

# Envy-Free Division of Sellable Goods

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## Abstract

We study the envy-free allocation of indivisible goods between two players. Our novel setting includes an option to sell each good for a fraction of the minimum value any player has for the good. To rigorously quantify the efficiency gain from selling, we reason about the *price of envy-freeness* of allocations of sellable goods — the ratio between the maximum social welfare and the social welfare of the best envy-free allocation. We show that envy-free allocations of sellable goods are significantly more efficient than their unsellable counterparts.

## 1 Introduction

After decades of unresolved communication problems (Kushilevitz and Nisan 1996), Alice and Bob have decided to get a divorce. Their worldly goods include a well-worn (shared) blackboard, a museum-quality collection of private keys, and a 19th century French vase. Can Alice and Bob divide these goods in a way that is fair to both sides?

To answer this question we must be more specific about what we mean by “fair”. The notion of *envy-freeness* provides a natural interpretation: Alice (weakly) prefers her own bundle of goods to Bob’s bundle, and Bob is likewise convinced that he got the better deal. In other words, when the allocation is envy free, neither Alice nor Bob is interested in swapping bundles.

While envy-freeness is a compelling ideal, envy may clearly be unavoidable when the goods are indivisible. But envy-freeness can nevertheless be achieved if we are willing to split one of the goods. This concession enables the famous *Adjusted Winner (AW)* protocol (Brams and Taylor 1996) — an envy-free protocol that has been patented by New York University and licensed to the law firm *Fair Outcomes, Inc.*

In their book, Brams and Taylor (1996, pp. 102–108) apply AW to the real divorce case of *Jolis vs. Jolis*, which was decided in 1981 (let us call the wife Alice Jolis, and the husband Bob Jolis). The marital property included a Paris apartment, a Paris studio, a New York City coop, a farm, cash and receivables, securities, a profit-sharing plan, and a life insurance policy. Deducing Alice and Bob’s values for these

goods (from available data) and applying AW yields an allocation that gives the studio, coop, farm, securities, and life insurance policy to Bob, the Paris apartment to Alice, and splits the cash (giving a 1/11 fraction to Bob and the rest to Alice). In this case AW split a good that happens to be divisible, but this is by no means guaranteed: had Alice and Bob expressed different preferences, AW could have split one of the indivisible goods, say, the Paris apartment. In practice, this would typically mean selling the Paris apartment and splitting the cash. However, for example, 40% of the market price of the Paris apartment may not be equal to 40% of Bob’s value for owning the entire apartment, invalidating the assumptions underlying AW and thus nullifying its guarantees. Moreover, if we are indeed allowed to sell goods, perhaps there is a better envy-free allocation?

This paper is motivated by the preceding observations and questions, which, we believe, call for an explicit model of the envy-free division of *sellable* goods.

### 1.1 Our Approach and Formal Model

We consider a setting with two players, Alice (denoted  $A$ ) and Bob (denoted  $B$ ). Our approach cannot give rise to non-trivial positive results when there are more than two players, as we discuss in Section 4. There is also a set of  $m$  indivisible goods to be allocated, denoted by  $[m] = \{1, \dots, m\}$ . For each  $j \in [m]$  and  $P \in \{A, B\}$ ,  $P$ ’s value for  $j$  is denoted  $v_P(j) \in [0, 1]$ .

We make two assumptions regarding the valuation functions:

1. *Additivity*: For  $P \in \{A, B\}$  and  $J \subseteq [m]$ ,  $v_P(J) = \sum_{j \in J} v_P(j)$ . In particular,  $v_P(\emptyset) = 0$ .
2. *Normalization*: For  $P \in \{A, B\}$ , it holds that  $v_P([m]) = \sum_{j \in [m]} v_P(j) = 1$ .

An *allocation* (partition of the goods between Alice and Bob)  $\mathcal{X} = \{X_A, X_B\}$  is *envy free* if  $v_A(X_A) \geq v_A(X_B)$  and  $v_B(X_B) \geq v_B(X_A)$ . We are also interested in the economic efficiency of allocations, which we measure via their (utilitarian) *social welfare*:  $\text{SW}(\mathcal{X}, v_A, v_B) = v_A(X_A) + v_B(X_B)$ .

Our main conceptual contribution is the idea that indivisible goods can be sold, and thereby converted into an infinitely divisible cash value (which Alice and Bob value equally). We assume that there is a universal constant  $c \in$

$(0, 1]$  such that the selling price of a set of goods  $J \subseteq [m]$  is  $c \cdot \sum_{j \in J} \min\{v_A(j), v_B(j)\}$ . The rationale is that Alice and Bob both value each good at least at its market value, because they can always sell a good they have obtained. But their value for a good can be strictly higher than its market value — this is captured by the constant  $c$ . While we assume, for ease of exposition, that the price is exactly a  $c$ -fraction of the minimum value, our results naturally hold if this expression is just a lower bound; this observation is important when different goods can be sold for different fractions of the minimum value. With this in mind, we define an *allocation with selling* as  $\mathcal{X} = \{X_A \cup C_A, X_B \cup C_B\}$ , where  $C_A$  and  $C_B$  are cash derived from the sale of a subset of the items, given to Alice and Bob, respectively.

