schedule for $G$ as follows: if $v_o$ sends token $a$ in round $r$ of $S'$ then $v$ will send token $a$ in round $r$ of $S$. Since $S$ only ever has vertices send one token per round, it is easy to see by induction over rounds that $S$ will successfully route along all paths. Moreover, $S$ takes as many rounds as $S'$ which by Rothvoß [61] we know takes $O(\text{con}(\mathcal{P}_U') + \text{dil}(\mathcal{P}_U')) = O(\text{con}(\mathcal{P}_U) + 2\text{dil}(\mathcal{P}_U)) = O(\text{con}(\mathcal{P}_U) + \text{dil}(\mathcal{P}_U))$. Thus, we let $\text{OPTROUTE}$ be the algorithm that returns $S$.

Now given $\mathcal{P}_U$, $\text{ROUTEPATHS}_m$ is as follows. Run $\text{OPTROUTE}$ and then perform a single computation. Notice that, as earlier mentioned, this algorithm delays computation until all tokens have been routed from sources to sinks.

**Lemma 34.** Given $\mathcal{P}_U$, $\text{ROUTEPATHS}_m$ in polynomial-time solves the Route and Compute Problem w.h.p. using $O(t_m(\text{con}(\mathcal{P}_U) + \text{dil}(\mathcal{P}_U)) + t_e)$ rounds.

**Proof.** By Lemma 33 $\text{OPTROUTE}$ takes $t_m(\text{con}(\mathcal{P}_U) + \text{dil}(\mathcal{P}_U))$ rounds to route all sources to sinks. Lastly, all sources are combined with sinks in the following computation and so $\text{ROUTEPATHS}_m$ successfully solves the Route and Compute Problem since every source has its token combined with another token. The polynomial runtime of our algorithm is trivial.
\( t_c \leq t_m \) We now present \textsc{RoutePaths}_c, the Route and Compute algorithm we use when \( t_c \leq t_m \). The main intuition we would like to formalize is that if \( t_c \leq t_m \) then greedily combining tokens is always advantageous; that is, any time a node has two tokens it is always better for the node to combine these two tokens than to send just one of them.

Given directed paths \( \tilde{P}_U \), \textsc{RoutePaths}_c is as follows. Initially, every sink is asleep and every other node is awake. For \( O(\text{dil}(\tilde{P}_U) \cdot t_m) \) rounds we repeat the following: if a node is not currently sending and has exactly one token then it forwards this token; if a node is not currently sending and has two or more tokens then it sleeps for the remainder of the \( O(\text{dil}(\tilde{P}_U) \cdot t_m) \) rounds. Lastly, every node combines any tokens it has for \( t_c \cdot \text{con}(\tilde{P}_U) \) rounds.

**Lemma 35.** Given \( \tilde{P}_U \), \textsc{RoutePaths}_c solves the Route and Compute Problem w.h.p. using \( O(t_c \cdot \text{con}(\tilde{P}_U) + t_m \cdot \text{dil}(\tilde{P}_U)) \) rounds.

**Proof.** We argue that every source’s token either ends at an asleep node with at least two tokens and no more than \( \text{con}(\tilde{P}_U) \) tokens. It follows that our computation at the end at least halves the number of tokens.

First notice that if a vertex falls asleep then it will receive at most \( \text{con}(\tilde{P}_S) \) tokens by the end of our algorithm since it is incident to at most this many paths. Moreover, notice that every token will either end at a sink or a sleeping vertex and every sleeping vertex is asleep because it has two or more tokens. It follows that every token is combined with at least one other token and so our schedule at least halves the total number of tokens.

The length of our schedule simply comes from noting that we have \( O(\text{dil}(\tilde{P}_U) \cdot t_m) \) forwarding rounds followed by \( \text{con}(\tilde{P}_U) \cdot t_c \) rounds of computation. Thus, we get a schedule of total length \( O(t_c \cdot \text{con}(\tilde{P}_S) + t_m \cdot \text{dil}(\tilde{P}_S)) \). A polynomial runtime is trivial.

\[ \square \]

### 6.2.3 Putting It All Together

Lastly, we conclude that \textsc{SolveTC} solves \textsc{Token Computation} up to a multiplicative \( O(\log n - \log \frac{\text{OPT}}{t_m}) \) factor.

