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When Can the Maximin Share Guarantee Be Guaranteed?

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Abstract

The fairness notion of *maximin share (MMS) guarantee* underlies a deployed algorithm for allocating indivisible goods under additive valuations. Our goal is to understand when we can expect to be able to give each player his MMS guarantee. Previous work has shown that such an *MMS allocation* may not exist, but the counterexample requires a number of goods that is exponential in the number of players; we give a new construction that uses only a linear number of goods. On the positive side, we formalize the intuition that these counterexamples are very delicate by designing an algorithm that provably finds an MMS allocation with high probability when valuations are drawn at random.

1 Introduction

We study the classic problem of *fairly* allocating *indivisible* goods among several players. This situation typically arises in inheritance cases, where a specific collection — containing, say, jewelry or artworks — is divided between several heirs, without the use of monetary payments. From the AI viewpoint, the overarching goal is to mediate such situations by constructing computer programs that can propose intelligent compromises, and, indeed, a large body of recent work in AI focuses on building the foundations necessary to achieve this goal (Bouveret and Lang 2008; Procaccia 2009; Cohler et al. 2011; Brams et al. 2012; Bei et al. 2012; Aumann, Dombb, and Hassidim 2013; Kurokawa, Lai, and Procaccia 2013; Brânzei and Miltersen 2013; Chen et al. 2013; Aziz et al. 2014; Karp, Kazachkov, and Procaccia 2014; Dickerson et al. 2014; Balkanski et al. 2014; Brânzei and Miltersen 2015; Li, Zhang, and Zhang 2015).

Formally, let the set of players be $N = \{1, \ldots, n\}$, and let the set of goods be G, with $|G| = m$. We denote the value of player $i \in N$ for good $g \in G$ by $V_i(g) \geq 0$. We assume that the valuations of the players are *additive*, that is, for a bundle of items $S \subseteq G$, we assume that $V_i(S) = \sum_{g \in S} V_i(g)$. We are interested in finding an *allocation* $A_1, \ldots, A_n$ — this is a partition of G where A_i is the bundle of goods allocated to player $i \in N$.

Let us now revisit the first sentence above — what do we mean by "fairly"? Before presenting the fairness notion we are interested in, let us briefly discuss two others. An allocation is *envy free* if for all $i, j \in N$, $V_i(A_i) \geq V_i(A_j)$; and it is *proportional* if for all $i \in N$, $V_i(A_i) \geq V_i(G)/n$. Note that, in our setting, any envy-free allocation is also proportional. While these notions are compelling — and provably feasible in some fair division settings, such as cake cutting (Brams and Taylor 1996; Procaccia 2013) — they cannot always be achieved in our setting (say for example when there are two players and one good).

We therefore focus on a third fairness notion: *maximin share (MMS)* guarantee, introduced by Budish (2011). The MMS guarantee of player $i \in N$ is

$$\text{MMS}(i) = \max_{S_1,\ldots,S_n} \min_{j \in N} V_i(S_j),$$

where $S_1, \ldots, S_n$ is a partition of the set of goods G; a partition that maximizes this value is known as an *MMS partition*. In words, this is the value player i can achieve by dividing the goods into n bundles, and receiving his least desirable bundle. Alternatively, this is the value i can *guarantee* by partitioning the items, and then letting all other players choose a bundle before he does. An *MMS allocation* is an allocation $A_1, \ldots, A_n$ such that for all $i \in N$, $V_i(A_i) \geq \text{MMS}(i)$. In contrast to work on maximizing the minimum value of any player (Bansal and Sviridenko 2006; Asadpour and Saberi 2007; Roos and Rothe 2010), MMS is a "Boolean" fairness notion. Also note that a proportional allocation is always an MMS allocation, that is, proportionality is a stronger fairness property than MMS.

It is tempting to think that in our setting (additive valuations), an MMS allocation always exists. In fact, extensive experiments by Bouveret and Lemaître (2014) did not yield a single counterexample. Alas, it turns out that (intricate) counterexamples do exist (Procaccia and Wang 2014). On the positive side, *approximate* MMS allocations are known to exist. Specifically, it is always possible to give each player a bundle worth at least $2/3$ of his MMS guarantee, that is, there exists an allocation $A_1, \ldots, A_n$ such that for all $i \in N$, $V_i(A_i) \geq \frac{2}{3}\text{MMS}(i)$ (Procaccia and Wang 2014). Furthermore, very recent work by Amanatidis et al. (2015) achieves the same approximation ratio in polynomial time.