To rigorously quantify the gain from selling, we use the notion of (utilitarian) *price of envy-freeness*, independently introduced by Caragiannis et al. (2012) and (in slightly different form) by Bertsimas et al. (2011). For valuation functions  $v_A, v_B$ , let  $\text{OPT}(v_A, v_B)$  be the social welfare of the welfare-maximizing allocation; i.e.,  $\text{OPT}(v_A, v_B) \triangleq \max_{\mathcal{X}} SW(\mathcal{X}, v_A, v_B)$ . Similarly, define  $\text{OPT}_{\text{EF}}(v_A, v_B)$  to be the social welfare of the welfare-maximizing *envy-free allocation without selling*, and  $\text{OPT}_{\text{EFS}}(v_A, v_B)$  to be the social welfare of the welfare-maximizing *envy-free allocation with selling*. The price of envy-freeness is the worst-case (over valuation functions  $v_A, v_B$ ) ratio  $\text{OPT}(v_A, v_B)/\text{OPT}_{\text{EF}}(v_A, v_B)$ . With selling, it is the worst-case ratio  $\text{OPT}(v_A, v_B)/\text{OPT}_{\text{EFS}}(v_A, v_B)$ . Hereinafter the valuation functions will always be clear from the context, so we simply write  $\text{OPT}$ ,  $\text{OPT}_{\text{EF}}$  and  $\text{OPT}_{\text{EFS}}$ .

We now formulate our primary research challenge:

*Show that the option to sell goods provides a major boost to efficiency by establishing that the price of envy-freeness with selling is significantly lower than the price of envy-freeness without selling.*

## 1.2 Our Results

Table 1 summarizes our results regarding the price of envy-freeness. The columns distinguish between two scenarios: (i) the general setting where an envy-free allocation without selling may not exist, and (ii) such an allocation does exist. The first row shows our bounds on the price of envy-freeness with selling (which are tight), the second row instantiates these bounds for  $c = 1$ , and the third row gives the price of envy-freeness without selling. In scenario (i), the price of envy-freeness without selling is  $\infty$  (or, alternatively, it is not well defined). In contrast, our analysis (Theorem 1) gives a bound of  $3/2$  for the case of  $c = 1$ . In scenario (ii), Caragiannis et al. (2009) show that the price of envy-freeness (without selling) is  $3/2$ ; our bound (Theorem 2) instantiates to  $6/5$  when  $c = 1$ . As  $c \rightarrow 0$ , our results match the bounds without selling.

We also investigate the problem of computing a social welfare maximizing allocation of sellable goods. While the problem is NP-complete (Theorem 3), we show that, when  $c = 1$ , it admits a fully polynomial time approximation scheme (Theorem 4).

	Is there an EF allocation?	
	No	Yes
Selling	$\max\{\frac{3-c}{c+c^2}, \frac{3}{1+c}\}$	$\max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\}$
Selling ( $c = 1$ )	$3/2$	$6/5$
No selling	$\infty$	$3/2$

Table 1: Summary of our results.

## 1.3 Additional Context and Significance in AI

The rigorous study of fair division dates back to the work of Steinhaus (1948). Over the years a deep theory has been developed by economists, mathematicians, and political scientists; see, e.g., the books by Brams and Taylor (1996) and Moulin (2003). More recently, the study of fair division has attracted significant attention from the AI community. This relatively newfound interest is partly motivated by the idea that fair division theory can inform the design of multiagent systems (Chevalyere et al. 2006).

The fair division literature makes a distinction between two typically disjoint cases, depending on whether the goods are divisible or indivisible. The divisible case usually involves a single, heterogeneous good, and the task of dividing this good is known as *cake cutting*. This setting has been extensively studied by AI researchers in recent years (Procaccia 2009; Chen et al. 2013; Caragiannis, Lai, and Procaccia 2011; Cohler et al. 2011; Brams et al. 2012; Bei et al. 2012; Kurokawa, Lai, and Procaccia 2013; Brânzei, Procaccia, and Zhang 2013); see the survey by Procaccia (2013) for an overview. In the context of indivisible goods, AI researchers have also studied issues like complexity, preference handling, and incentives (Bouveret and Lang 2008; 2011; Kalinowski et al. 2013).

Our work attempts to bridge these two worlds, by essentially allowing the division of indivisible goods — at a cost. In this sense, our paper is somewhat related to a line of work on reaching envy-free states through distributed negotiation over indivisible goods (Chevalyere et al. 2007; Chevalyere, Endriss, and Maudet 2007; 2010), because these papers allow players to pay each other in order to achieve envy-freeness (as long as the sum of payments is zero). That said, our motivation, questions, approach, and results are all fundamentally different.

