has been allocated. Notice that the randomized upper bound of Theorem 2.1 carries over, because the algorithm ignores the values shown to it. So does the lower bound of Theorem 2.13, because it holds *even* against more powerful algorithms. However, the game between the algorithm and the adversary is now an extensive-form game of *incomplete* information, where randomization can potentially help. This turns out to be the case, and, in fact, deterministic algorithms cannot have vanishing envy in this setting. For more details, see the full version of this paper.

Is low envy fair enough? We have focused with single-minded determination on a single goal — that of minimizing envy. A possible concern is that low envy, in and of itself, is not sufficient to lead to intuitively fair outcomes, as has been observed in various contexts [7, 10]. Be that as it may, even if one is interested in a combination of low envy and other properties (Pareto efficiency comes to mind), our results establish a baseline for what one could hope for, and are therefore a crucial first step in any such investigation.

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