These theoretical results have already made a significant real-world impact through *Spliddit* (`www.spliddit.org`), a not-for-profit fair division website (Goldman and Procaccia 2014). Since its launch in November 2014, Spliddit has attracted more than 55,000 users. The website currently offers five applications, for dividing goods, rent, credit, chores, and fare. Spliddit's algorithm for dividing goods, in particular, elicits additive valuations (which is easy to do), and maximizes social welfare (the total value players receive) subject to the highest feasible level of fairness among envy-freeness, proportionality, and MMS. If envy-freeness and proportionality are infeasible, the algorithm computes the maximum α such that all players can receive an α fraction of their MMS guarantee; since $\alpha \geq 2/3$ (Procaccia and Wang 2014), the solution is, in a sense, provably fair. The website summarizes the method's fairness guarantees as follows:

> *"We guarantee each participant at least two thirds of her maximin share. In practice, it is extremely likely that each participant will receive at least her full maximin share."*

Our goal in this paper is to better understand the second sentence of this quote: When is it possible to find an (exact) MMS allocation? And how "likely" is it?

Our results. Our first set of results has to do with the following question: what is the maximum $f(n)$ such that every instance with n players and $m \leq f(n)$ goods admits an MMS allocation? The previously known counterexample to the existence of MMS allocations uses a huge number of goods — n^n, to be exact (Procaccia and Wang 2014). Hence, $f(n) \leq n^n - 1$. Our first major result drastically improves this upper bound: an MMS allocation may not exist even when the number of goods is *linear* in the number of players.

Theorem 2.1. *For all $n \geq 3$, there is an instance with n players and $m \leq 3n+4$ goods such that an MMS allocation does not exist.*

That is, $f(n) \leq 3n + 3$. On the other hand, Bouveret and Lemaître (2014) show that $f(n) \geq n + 3$. As a bonus result, we show in the full version of the paper[1] that $f(n) \geq n + 4$.

The counterexamples to the existence of MMS allocations are extremely delicate, in the sense that an MMS allocation does exist if the valuations are even slightly perturbed. In addition, as mentioned above, randomly generated instances did not contain any counterexamples (Bouveret and Lemaître 2014). We formalize these observations by considering the regime where for each $i \in N$ there is a distribution $\mathcal{D}_i$ such that the values $V_i(g)$ are drawn independently from $\mathcal{D}_i$.

Theorem 3.1 *Assume that for all $i \in N$, $\mathbb{V}[\mathcal{D}_i] \geq c$ for a constant $c > 0$. Then for all $\varepsilon > 0$ there exists $K = K(c, \varepsilon)$ such that if $\max(n, m) \geq K$, then the probability that an MMS allocation exists is at least $1 - \varepsilon$.*

In words, an MMS allocation exists with high probability as the number of players *or* the number of goods goes to infinity. It was previously known that an envy-free allocation (and, hence, an MMS allocation) exists with high probability when $m \in \Omega(n \ln n)$ (Dickerson et al. 2014). Our analysis therefore focuses on the case of $m \in O(n \ln n)$. In this case, an envy-free allocation is unlikely to exist (such an allocation certainly does not exist when $m < n$), but (as we show) the existence of an MMS allocation is still likely. Specifically, we develop an allocation algorithm and show that it finds an MMS allocation with high probability. The algorithm's design and analysis leverage techniques for matching in random bipartite graphs.

[1] Available from `http://procaccia.info/research`.

2 Dependence on the Number of Goods

The main result of this section is the following theorem:

Theorem 2.1. *For all $n \geq 3$, there is an instance with n players and $m \leq 3n+4$ goods such that an MMS allocation does not exist.*

Note that when $n = 2$, an MMS allocation is guaranteed to exist: simply let player 1 divide the goods into two bundles according to his MMS partition, and let player 2 choose. Player 1 then obviously receives his MMS guarantee, whereas player 2 receives a bundle worth at least $V_2(G)/2 \geq \text{MMS}(2)$. The result of Procaccia and Wang (2014) shows that an MMS allocation may not exist even when $n = 3$ and $m = 12$ which proves the theorem for $n = 3$, but, as noted in Section 1, their construction requires n^n goods in general.