## 2 Bounds on the Price of Envy-Freeness

To gain some intuition for price of fairness bounds, we start by discussing the price of envy-freeness *without* selling. As mentioned above, Caragiannis et al. (2009) restrict their attention to instances where an envy-free allocation does exist, and — for these instances — show that  $\text{OPT}/\text{OPT}_{\text{EF}} < 3/2$ . The proof is simple. First, notice that a player is not envious if and only if his or her bundle is worth at least  $1/2$  (because the sum of values for the two bundles is 1). Now, if the optimal allocation is envy-free we are done. If not, on the one hand either Alice or Bob is envious under the optimal solution, so  $\text{OPT} < 1 + 1/2 = 3/2$ ; and on the other hand,  $\text{OPT}_{\text{EF}} \geq 1/2 + 1/2 = 1$ . Crucially, Caragiannis et

al. (2012) also show that this bound is tight; i.e., for every  $\epsilon > 0$  there is an example where the ratio is  $3/2 - \epsilon$ .

In contrast, when the goods are sellable, there always exists an envy-free allocation, so the price of envy-freeness is always well defined. Interestingly, it turns out that the price of envy-freeness (with selling) when an envy-free allocation without selling is assumed to exist (Theorem 2) is significantly lower than the price of envy-freeness in the general case (Theorem 1), so we present these results as two separate theorems, starting with the (simpler, surprisingly) general case.

**Theorem 1.** *When an envy-free solution without selling does not exist for a particular instance, for any  $c \in (0, 1]$ ,*

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq \max\left\{\frac{3-c}{c+c^2}, \frac{3}{1+c}\right\}.$$

Before proving the theorem, we demonstrate that the bound is tight. For  $c \in (0, 3/4]$  it holds that  $\frac{3-c}{c+c^2} \geq \frac{3}{1+c}$ . Let there be two goods, and let  $v_A(1) = 1 - c/2 - \epsilon$ ,  $v_A(2) = c/2 + \epsilon$ ,  $v_B(1) = 1/2 + \epsilon$ ,  $v_B(2) = 1/2 - \epsilon$ , for an arbitrarily small  $\epsilon > 0$ . We have  $\text{OPT} = 3/2 - c/2 - 2\epsilon$ . The only way to get an envy-free allocation is to sell both goods, and split the cash equally. The value of the resulting allocation is just the amount of cash:  $\text{OPT}_{\text{EFS}} = c \cdot (1/2 + \epsilon + c/2 + \epsilon) \approx (c + c^2)/2$ . Then  $\text{OPT}/\text{OPT}_{\text{EFS}} \approx (3 - c)/(c + c^2)$ .

For  $c \in [3/4, 1]$ ,  $\frac{3}{1+c} \geq \frac{3-c}{c+c^2}$ . Let  $v_A(1) = 1$ ,  $v_A(2) = 0$ ,  $v_B(1) = 1/2 + \epsilon$ ,  $v_B(2) = 1/2 - \epsilon$ , for an arbitrarily small  $\epsilon > 0$ . For this instance,  $\text{OPT} = 3/2 - \epsilon$ . To obtain an EF allocation, good 1 must be sold — we then allocate good 2 to Bob and divide that cash so that Alice and Bob are both satisfied. Therefore,  $\text{OPT}_{\text{EFS}} = c \cdot (1/2 + \epsilon) + (1/2 - \epsilon) \approx (c + 1)/2$ , and  $\text{OPT}/\text{OPT}_{\text{EFS}} \approx 3/(1 + c)$ .

*Proof of Theorem 1.* Suppose we start with a social welfare maximizing allocation  $\mathcal{S} = \{S_A, S_B\}$ , and assume without loss of generality that  $v_A(S_A) \geq v_B(S_B)$ . It holds that  $v_A(S_A) \geq v_B(S_A)$  and  $v_B(S_B) \geq v_A(S_B)$ , because to maximize social welfare each good is allocated to the player who values it more highly. Since no envy-free solution exists without selling, we have that  $1 \geq v_A(S_A) \geq v_B(S_A) > 1/2 > v_B(S_B) \geq v_A(S_B)$ . Thus, selling only  $S_B$  would leave the player that does not receive  $S_A$  envious. Hence, to obtain an envy-free allocation, it may be necessary to sell the entirety of  $S_A$ , in the worst case that it is a single good.

It may also be required to sell  $S_B$  to guarantee the existence of an envy-free allocation. If only  $S_A$  is sold, it is possible that the player that does not receive  $S_B$  will be envious, if  $v_A(S_B) > c \cdot v_B(S_A)$ .

*Case 1:*  $v_A(S_B) \leq c \cdot v_B(S_A)$ . In this case we only sell  $S_A$ . Alice's total remaining value is

$$\begin{aligned} v_A(S_B) + c \cdot v_B(S_A) &= 1 - v_A(S_A) + c \cdot v_B(S_A) \\ &\leq 1 - (1 - c) \cdot v_B(S_A), \end{aligned}$$

and Bob's total remaining value is  $1 - (1 - c) \cdot v_B(S_A)$ . We first show that there is enough cash from selling  $S_A$  to satisfy both Alice and Bob in an envy-free allocation.