**Theorem 36.** \textsc{SolveTC} is a polynomial-time \( O(\log n \cdot \log \frac{\text{OPT}}{t_m}) \)-approximation for \textsc{Token Computation} w.h.p.

**Proof.** By Lemma 32 we know that the paths returned by \textsc{GetDirectedPaths}, \( \tilde{P}_U \) are such that \( \text{con}(\tilde{P}_U) \leq O\left(\frac{\text{OPT}}{\min(t_c,t_m)} \cdot \log \frac{\text{OPT}}{t_m}\right) \) and \( \text{dil}(\tilde{P}_U) \leq 8\text{OPT} \cdot \frac{t_m}{t_m} \) and the paths returned have unique sinks and sources in \( W \) and there are at least \( |W|/12 \) paths w.h.p.

If \( t_c > t_m \) then \textsc{RoutePaths}_m is run which by Lemma 34 solves the Route and Compute Problem in \( O(t_m \cdot \text{con}(\tilde{P}_U) + t_m \cdot \text{dil}(\tilde{P}_U) + t_c) \) rounds which is

\[
\begin{align*}
\leq O\left(t_m \cdot \frac{\text{OPT}}{\min(t_c,t_m)} \cdot \log \frac{\text{OPT}}{t_m} + t_m \cdot 8\text{OPT} \cdot \frac{t_m}{t_m} + t_c\right) \\
= O\left(\text{OPT} \cdot \log \frac{\text{OPT}}{t_m} + t_c\right)
\end{align*}
\]

If \( t_c \leq t_m \) then \textsc{RoutePaths}_c is run to solve the Route and Compute Problem which by Lemma 35 takes \( O(t_c \cdot \text{con}(\tilde{P}_U) + t_m \cdot \text{dil}(\tilde{P}_U)) \) rounds which is
\[ \leq O \left( t_c \cdot \frac{4OPT}{\min(t_c, t_m)} \cdot \log \frac{OPT}{t_m} + t_m \cdot \frac{8OPT}{t_m} \right) \]

\[ = O \left( OPT \cdot \log \frac{OPT}{t_m} \right) \]

Thus, in either case we produce a schedule that takes at most \( O \left( OPT \cdot \log \frac{OPT}{t_m} + t_c \right) \) rounds to solve the Route and Compute Problem on at least \(|W|/12\) paths in each iteration. Since solving the Route and Compute Problem reduces the total number of tokens by a constant fraction on the paths over which it is solved, and we have at least \(|W|/12\) paths in each iteration w.h.p., by a union bound, every iteration reduces the total number of tokens by a constant fraction w.h.p. Thus, the concatenation of the \( O(\log n) \) schedules produced, each of length \( O(OPT \cdot \log \frac{OPT}{t_m} + t_c) \), is sufficient to reduce the total number of tokens to 1.

Thus, \textsc{SolveTC} produces a schedule that solves the problem of \textsc{Token Computation} in \( O(OPT \cdot \log n \log \frac{OPT}{t_m} + t_c \cdot \log n) \) rounds. However, notice that \( t_c \cdot \log n \leq OPT \) (since the optimal schedule must perform at least \( \log n \) serialized computations) and so the produced schedule is of length \( O(OPT \cdot \log n \log \frac{OPT}{t_m} + t_c \cdot \log n) \leq O(OPT \cdot \log n \log \frac{OPT}{t_m}) \). Lastly, a polynomial runtime is trivial given the polynomial runtime of our subroutines.

\section{Future Work}

There are many promising directions for future work. First, as Section 6.1 illustrated, the extremes of our problem—when \( t_c \ll t_m \) and when \( t_m \ll t_c \)—are trivial to solve. However, our reduction in Section 5 demonstrates that for \( t_c = 1 \) and \( t_m \) taking on values of about \( \Delta \), our problem is hard to approximate. Therefore, determining precisely what values of \( t_m \) and \( t_c \) make our problem hard to approximate is an open question. Similar issues have been studied in the context of random constraint satisfaction problems [3].