Because the new construction that proves Theorem 2.1 is somewhat intricate, we relegate the detailed proof to the full version of the paper. Here we explicitly provide the special case of $n = 4$. To this end, let us define the following two matrices, where ε is a very small positive constant ($\varepsilon = 1/16$ will suffice).

$$S = \begin{bmatrix} \frac{7}{8} & 0 & 0 & \frac{1}{8} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{8} \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & \varepsilon^4 & 0 & -\varepsilon^4 \\ \varepsilon^3 & 0 & -\varepsilon^3 + \varepsilon^2 & -\varepsilon^2 \\ 0 & -\varepsilon^4 + \varepsilon & 0 & \varepsilon^4 - \varepsilon \\ -\varepsilon^3 & -\varepsilon & \varepsilon^3 - \varepsilon^2 & \varepsilon^2 + \varepsilon \end{bmatrix}$$

Let $M = S + T$. Crucially, the rows and columns of M sum to 1. Let G contain goods that correspond to the nonzero elements of M, that is, for every entry $M_{i,j} > 0$ we have a good $g_{i,j}$; note that $|G| = 14 \leq 3n + 4$.

Next, partition the 4 players into $P = \{1, 2\}$ and $Q = \{3, 4\}$. Define the valuations of the players in P as follows where $0 < \tilde{\varepsilon} \ll \varepsilon$ ($\tilde{\varepsilon} = 1/64$ will suffice).

$$M + \begin{bmatrix} 0 & 0 & 0 & -\tilde{\varepsilon} \\ 0 & 0 & 0 & -\tilde{\varepsilon} \\ 0 & 0 & 0 & -\tilde{\varepsilon} \\ 0 & 0 & 0 & 3\tilde{\varepsilon} \end{bmatrix}$$

That is, the values of the rightmost column are perturbed. For example, for $i \in P$, $V_i(g_{1,4}) = 1/8 - \varepsilon^4 - \tilde{\varepsilon}$. Similarly, for players in Q, the values of the bottom row are perturbed:

$$M + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\tilde{\varepsilon} & -\tilde{\varepsilon} & -\tilde{\varepsilon} & 3\tilde{\varepsilon} \end{bmatrix}$$

It is easy to verify that the MMS guarantee of all players is 1. Moreover, the unique MMS partition of the players in P (where every subset has value 1) corresponds to the columns of M, and the unique MMS partition of the players in Q corresponds to the rows of M. If we divide the goods by columns, one of the two players in Q will end up with a bundle of goods worth at most $1 - \tilde{\varepsilon}$ — less than his MMS value of 1. Similarly, if we divide the goods by rows, one of the players in P will receive a bundle worth only $1 - \tilde{\varepsilon}$. Any other partition of the goods will ensure that some party does not achieve their MMS value due to the relative size of $\tilde{\varepsilon}$.

3 Random Valuations

The counterexamples to the existence of MMS allocations — Theorem 2.1 and the construction of Procaccia and Wang (2014) — are very sensitive: tiny random perturbations are extremely likely to invalidate them. Our goal in this section is to prove MMS allocations do, in fact, exist with high probability, if a small amount of randomness is present.

To this end, let us consider a probabilistic model with the following features:

1. For all $i \in N$, $\mathcal{D}_i$ denotes a probability distribution over $[0, 1]$.
2. For all $i \in N, g \in G$, $V_i(g)$ is randomly sampled from $\mathcal{D}_i$.
3. The set of random variables $\{V_i(g)\}_{i \in N, g \in G}$ is mutually independent.

We will establish the following theorem:

Theorem 3.1. *Assume that for all $i \in N$, $\mathbb{V}[\mathcal{D}_i] \geq c$ for a constant $c > 0$. Then for all $\varepsilon > 0$ there exists $K = K(c, \varepsilon)$ such that if $\max(n, m) \geq K$, then the probability that an MMS allocation exists is at least $1 - \varepsilon$.*

In words, as long as each $\mathcal{D}_i$ has constant variance, if either the number of players or the number of goods goes to infinity, there exists an MMS allocation with high probability. In parallel, independent work, Amanatidis et al. (2015) establish (as one of several results) a special case of Theorem 3.1 where each $\mathcal{D}_i$ is the uniform distribution over $[0, 1]$. Dealing with arbitrary distributions presents significant technical challenges, and is also important in terms of explaining the abovementioned experiments, which cover a wide range of distributions. Yet the result of Amanatidis et al. is not completely subsumed by Theorem 3.1, as they carefully analyze the rate of convergence to 1.