Suppose that  $v_B(S_B) \geq c \cdot v_B(S_A)$ ; i.e., Bob does not need to be given any cash to be envy-free. Alice's total (remaining) value is  $v_A(S_B) + c \cdot v_B(S_A) \leq 2c \cdot v_B(S_A)$ , so giving her the cash will make her envy-free.

Otherwise, suppose Bob is not immediately envy-free after selling  $S_A$ . It is easy to see there is enough cash to satisfy both Alice and Bob, since the following equation can be obtained by rearranging the terms of the identity  $v_B(S_A) + v_B(S_B) = 1$ :

$$\begin{aligned} c \cdot v_B(S_A) &= \frac{1}{2}(1 - (1 - c) \cdot v_B(S_A)) \\ &\quad + \left(\frac{1}{2}(1 - (1 - c) \cdot v_B(S_A)) - v_B(S_B)\right). \end{aligned}$$

We conclude that in Case 1 we can obtain an envy-free allocation by selling only  $S_A$  and letting Bob keep  $S_B$ . This yields

$$\text{OPT}_{\text{EFS}} \geq c \cdot v_B(S_A) + v_B(S_B) = 1 - (1 - c) \cdot v_B(S_A).$$

Since  $\text{OPT} \leq 2 - v_B(S_A)$ , and letting  $x \triangleq v_B(S_A)$ ,

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq \frac{2 - x}{1 - (1 - c) \cdot x} \triangleq g(x, c).$$

We want the maximum of  $g$  over the entire range  $v_B(S_A) \in [1/2, 1]$ . The partial derivative of  $g$  with respect to  $x$  is

$$\frac{\partial g(x, c)}{\partial x} = \frac{1 - 2c}{(1 - (1 - c) \cdot x)^2}.$$

Thus, the maximum occurs at  $g(1/2, c)$  for all  $c \in [1/2, 1]$ , since for such  $c$ ,  $g$  is non-increasing as  $x$  increases, and at  $g(1, c)$  for all  $c \in (0, 1/2]$ , since for these values,  $g$  is non-decreasing as  $x$  increases. Hence,

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq \max\left\{\frac{1}{c}, \frac{\frac{3}{2}}{1 - (1 - c) \cdot \frac{1}{2}}\right\} = \max\left\{\frac{1}{c}, \frac{3}{1 + c}\right\}.$$

*Case 2:*  $v_A(S_B) > c \cdot v_B(S_A)$ . It may be required to sell both  $S_A$  and  $S_B$  to guarantee the existence of an envy-free solution, if both are single goods. Take the allocation in which everything is sold and the resulting cash is split evenly. We have already established that  $v_B(S_A) > 1/2$ . As a result,

$$\text{OPT} = 2 - (v_A(S_B) + v_B(S_A)) < 2 - c/2 - 1/2,$$

and

$$\begin{aligned} \text{OPT}_{\text{EFS}} &\geq c \cdot (v_A(S_B) + v_B(S_A)) > c \cdot (1 + c) \cdot v_B(S_A) \\ &> c \cdot (1 + c) \cdot \frac{1}{2}. \end{aligned}$$

Thus,  $\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} < \frac{3-c}{c+c^2}$ .

*Wrapping up.* Since  $\frac{3-c}{c+c^2} \geq \frac{1}{c}$  for all  $c \leq 1$ , we proved  $\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq \max\left\{\frac{1}{c}, \frac{3-c}{c+c^2}, \frac{3}{1+c}\right\} = \max\left\{\frac{3-c}{c+c^2}, \frac{3}{1+c}\right\}$ .  $\square$

Next, we assume that an envy-free allocation without selling exists. Recall that in this case, even without selling the price of envy-freeness is  $3/2$  (so Theorem 1 does not give a better bound). However, we are able to show that, with selling, the price of envy-freeness is significantly lower. In particular, for  $c = 1$ , the price of envy-freeness is only  $6/5$ .

**Theorem 2.** *Suppose an envy-free solution without selling exists for a particular instance. Then for  $c \in (0, 1]$ ,*

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq \max\left\{\frac{3-2c}{2-c}, \frac{6}{4+c}\right\}.$$

Once again, before proving the theorem we give an example showing that the bound is tight. For  $c \in [1/2, 1]$  it holds that  $\frac{6}{4+c} \geq \frac{3-2c}{2-c}$ . Let there be four goods, with values given by the following table:

	1	2	3	4
$v_A$	$\frac{1-\epsilon}{2}$	$\frac{1-\epsilon}{2}$	$\epsilon$	0
$v_B$	$1/4 + \epsilon$	$1/4 + \epsilon$	$1/4 - \epsilon$	$1/4 - \epsilon$

$\epsilon > 0$  is arbitrarily small. Note that an envy-free allocation without selling exists (Alice gets 1 and 3 and Bob gets 2 and 4). Moreover, we have that  $\text{OPT} \approx 3/2$ . The optimal envy-free allocation with selling would sell either 1 or 2, resulting in  $\text{OPT}_{\text{EFS}} \approx 1 + c/4$ . Thus,  $\text{OPT}/\text{OPT}_{\text{EFS}} \approx 6/(4+c)$ .