Next, because it is not always the case that there exists a centralized coordinator to produce a schedule, we hope to eventually give an analysis of our problem in a distributed setting. While this has received some attention in the past (see Section 2), no past work in this setting takes computation into account. Even more broadly, we hope to analyze formal models of distributed computation in which nodes are not assumed to have unbounded computational resources and computation takes a non-trivial amount of time.

We also note that there is a gap between our hardness of approximation and the approximation guarantee of our algorithm. Whether our problem is \( \Omega(\log n) \) hard to approximate or \( O(1) \)-approximable by a polynomial-time algorithm is a question to be decided by future work.

Furthermore, we are interested in studying technical challenges similar to those studied in approximation algorithms for network design. For instance, we are interested in the problem in which each edge has a cost and one must build a network subject to some budget constraints which has as efficient a \textsc{Token Computation} schedule as possible.

Lastly, there are many natural generalizations of our problem. For instance, consider the problem in which nodes can aggregate an arbitrary number of tokens together, but the time to aggregate multiple tokens is, e.g., a submodular function of the number of tokens aggregated. Additionally, one could consider a directed version of this problem to capture settings in which it is only the case that subordinates send information to superiors to aggregate. We note that,
in effect, our optimal construction behaves like a directed network, but this is not the case when solving Token Computation on general graphs. These new directions offer not only compelling theoretical challenges but may be of practical interest.

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A Formal Definition of the Token Computation Problem

Formally, the Token Computation problem is as follows. The input to the problem is a Token Network network specified by graph \( G = (V, E) \), parameters \( t_c, t_m \in \mathbb{N} \) and an initial configuration of how many token each node has given by \( I : V \to \mathbb{N} \) (throughout this paper we have assumed \( I(v) = 1 \) for all \( v \in V \)). An algorithm for this problem must provide a schedule, \( S : V \times [l] \to V \cup \{\text{idle, busy}\} \) where we refer to \( |S| := l \) as the length of the schedule. Intuitively, a schedule \( S \) directs each node when to compute and when to communicate as follows:

- \( S(v, r) = v' \neq v \) indicates that \( v \) begins passing a token to \( v' \) in round \( r \) of \( S \);
- \( S(v, r) = v \) indicates that \( v \) begins combining two token in round \( r \) of \( S \);
- \( S(v, r) = \text{idle} \) indicates that \( v \) does nothing in round \( r \);
- \( S(v, r) = \text{busy} \) indicates that \( v \) is currently communicating or computing.

Moreover, we define the number of computations that \( v \) has performed up to round \( r \) as follows.

\[
C_S(v, r) := \sum_{r' \in [r-t_c]} 1(S(v, r') = v) \tag{8}
\]

We let the number of messages that \( v \) has received up to round \( r \) be defined as follows.

\[
R_S(v, r) := \sum_{r' \in [r-t_m]} \sum_{v' \neq v} 1(S(v', r') = v) \tag{9}
\]

We let the number of messages that \( v \) has sent up to round \( r \) be defined as follows.

\[
M_S(v, r) := \sum_{r' \in [r-t_m]} \sum_{v' \neq v} 1(S(v', r') = v') \tag{10}
\]

Lastly, we define the number of token a node has in round \( r \) of \( S \) as follows.

\[
tokens_S(v, r) := I(v) + R_S(v, r) - M_S(v, r) - C_S(v, r) \tag{11}
\]

A schedule, \( S \), is valid for Token Network \((G, t_c, t_m)\) if:

1. Valid communication: If \( S(v, r) = v' \neq v \) then \((v, v') \in E, S(v, r') = \text{busy} \) for \( r' \in [r+1, r+t_m] \) and \( tokens_S(v, r) \geq 1 \);
2. Valid computation: If \( S(v, r) = v \) then \( S(v, r') = \text{busy} \) for \( r' \in [r+1, r+t_c] \) and \( tokens_S(v, r) \geq 2 \);
3. Full aggregation: \( \sum_{v \in V} tokens_S(v, |S|) = 1 \).