Our starting point is a result by Dickerson et al. (2014), who study the existence of envy-free allocations. They show that an envy-free allocation exists with high probability as $m \to \infty$, as long as $n \in O(m/\ln m)$, and the distributions $\mathcal{D}_i$ satisfy the following conditions for all $i, j \in N$:

1. $\mathbb{P}\left[\arg\max_{k \in N} V_k(g) = \{i\}\right] = 1/n$.
2. There exist constants μ, μ^* such that

$$0 < \mathbb{E}\left[V_i(g) \;\middle|\; \arg\max_{k \in N} V_k(g) = \{j\}\right] \leq \mu < \mu^* \leq \mathbb{E}\left[V_i(g) \;\middle|\; \arg\max_{k \in N} V_k(g) = \{i\}\right].$$

The proof uses a naïve allocation algorithm: simply give each good to the player who values it most highly. The first condition then implies that each player receives roughly $1/n$ of the goods, and the second condition ensures that each player has higher expected value for each of his own goods compared to goods allocated to other players.

It turns out that, via only slight modifications, their theorem can largely work in our setting. That is, alter their allocation algorithm to give a good g to a player i who believes g is in the top $1/n$ of their probability distribution $\mathcal{D}_i$. If there are multiple such players, choose one uniformly at random and if no such player exists, give it to any player uniformly at random.

This procedure is fairly straightforward for continuous probability distributions. For example, if player i's distribution $\mathcal{D}_i$ is uniform over the interval $[0, 1]$ then he believes g is in the top $1/n$ of $\mathcal{D}_i$ if $V_i(g) \geq (n-1)/n$. However, distributions with atoms require more care. For example, suppose D_i is $1/3$ with probability $7/8$ and uniform over $[1/2, 1]$ with probability $1/8$. Then if $n = 3$, i believes g is in the top $1/n$ of $\mathcal{D}_i$ if $V_i(g) > 1/3$ or if $V_i(g) = 1/3$ he should believe it is in his top $1/n$ only $1/n - 1/8 = 5/24$ of the time. To implement such a procedure, when sampling from $\mathcal{D}_i$, we should first sample from the uniform distribution over $[0, 1]$. If our sampled value is at least $(n - 1)/n$ we will say i has drawn from his top $1/n$. We then convert our sampled value to a sampled value from $\mathcal{D}_i$ by applying the inverse CDF.

Utilizing the observation that any envy-free allocation is also an MMS allocation we can then restate the result of Dickerson et al. (2014) as the following lemma, whose proof is relegated to the full version of the paper.

Lemma 3.2 ((Dickerson et al. 2014))**.** *Assume that for all $i \in N$, $\mathbb{V}[\mathcal{D}_i] \geq c$ for a constant $c > 0$. Then for all $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that if $m \geq K$ and $m \geq \alpha n \ln n$, for some $\alpha = \alpha(c)$, then the probability that an MMS allocation exists is at least $1 - \varepsilon$.*

Note that the statement of Lemma 3.2 is identical to that of Theorem 3.1, except for two small changes: only m is assumed to go to infinity, and the additional condition $m \geq \alpha n \ln n$. So it only remains to deal with the case of $m < \alpha n \ln n$. We can handle this scenario via consideration of the case $m < n^{8/7}$ — formalized in the following lemma.

Lemma 3.3. *For all $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that if $n \geq K$ and $m < n^{8/7}$, then the probability that an MMS allocation exists is at least $1 - \varepsilon$.*

Note that this lemma actually does not even require the minimum variance assumption, that is, we are proving a stronger statement than is needed for Theorem 3.1.

It is immediately apparent that when the number of goods is relatively small, we will not be able to prove the existence of MMS allocations via the existence of envy-free allocations. For example, envy-free allocations certainly do not exist if $m < n$, and are provably highly unlikely to exist if $m = n + o(n)$ (Dickerson et al. 2014). Our approach, to which we devote the remainder of this section, is significantly more intricate.

3.1 Proof of Lemma 3.3

We assume that $m > n$, because an MMS allocation always exists when $m \leq n$. We will require the following notions and lemma.

Definition 3.4. A *ranking* of the goods G for some player $i \in N$ is the order of the goods by value from most valued to least. Ties are broken uniformly at random. Furthermore, a good g's *rank* for a player i is the position of g in i's ranking.

An important observation of the rankings that we will use often throughout this section is that the players' rankings are independent of each other.

Definition 3.5. Suppose $X \subseteq N$ and $Y \subseteq G$ where $|X| \leq |Y|$. Let

$$s = |X|\lceil |Y|/|X| \rceil - |Y|,$$

and Γ be the bipartite graph where:

1. L represents the vertices on the left, and R on the right.
2. L is comprised of $\lfloor |Y|/|X| \rfloor$ copies of the first s players of X and $\lceil |Y|/|X| \rceil$ copies of the other players.
3. $R = Y$.
4. The i^{th} copy of a player has an edge to a good g iff g's rank is in $((i-1)\Delta, i\Delta]$ in the player's ranking where $\Delta = \ln^3 n$.