For  $c \in (0, 1/2]$ , the values are given by the following table:

	1	2	3	4
$v_A$	$\frac{1-\epsilon}{2}$	$\frac{1-\epsilon}{2}$	$\epsilon$	0
$v_B$	$\frac{1/2}{2-c} - \epsilon$	$\frac{1/2}{2-c} - \epsilon$	$\frac{(1-c)/2}{2-c} + \epsilon$	$\frac{(1-c)/2}{2-c} + \epsilon$

The social welfare maximizing solution, where Alice receives 1 and 2 and Bob receives 3 and 4, yields a total value of  $\text{OPT} \approx (3-2c)/(2-c)$ . The envy-free with selling solution sells 1 or 2, and yields  $\text{OPT}_{\text{EFS}} = 1$ . The ratio is therefore roughly  $(3-2c)/(2-c)$ .

We are now ready to prove the theorem. The proof is quite long and intricate, so the proofs of several lemmas are omitted. All omitted proofs can be found in the extended version of the paper.<sup>1</sup>

*Proof of Theorem 2.* Let  $\mathcal{X} = \{X_A \cup C_A, X_B \cup C_B\}$  be an allocation where  $X_A$  and  $X_B$  are disjoint subsets of the goods and  $C_A + C_B$  is the cash obtained from selling  $[m] \setminus (X_A \cup X_B)$ . We let

$$\underline{X}_A \triangleq \{j \in X_A : v_A(j) < v_B(j)\},$$

$$\underline{X}_B \triangleq \{j \in X_B : v_B(j) < v_A(j)\},$$

and define their complements,  $\overline{X}_A \triangleq X_A \setminus \underline{X}_A$ , and  $\overline{X}_B \triangleq X_B \setminus \underline{X}_B$ .

We have assumed that an envy-free allocation without selling exists, so let  $\mathcal{Y} = \{Y_A, Y_B\}$  be a social welfare maximizing envy-free allocation without selling; that is,  $\text{OPT}_{\text{EF}} = \text{SW}(\mathcal{Y}, v_A, v_B)$ . Note that  $\text{OPT}_{\text{EF}} \geq 1$ , because  $v_A(Y_A) \geq 1/2$  and  $v_B(Y_B) \geq 1/2$  due to envy-freeness. Without loss of generality, assume that  $v_A(\underline{Y}_A) \leq v_B(\underline{Y}_B)$ .

Note that when  $v_A(\overline{Y}_A) \geq 1/2$ ,  $\mathcal{Y}$  satisfies  $\underline{Y}_A = \emptyset$  (so that  $v_A(\underline{Y}_A) = v_B(\underline{Y}_A) = 0$ ), since giving those goods to Alice is not necessary for envy-freeness, and Alice values them strictly less than Bob.

Next,  $\mathcal{S} = \{S_A, S_B\}$  will refer to a particular social welfare maximizing allocation:  $S_A = \overline{Y}_A \cup \underline{Y}_B$ , and  $S_B = \overline{Y}_B \cup \underline{Y}_A$ . Note that when  $v_B(\underline{Y}_B) = 0$ ,  $v_A(\overline{Y}_A) + v_B(\underline{Y}_B) = 0$ , so  $\text{SW}(\mathcal{S}) = \text{SW}(\mathcal{Y})$ , in which case the price of fairness is 1. This is not an interesting case so assume henceforth that

the price of envy-freeness is strictly greater than one; i.e., assume  $v_B(\underline{Y}_B) > 0$ .

Our first lemma is used throughout the theorem's proof.

**Lemma 1.**  $v_A(S_A) \geq \frac{1}{2}$  and  $v_B(S_B) < \frac{1}{2}$ .

Consider the following two allocations. They both begin with the allocation  $\mathcal{S}$ , and involve Alice selling one of  $\overline{Y}_A$  or  $\underline{Y}_B$ .

*Allocation 1.* Alice sells  $\underline{Y}_B$  and gives Bob cash worth

$$C_B^1 \triangleq \max \left\{ 0, \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\underline{Y}_B) - v_B(\overline{Y}_B) - v_B(\underline{Y}_A) \right\}.$$

Let  $C_A^1 \triangleq c \cdot v_B(\underline{Y}_B) - C_B^1$  be Alice's remaining cash. Then define  $\mathcal{Z}^1 \triangleq \{\overline{Y}_A \cup C_A^1, \overline{Y}_B \cup \underline{Y}_A \cup C_B^1\}$ .

Note that Alice's value for all the remaining goods and the cash from the sale is  $1 - v_A(\underline{Y}_B) + c \cdot v_B(\underline{Y}_B) \leq 1 - (1-c) \cdot v_B(\underline{Y}_B)$ , while Bob's value for everything is  $1 - (1-c) \cdot v_B(\underline{Y}_B)$ .