An algorithm solves Token Computation if when given an input Token Network it outputs a valid schedule.
B  Additional Related Work

B.1  Additional Related Theoretical Work

Approximation Algorithms for Network Design  There are several decades of work on approximation algorithms for network design. Broadly, this work considers how, given a cost for each edge in a graph, one can buy a subgraph satisfying certain properties of minimal cost. For example, Goemans et al. [26] considers how to buy edges to satisfy certain connectivity constraints. For a nice survey see Gupta and Könemann [30].

Distributed Graph Algorithms for Computing Functions  There is also a wealth of distributed algorithms for computing functions [8, 31, 38, 48, 55]. In this setting, there is no centralized algorithm coordinating nodes as in our setting; rather, nodes themselves are responsible for dictating their own behaviour given a restricted notion of the structure of the network. The models considered in these spaces—such as the LOCAL [45, 55] and CONGEST [55] models—also assume that nodes have unbounded computation and so fail to capture how communication and computation interact as our Token Network model does. We note that our algorithm for the Token Computation problem applies to the Token Computation problem where nodes communicate as in the LOCAL or CONGEST model but still take time to do computation. Although nodes can send multiple messages to neighbors in each round in these models, it is not hard to see that nodes could always send a single value when solving the Token Computation problem without increasing the total length of the output schedule.

Games on Networks  A bit farther afield, there has been significant research related to strategic games, information structure, and optimal language for communicating on networks. Some previous work considers the setting where agents in a network participate in a strategic game with a common overarching goal by sharing private information along edges in the network [13]. In our setting, however, we consider centralized solutions to the aggregation problem, meaning agents do not play strategic, partial-information games with one another. Multiple groups have examined the relative benefits of fully centralized knowledge, where every agent in the network receives the same signal, versus fully decentralized knowledge, where every agent receives a different signal, and determine in which settings each paradigm does better [17, 57]. We only consider the fully decentralized information model, as the fully centralized knowledge model would render the aggregation problem trivial. Additionally, in the spirit of Arrow [5], some research groups have considered the optimal language for communicating within organizations, but we assume that there is perfect transmission of information between pairs of agents [18].

B.2  Additional Related Applied Work

There is a significant body of related work in resource-aware scheduling and sensor networks that considers both the relative costs of communication and computation, often bundled together in an energy cost. However, these studies have been largely empirical rather than theoretical, and much of the work considers distributed algorithms.

Resource-Aware Scheduling  In the distributed computation space, people have considered resource-aware scheduling on a completely connected topology with different nodes having different
loads. Although this problem considers computation-aware communication, these studies are much more empirical than theoretical, and only consider distributed solutions as opposed to centralized algorithms [46, 70].

**Sensor Networks** Members of the sensor networks community have studied the problem of minimizing an energy cost, which succinctly combines the costs of communication and computation. However, sensor networks involve rapidly-changing, non-static topologies [1, 2], which means that their objective is not to construct a fixed, optimal topology, but rather to develop adaptive algorithms for minimizing total energy cost with respect to an objective function [4].

C **Deferred Proofs**

**Lemma 37.** \( \text{OptComplete} \) is a polynomial-time algorithm.

*Proof.* First notice that by dynamic programming \( T(R) \) can be computed in polynomial time for \( R = \text{poly}(n) \). Next, notice that \( |T(R)| < |T(R + t_c + t_m)| \) since \( T(R + t_c + t_m) \) contains \( T(R) \) as a subtree but also contains at least one additional node. As such, we have that \( R^* \leq n(t_c + t_m) = \text{poly}(n) \).

Since \( T(R) \) can be computed efficiently and \( R^*(n) \) is at most \( \text{poly}(n) \) by naively searching over values of \( R^*(n) \) we can compute \( T_n^* = T(R^*(n)) \). Embedding a graph of \( n \) nodes into \( K_n \) is trivially achievable in \( \text{poly}(n) \) time. Lastly, the greedy aggregation schedule is also trivially computable in polynomial time. \( \Box \)