Note that $|L| = |R|$ since if we let $x = |X|$ and $y = |Y|$ (and therefore $s = x\lceil y/x \rceil - y$). Then

$$\begin{aligned}|L| &= s\lfloor y/x \rfloor + (x-s)\lceil y/x \rceil \\ &= x\lceil y/x \rceil - s\left(\lceil y/x \rceil - \lfloor y/x \rfloor\right).\end{aligned}$$

If x divides y, then we have that $\lceil y/x \rceil = \lfloor y/x \rfloor = \frac{y}{x}$ and so $|L| = y$. If, on the other hand, x does not divide y, then we have that $\lceil y/x \rceil - \lfloor y/x \rfloor = 1$ and so we have

$$\begin{aligned}|L| &= x\lceil y/x \rceil - s \\ &= x\lceil y/x \rceil - (x\lceil y/x \rceil - y) \\ &= y.\end{aligned}$$

Therefore, in either case, $|L| = y = |Y| = |R|$.

The *matched draft* on X and Y is the process of constructing Γ and producing an allocation corresponding to a perfect matching of Γ. That is, if a perfect matching exists then a player in X is given all goods the copies of it are matched to. In the event that no perfect matching exists, the matched draft is said to fail.

Lemma 3.6. *Suppose of the $m < n^{8/7}$ goods $x = \gamma\lfloor m/n \rfloor$ are randomly chosen and removed, where $\gamma \leq n^{1/3}$, and the remaining $\tilde{m} := m - x$ goods are allocated via a matched draft to $\tilde{n} := n - \gamma$ players. Then this matched draft succeeds with probability $\to 1$ as $n \to \infty$ (note that as $n \to \infty$, so too do $\tilde{n}, \tilde{m}$).*

Proof. Define d as the minimum degree of a vertex of L in Γ and $D = 2\lg n \ln n$. Then we have

$$\begin{aligned}&\mathbb{P}\left[\text{matched draft fails}\right] \\ &= \mathbb{P}\left[\text{matched draft fails} \mid d < D\right]\mathbb{P}\left[d < D\right] \\ &\quad + \mathbb{P}\left[\text{matched draft fails} \mid d \geq D\right]\mathbb{P}\left[d \geq D\right] \\ &\leq \mathbb{P}\left[d < D\right] + \mathbb{P}\left[\text{matched draft fails} \mid d \geq D\right].\end{aligned}$$

Let us consider these two terms separately and show they $\to 0$ as $n \to \infty$.

If $x = 0$ we have that $\mathbb{P}[d < D] = 0$ for sufficiently large n, so let us assume $x > 0$. Denoting by p_D^{ij} the probability that player i has less than D of the goods ranked in positions $((j-1)\Delta, j\Delta]$ remaining, we have

$$\mathbb{P}\left[d < D\right] \leq \sum_{i=1}^{\tilde{n}} \sum_{j=1}^{\tilde{m}/\tilde{n}} p_D^{ij}.$$

The right hand side is equal to $\tilde{m}$ times the probability that player 1 has less than D of the goods ranked in the top Δ positions remaining, which is equal to $\tilde{m}$ times the probability that of the x randomly chosen goods, more than $\Delta - D$ are ranked in the top Δ positions for player 1.

Now let the random variable X denote the number of the x random goods ranked in the top Δ for player 1. Clearly $\mathbb{E}[X] = \frac{\Delta x}{\tilde{m}}$. Thus by Markov's inequality we have that

$$\begin{aligned}&\mathbb{P}\left[X > \Delta - D\right] \\ &\leq \frac{\mathbb{E}[X]}{\Delta - D} \\ &= \left(\frac{\Delta x}{\tilde{m}}\right)\frac{1}{\Delta - D} \\ &= \left(\frac{(\ln^3 n)(\gamma\lfloor m/n \rfloor)}{m - \gamma\lfloor m/n \rfloor}\right)\frac{1}{\ln^3 n - 2\lg n \ln n} \\ &\leq \left(\frac{n^{10/21}\ln^3 n}{n - n^{10/21}}\right)\frac{1}{\ln^3 n - 2\lg n \ln n} \\ &\to 0.\end{aligned}$$