*Allocation 2.* Alice sells  $\overline{Y}_A$  and gives Bob cash worth

$$C_B^2 \triangleq \max \left\{ 0, \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\overline{Y}_A) - v_B(\overline{Y}_B) - v_B(\underline{Y}_A) \right\}.$$

Let  $C_A^2 \triangleq c \cdot v_B(\overline{Y}_A) - C_B^2$ . Then define  $\mathcal{Z}^2 \triangleq \{\underline{Y}_B \cup C_A^2, \overline{Y}_B \cup \underline{Y}_A \cup C_B^2\}$ .

Observe that Alice's value for all the remaining goods and the cash from the sale is  $1 - v_A(\overline{Y}_A) + c \cdot v_B(\overline{Y}_A) \leq 1 - (1-c) \cdot v_B(\overline{Y}_A)$ , while Bob's value for everything is  $1 - (1-c) \cdot v_B(\overline{Y}_A)$ .

The next two lemmas show that, if the two allocations  $\mathcal{Z}^1$  and  $\mathcal{Z}^2$  have enough cash, then they are envy free and provide certain guarantees with respect to social welfare.

**Lemma 2.** Assume that  $C_B^1 \leq c \cdot v_B(\underline{Y}_B)$  (that is, there is enough cash to give to Bob under Allocation 1), and  $\text{OPT} > \frac{6}{4+c}$ . Then the allocation  $\mathcal{Z}^1$  is envy free, and  $\text{SW}(\mathcal{Z}^1) = v_A(\overline{Y}_A) + v_B(\underline{Y}_A) + v_B(\overline{Y}_B) + c \cdot v_B(\underline{Y}_B)$ .

*Proof.* The statement about the value of  $\text{SW}(\mathcal{Z}^1)$  is trivial. Turning to envy-freeness, clearly Bob has no envy in this transaction, since his value is

$$v_B(\overline{Y}_B) + v_B(\underline{Y}_A) + v_B(C_B^1) \geq \frac{1 - (1-c) \cdot v_B(\underline{Y}_B)}{2};$$

i.e., half his total (remaining) value. We need to show that Alice is not envious.

*Case 1:*  $C_B^1 = 0$ . Note that this case cannot happen if  $c = 1$ , since then  $1/2 - v_B(\overline{Y}_B) - v_B(\underline{Y}_A) \leq 0$ , contradicting Lemma 1. Thus, suppose  $c \in (0, 1)$ .

We show that if Alice receives  $\overline{Y}_A$  and all the cash, then she would not envy Bob. Assume for the sake of contradiction that Alice would actually envy Bob:  $v_A(\underline{Y}_A) + v_A(\underline{Y}_B) > v_A(\overline{Y}_A) + c \cdot v_B(\underline{Y}_B)$ .

<sup>1</sup>Available at: <http://cs.cmu.edu/~arielpro/papers>

Note that, by Lemma 1,  $v_B(\overline{Y_A}) + v_B(\underline{Y_B}) = v_B(S_A) > 1/2$ . Since we are assuming  $\text{OPT} > \frac{6}{4+c}$ , and

$$\begin{aligned} \frac{6}{4+c} < \text{OPT} &= 2 - (v_B(\overline{Y_A}) + v_B(\underline{Y_B}) + v_A(\underline{Y_A}) + v_A(\overline{Y_B})) \\ &< \frac{3}{2} - (v_A(\underline{Y_A}) + v_A(\overline{Y_B})), \end{aligned}$$

it follows that  $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) < \frac{3c}{2(4+c)}$ . By our earlier assumption that Alice would envy Bob,

$$\begin{aligned} \frac{3c}{2(4+c)} &> v_A(\underline{Y_A}) + v_A(\overline{Y_B}) \\ &> v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) \\ &\geq v_B(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) \\ &= v_B(\overline{Y_A}) + v_B(\underline{Y_B}) - (1-c) \cdot v_B(\underline{Y_B}) \\ &> \frac{1}{2} - (1-c) \cdot v_B(\underline{Y_B}). \end{aligned}$$

Thus,  $v_B(\underline{Y_B}) > \frac{2-c}{(1-c)(4+c)}$ . This value is strictly greater than  $1/2$  for all  $c \in (0, 1)$ , which is a contradiction to the envy-freeness of  $\mathcal{Y}$ , since then  $v_A(\underline{Y_B}) \geq v_B(\underline{Y_B}) > 1/2$ .

Hence, when  $C_B^1 = 0$ ,  $v_A(\underline{Y_A}) + v_A(\overline{Y_B}) \leq v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B})$ , so Alice does not envy Bob.

Case 2:  $C_B^1 > 0$ . Alice's value is

$$\begin{aligned} &v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) - \left( \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\underline{Y_B}) \right) \\ &\quad + v_B(\overline{Y_B}) + v_B(\underline{Y_A}) \\ &= v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) - \left( \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\underline{Y_B}) \right) \\ &\quad + (1 - v_B(\underline{Y_B}) - v_B(\overline{Y_A})) \\ &= \frac{1}{2} - \frac{(1-c)}{2} \cdot v_B(\underline{Y_B}) + v_A(\overline{Y_A}) - v_B(\overline{Y_A}) \\ &\geq \frac{1}{2} \left( 1 - (1-c) \cdot v_B(\underline{Y_B}) \right), \end{aligned}$$

where the last line follows from the assumption that  $v_A(\overline{Y_A}) \geq v_B(\overline{Y_A})$ . The right hand side is at least half of Alice's total (remaining) value.  $\square$

The proofs of the following two lemmas, 3 and 4, are relegated to the full version of the paper.