Next let us consider $\mathbb{P}[\text{matched draft fails} \mid d \geq D]$. We would like to appeal to the plethora of results on perfect matchings in bipartite Erdös-Rényi graphs (Bollobás 2001) or random bipartite k-out graphs (McDiarmid 1980), but due to the lack of independence on the edge existences we do not satisfy a crucial assumption of much of this literature, and more importantly its proofs. We will therefore prove this in full here via an approach that allows us to ignore the dependence. We will utilize Hall's theorem and denote by $N(X)$ the set of neighbors of X in the bipartite graph Γ.

$$\begin{aligned}&\mathbb{P}\left[\text{matched draft fails} \mid d \geq D\right] \\ &= \mathbb{P}\left[\exists\, X \subseteq L \text{ s.t. } |X| < |N(X)| \mid d \geq D\right] \\ &\leq \sum_{X \subseteq L} \mathbb{P}\left[|X| < |N(X)| \mid d \geq D\right] \\ &\leq \sum_{i=D}^{\tilde{m}} \sum_{\substack{X \subseteq L \\ |X|=i}} \sum_{\substack{Y \subseteq R \\ |Y|=i-1}} \mathbb{P}\left[N(X) \subseteq Y \mid d \geq D\right].\end{aligned}$$

If the edges of Γ were independent then we would find that for $|X| = i$ and $|Y| = i - 1$,

$$\mathbb{P}\left[N(X) \subseteq Y\right] = \left(\frac{i-1}{\tilde{m}}\right)^{\sum_{x \in X}|N(x)|},$$

and more importantly

$$\mathbb{P}\left[N(X) \subseteq Y \mid d \geq D\right] \leq \left(\frac{i-1}{\tilde{m}}\right)^{iD}. \qquad (1)$$

Via our independence assumptions in our randomized setting there is only one form of dependence in the edges of Γ. Specifically, if we take all copies of any player $i \in L$, then their neighbors in R never intersect. Though this does indeed introduce dependence into our system, note that we still have that Equation (1) as the dependence only lowers the probability of $N(X)$ "fitting" into Y. We therefore find

$$\begin{aligned}
&\mathbb{P}\left[\text{matched draft fails} \mid d \geq D\right] \\
&\leq \sum_{i=D}^{\tilde{m}} \sum_{\{X \subseteq L \mid |X|=i\}} \sum_{\{Y \subseteq R \mid |Y|=i-1\}} \left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&= \sum_{i=D}^{\tilde{m}} \binom{\tilde{m}}{i}\binom{\tilde{m}}{i-1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&\leq \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \binom{\tilde{m}}{i}\binom{\tilde{m}}{i-1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&\quad + \sum_{i=\lceil \tilde{m}/2 \rceil}^{\tilde{m}} \binom{\tilde{m}}{\tilde{m}-i}\binom{\tilde{m}}{\tilde{m}-i+1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&= \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \binom{\tilde{m}}{i}\binom{\tilde{m}}{i-1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&\quad + \sum_{j=0}^{\lfloor \tilde{m}/2 \rfloor} \binom{\tilde{m}}{j}\binom{\tilde{m}}{j+1}\left(\frac{\tilde{m}-j-1}{n}\right)^{(\tilde{m}-j)D}.
\end{aligned}$$

We now show both of these terms separately $\to 0$ as $n \to \infty$.

First,

$$\begin{aligned}
&\sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \binom{\tilde{m}}{i}\binom{\tilde{m}}{i-1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&\leq \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}e}{i}\right)^{i}\left(\frac{\tilde{m}e}{i-1}\right)^{i-1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&\leq \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}e}{i-1}\right)^{2i-1}\left(\frac{i-1}{\tilde{m}}\right)^{iD} \\
&= \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{i-1}{\tilde{m}}\right)^{i(D-2)+1} e^{2i-1} \\
&\leq \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \frac{e^{2i-1}}{2^{i(D-2)+1}} \\
&\leq \sum_{i=D}^{\lfloor \tilde{m}/2 \rfloor} \frac{4e^2}{2^D} \\
&\leq \frac{2e^2 n^{8/7}}{n^{2\ln n}} \\
&\to 0,
\end{aligned}$$

where the first inequality follows from the fact that $\binom{a}{b} \leq \left(\frac{ae}{b}\right)^b$ for $b > 0$, and the third inequality follows from the fact that $i \leq \lfloor \tilde{m}/2 \rfloor$.

Second,

$$\begin{aligned}
&\sum_{j=0}^{\lfloor \tilde{m}/2 \rfloor} \binom{\tilde{m}}{j}\binom{\tilde{m}}{j+1}\left(\frac{\tilde{m}-j-1}{\tilde{m}}\right)^{(\tilde{m}-j)D} \\
&\leq \tilde{m}\left(\frac{\tilde{m}-1}{\tilde{m}}\right)^{\tilde{m}D} \\
&\quad + \sum_{j=1}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}e}{j}\right)^{j}\left(\frac{\tilde{m}e}{j+1}\right)^{j+1}\left(\frac{\tilde{m}-j-1}{\tilde{m}}\right)^{(\tilde{m}-j)D} \\
&\leq \tilde{m}\left(1-\frac{1}{\tilde{m}}\right)^{\tilde{m}D} \\
&\quad + \sum_{j=1}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}e}{j}\right)^{2j+1}\left(1-\frac{j+1}{\tilde{m}}\right)^{(\tilde{m}-j)D} \\
&\leq \tilde{m}e^{-D} + \sum_{j=1}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}e}{j}\right)^{2j+1} e^{-D(j+1)(\tilde{m}-j)/\tilde{m}} \\
&\leq \tilde{m}e^{-D} + \sum_{j=1}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}e}{j}\right)^{2j+1} e^{-D(j+1)/2} \\
&\leq \tilde{m}e^{-D} + \sum_{j=1}^{\lfloor \tilde{m}/2 \rfloor} \left(\frac{\tilde{m}^2e^2}{j^2e^{D/2}}\right)^{j+1} \\
&\leq \frac{n^{8/7}}{n^{2\lg n}} + \sum_{j=1}^{\lfloor n^{8/7}/2 \rfloor} \left(\frac{(n^{8/7})^2e^2}{j^2n^{\lg n}}\right)^{j+1} \\
&\leq \frac{n^{8/7}}{n^{2\lg n}} + \lfloor n^{8/7}/2 \rfloor \left(\frac{(n^{8/7})^2e^2}{n^{\lg n}}\right) \\
&\to 0,
\end{aligned}$$

where the first inequality follows from $\binom{a}{b} \leq \left(\frac{ae}{b}\right)^b$ for $b > 0$ and the third inequality follows from $1 + x \leq e^x$ for all x.

Thus, we find that as $n \to \infty$ the matched draft succeeds with probability $\to 1$. ■

We are now ready to prove the lemma.

Proof of Lemma 3.3. Recall that we may assume that $m > n$. We will ensure every player has at most one less good than any other player. Let s then represent the number of players that receive one less good than any other player, that is,

$$s = n\lceil m/n \rceil - m.$$

We consider two separate cases here.

Case 1: $s \leq n^{1/3}$. In this scenario we do the following.

1. If possible, give each of the first s players their top $\lfloor m/n \rfloor$ goods. Otherwise, fail to produce any allocation.
2. Hold a matched draft for the remaining $(n-s)\lceil m/n \rceil$ goods and $n-s$ players.

We first show that as $n \to \infty$ this procedure successfully produces an allocation with probability $\to 1$.

Consider the probability that the first step of the procedure successfully completes. That is, the first s players each get their top $\lfloor m/n \rfloor$ goods. Similarly to a birthday paradox like argument we get that this occurs with probability at least

$$\prod_{i=1}^{s\lfloor m/n \rfloor} \left(1 - \frac{i-1}{m}\right) > \left(1 - \frac{sm/n}{m}\right)^{sm/n} \geq \left(1 - \frac{1}{n^{2/3}}\right)^{n^{10/21}}.$$

But as

$$\lim_{x\to\infty} \left(1 - \frac{1}{\omega(x)}\right)^{x} = 1$$

we find that this too goes to 1 as $n \to \infty$.

Now consider the second step of the procedure. By Lemma 3.6 with $\gamma = s$, we know that this succeeds with probability 1 as $n \to \infty$. Therefore the entire procedure will successfully complete with probability $\to 1$ as $n \to \infty$.

Therefore, to prove the theorem, it suffices to show that if the procedure successfully completes, then we have an MMS allocation. Since for every player any MMS partition must include a subset with at most $\lfloor m/n \rfloor$ goods and the first s players are given their top $\lfloor m/n \rfloor$ goods, they must receive their MMS value.