**Lemma 3.** Assume that  $C_B^2 \leq c \cdot v_B(\overline{Y_A})$  (that is, there is enough cash to give to Bob under Allocation 2), and  $\text{OPT} \geq \frac{6}{4+c}$ . Then the allocation  $\mathcal{Z}^2$  is envy free, and  $\text{SW}(\mathcal{Z}^2) = v_A(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}) + c \cdot v_B(\overline{Y_A})$ .

At this point, it will be useful to define the maximum of Allocations 1 and 2, with respect to social welfare, as  $\mathcal{Z}$ . That is,

$$\mathcal{Z} \triangleq \begin{cases} \mathcal{Z}^1 & \text{if } v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) > v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}), \\ \mathcal{Z}^2 & \text{otherwise.} \end{cases}$$

We will use  $C_A^*$  to refer to the cash received by Alice in  $\mathcal{Z}$ , and  $C_B^*$  for the cash Bob receives in  $\mathcal{Z}$ .

**Lemma 4.** When  $\text{OPT} \geq \frac{6}{4+c} \cdot \text{OPT}_{\text{EF}}$ ,  $\mathcal{Z}$  has sufficient cash:  $C_A^*$  and  $C_B^*$  are both non-negative.

With all the lemmas in place, we can now complete the proof of Theorem 2. Observe that the lemmas imply  $\mathcal{Z}$  is envy-free. Also note that  $\text{OPT}_{\text{EF}} \geq 1$ , so assuming  $\text{OPT} > \max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\} \cdot \text{OPT}_{\text{EF}}$  allows Lemmas 2, 3, and 4 to apply.

Since the maximum of two numbers is at least their average, and

$$v_B(\underline{Y_A}) + v_B(\overline{Y_B}) = \text{OPT} - v_A(\overline{Y_A}) - v_A(\underline{Y_B}) \geq \text{OPT} - 1,$$

we have that

$$\begin{aligned} \text{OPT}_{\text{EFS}} &\geq \text{SW}(\mathcal{Z}) \\ &= \max\{v_A(\overline{Y_A}) + c \cdot v_B(\underline{Y_B}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B}), \\ &\quad v_A(\underline{Y_B}) + c \cdot v_B(\overline{Y_A}) + v_B(\underline{Y_A}) + v_B(\overline{Y_B})\} \\ &\geq \frac{1}{2} \left( v_A(\overline{Y_A}) + v_A(\underline{Y_B}) + c \cdot (v_B(\overline{Y_A}) + v_B(\underline{Y_B})) \right. \\ &\quad \left. + 2(v_B(\underline{Y_A}) + v_B(\overline{Y_B})) \right) \\ &= \frac{1}{2} \left( \text{OPT} - v_B(\underline{Y_A}) - v_B(\overline{Y_B}) \right. \\ &\quad \left. + c \cdot (1 - v_B(\underline{Y_A}) - v_B(\overline{Y_B})) \right. \\ &\quad \left. + 2(v_B(\underline{Y_A}) + v_B(\overline{Y_B})) \right) \\ &= \frac{1}{2} \left( \text{OPT} + c + (1-c)(v_B(\underline{Y_A}) + v_B(\overline{Y_B})) \right) \\ &\geq \frac{1}{2} \left( (2-c) \cdot \text{OPT} + 2c - 1 \right). \end{aligned}$$

This implies

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq \frac{2 \cdot \text{OPT}}{(2-c) \cdot \text{OPT} + 2c - 1} \triangleq f(\text{OPT}, c).$$

For every  $c$ , we want to re-write this bound to be only in terms of  $c$  and not  $\text{OPT}$ . Note that it suffices to consider  $\text{OPT} \in (\max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\}, 3/2)$ . This follows from the reasoning from the beginning of Section 2: if at least one player is envious, then  $\text{OPT} < 1 + 1/2 = 3/2$ . Since we want the bound to hold across the entire possible range of  $\text{OPT}$ , we need to take the maximum of  $f(\text{OPT}, c)$  over the range. Thus, we take the derivative, with respect to  $\text{OPT}$ , of  $f(\text{OPT}, c)$ .

$$\frac{\partial f(\text{OPT}, c)}{\partial \text{OPT}} = \frac{4c - 2}{((2-c) \cdot \text{OPT} + 2c - 1)^2}.$$

For  $c \in [1/2, 1]$ ,  $\frac{\partial f}{\partial \text{OPT}} \geq 0$ , so

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq f\left(\frac{3}{2}, c\right) = \frac{6}{4+c}.$$

For  $c \in (0, 1/2]$ ,  $\frac{\partial f}{\partial \text{OPT}} \leq 0$  and  $\max\{\frac{3-2c}{2-c}, \frac{6}{4+c}\} = \frac{3-2c}{2-c}$ , so

$$\frac{\text{OPT}}{\text{OPT}_{\text{EFS}}} \leq f\left(\frac{3-2c}{2-c}, c\right) = \frac{3-2c}{2-c}. \quad \square$$

### 3 An Algorithmic Retrospective

Our theorems in Section 2 are *existence* results: they state that there always exists an envy-free allocation of sellable goods that yields a certain fraction of the optimal social welfare. The proof of Theorem 1 constructs an allocation that achieves the stated bound by selling portions of the social welfare maximizing allocation  $\mathcal{S}$ . The allocation  $\mathcal{S}$  is easy to compute (just give each good to the player that values it more). In contrast, from a computational viewpoint, the guarantees of Theorem 2 may be hard to achieve, as the proof requires Alice to sell bundles from a welfare-maximizing *envy-free* (without selling) allocation  $\mathcal{Y}$ , which is far trickier to compute (Bouveret and Lang 2008).

The optimal envy-free allocation with selling is at least as good as the allocations constructed in the two proofs. And, in theory, the option to sell goods may actually make its computation easy. Our next result shows that this is not the case.<sup>2</sup> To be more formal, let us define the MAX-EFS( $c$ ) problem as follows: the input is the set of goods  $[m]$  that can be sold for a  $c$ -fraction of the minimum value, the valuation functions  $v_A$  and  $v_B$ , and  $k \in \mathbb{R}^+$ ; the question is whether there is an envy-free allocation with social welfare at least  $k$ .

**Theorem 3.** *For any  $c \in (0, 1]$ , the MAX-EFS( $c$ ) problem is NP-complete.*

Intuitively, though, what makes the problem hard is that, starting from a social welfare maximizing allocation (which is not envy free), an optimal solution would have to sell a set of goods that is sufficient to satisfy the envious player, while losing as little value as possible. For  $c = 1$ , this can be formulated as a MIN-KNAPSACK problem, which admits a fully polynomial time approximation scheme (FPTAS) (Kellerer, Pferschy, and Pisinger 2004). Leveraging this insight, we establish following result.

**Theorem 4.** *MAX-EFS(1) admits an FPTAS; i.e., there is an algorithm that, for any  $\epsilon > 0$ , returns an envy-free allocation  $\mathcal{X}$  (which possibly includes cash) such that  $\text{SW}(\mathcal{X}, v_A, v_B) \geq (1 - \epsilon) \text{OPT}_{\text{EFS}}(v_A, v_B)$ , and runs in polynomial time in the parameters of the problem and  $1/\epsilon$ .*

Moreover, it is easy to see that for any value of  $c$ , MAX-EFS( $c$ ) can be formulated as an integer linear program (ILP), which can be solved using a variety of practical algorithms. To conclude, we do not view Theorem 3 as a serious obstacle to solving fair division problems in our framework, and, in particular, to achieving the guarantees given by Theorems 1 and 2.

### 4 Discussion

All of our results focus on the case of two players. This is because, when there are three or more players, the price of envy-freeness with selling is unbounded. To see why, let there be two goods and three players  $A, B, C$ ; set  $v_A(1) =$

<sup>2</sup>A related computational problem is whether  $\text{OPT}_{\text{EFS}} = \text{OPT}$ . This question does turn out to be tractable for  $c = 1$ ; the proof is implicit in the arguments in the extended version of the paper. In contrast, the question of whether  $\text{OPT}_{\text{EF}} = \text{OPT}$  is NP-complete.

$v_B(1) = 1 - \epsilon$ ,  $v_C(1) = 0$ ,  $v_A(2) = v_B(2) = \epsilon$ , and  $v_C(2) = 1$ . Both goods needs to be sold to prevent envy, but this only generates  $\epsilon$  cash. In contrast, the case of two players (which is of special significance) gives rise to a rich collection of insights.

Another assumption — this one implicit — worth discussing is the conversion between value and cash. We have assumed that Alice and Bob’s valuations for the complete bundle of goods are normalized to 1. This assumption is also made in many other fair division papers that reason about utilitarian social welfare (see, e.g., (Caragiannis et al. 2009; Cohler et al. 2011; Brams et al. 2012)). In practice, this could mean assigning Alice and Bob the same number of points to distribute between goods. But the conversion to cash means that the normalized valuation of a good can be compared to its market value. This is clearly possible, for example, if Alice and Bob have the same *actual* value for the complete bundle of goods. In any case, the fact that  $c$  can be any number in  $(0, 1]$  gives us the flexibility to handle a range of conversion schemes, possibly at the cost of slightly weaker guarantees.

Finally, note that the Adjusted Winner protocol (Brams and Taylor 1996), discussed in Section 1, actually guarantees another fairness property called equitability: the players have equal values for their own bundles of goods. In principle, one can ask the same questions we have answered above about the price of equitability instead of envy-freeness — but that would be a bit repetitive!

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