Let us turn our attention then to the remaining $n-s$ players. Upon successful completion of the matched draft, we know that all of these players will receive goods ranked in their top $\Delta\lceil m/n \rceil$. We claim that for sufficiently large n any player's MMS partition must include a subset of at most $\lceil m/n \rceil$ goods where each good is ranked lower than $\Delta\lceil m/n \rceil$. Suppose this were not true for purposes of contradiction. Then each of the n subsets in an offending player's MMS partition must include either one of the top $\Delta\lceil m/n \rceil$ goods or $\lceil m/n \rceil + 1$ goods. We then see that for sufficiently large n, the number of such subsets is bounded by

$$\begin{aligned}
&\Delta\lceil m/n \rceil + \frac{m - \Delta\lceil m/n \rceil}{\lceil m/n \rceil + 1} \\
&= \Delta\lceil m/n \rceil \\
&\quad + \frac{s(\lceil m/n \rceil - 1) + (n-s)\lceil m/n \rceil - \Delta\lceil m/n \rceil}{\lceil m/n \rceil + 1} \\
&= \frac{\Delta\lceil m/n \rceil^2 + n\lceil m/n \rceil - s}{\lceil m/n \rceil + 1} \\
&\leq \frac{\lceil m/n \rceil}{\lceil m/n \rceil + 1} n + \Delta\lceil m/n \rceil \\
&\leq \frac{n^{1/7}}{n^{1/7} + 1} n + n^{1/7} \ln^3 n \\
&< n.
\end{aligned}$$

Thus the offending player cannot produce such an MMS partition which proves the claim.

Now note that the $n-s$ players of interest have MMS partitions that include the same number of goods they received, but all of which are worth strictly less than every good in their bundle. They therefore must have achieved their MMS value.

Case 2: $s > n^{1/3}$. In this scenario we simply run a matched draft. Similarly to the previous case we know from Lemma 3.6 with $\gamma = 0$ that all the players will receive goods ranked in their top $\Delta\lceil m/n \rceil$ with probability $\to 1$ as $n \to \infty$.

In this case for sufficiently large n any player's MMS partition must include a subset of at most $\lfloor m/n \rfloor$ goods where each good is ranked lower than $\Delta\lceil m/n \rceil$. Again, suppose this were not true for purposes of contradiction. Then each of the n subsets in a player's MMS partition must include either one of the top $\Delta\lceil m/n \rceil$ goods or $\lfloor m/n \rfloor + 1 = \lceil m/n \rceil$ goods (in this case $m \not\equiv 0 \pmod{n}$). We then see that for sufficiently large n, the number of subsets is at most

$$\begin{aligned}
&\Delta\lceil m/n \rceil + \frac{m - \Delta\lceil m/n \rceil}{\lceil m/n \rceil} \\
&= \Delta\lfloor m/n \rfloor + \frac{s(\lceil m/n \rceil - 1) + (n-s)\lceil m/n \rceil}{\lceil m/n \rceil} \\
&= n + \Delta\lfloor m/n \rfloor - \frac{s}{\lceil m/n \rceil} \\
&\leq n + n^{1/7} \ln^3 n - \frac{n^{1/3}}{n^{1/7}} \\
&< n.
\end{aligned}$$

Via logic similar to the previous case, we conclude that all players must have achieved their MMS value. ■

4 Discussion

Theorem 3.1, together with the extensive experiments of Bouveret and Lemaître (2014), tells us that an MMS allocation is very likely to exist *ex post*, that is, after the players report their preferences. But, unfortunately, Theorem 2.1 implies that an MMS allocation cannot be *guaranteed* even if the number of goods is quite small.

While Theorem 2.1 essentially settles one of the main open problems of Procaccia and Wang (2014), it sheds no light on the other: For each number of players n, what is the maximum $g(n) \in (0,1)$ such that it is always possible to achieve a $g(n)$-approximate MMS allocation, that is, an allocation satisfying $V_i(A_i) \geq g(n) \cdot \text{MMS}(i)$ for all players i. Procaccia and Wang prove that

$$g(n) \geq \frac{2\lfloor n \rfloor_{\text{odd}}}{3\lfloor n \rfloor_{\text{odd}} - 1},$$

where $\lfloor n \rfloor_{\text{odd}}$ is the largest odd n' such that $n' \leq n$. In particular, for all n we have that $g(n) > 2/3$, and $g(3) \geq 3/4$. Amanatidis et al. (2015) establish (among their other results) an improved bound of $g(3) \geq 7/8$, but do not improve the general lower bound. On the other hand, counterexamples to the existence of MMS allocations — the construction of Procaccia and Wang (2014), and the proof of Theorem 2.1 — only imply that $g(n) \leq 1 - o(1)$, that is, they give an upper bound that is extremely close to 1. The challenge of closing this gap is, in our view, both technically fascinating and practically significant.

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