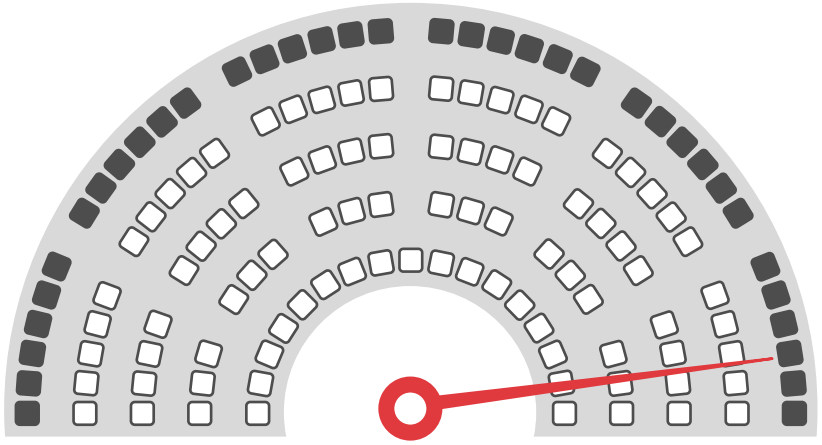


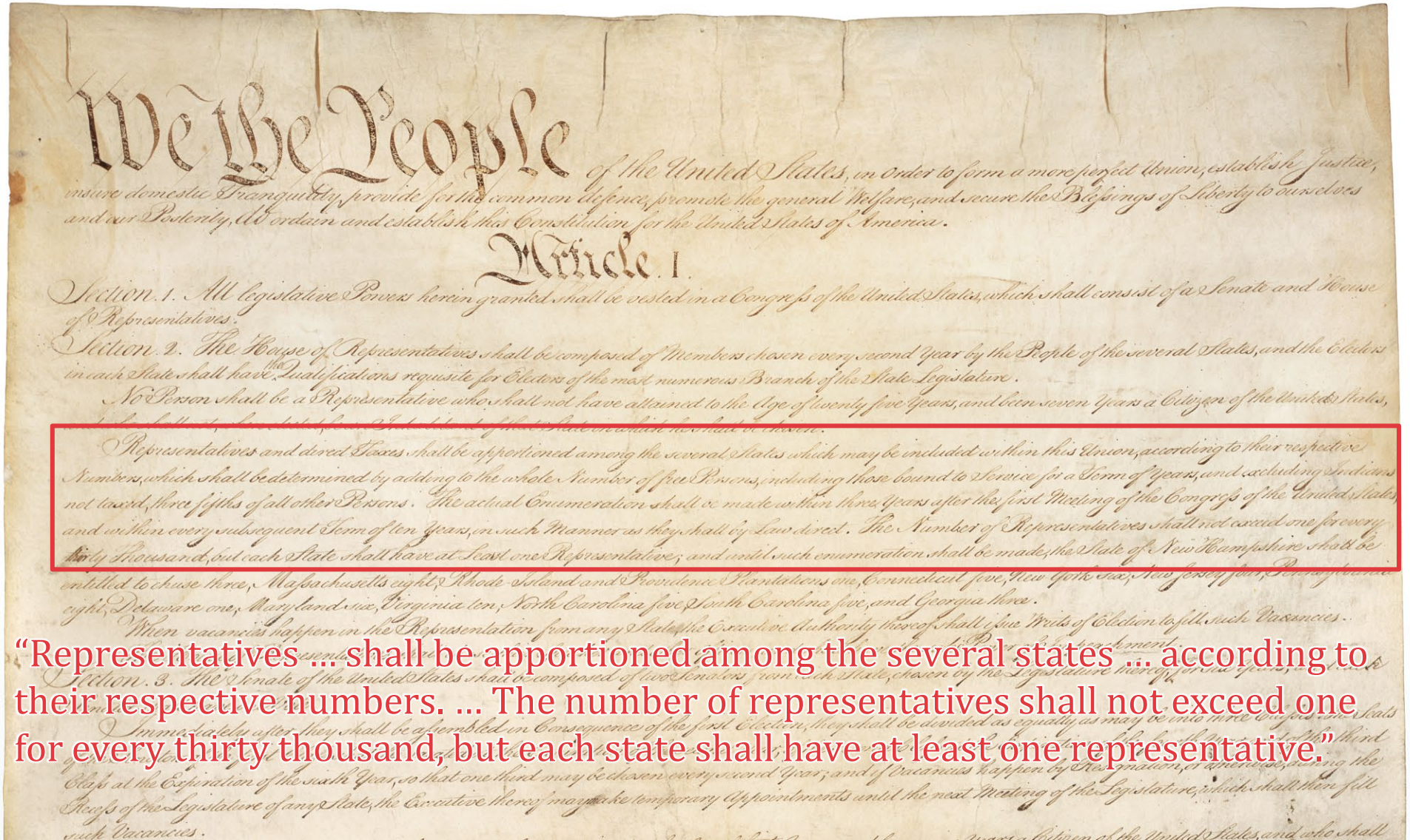
Optimized Democracy

Fall 2026 | Lecture 18

Fair Redistricting

Ariel Procaccia | Harvard University

THE CONSTITUTION



"Representatives … shall be apportioned among the several states … according to their respective numbers. … The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative."

THE CONSTITUTION

Section. 3. The Senate of the United States shall be composed of two Senators from each State, chosen by the Legislature thereof, for six Years; and each Senator shall have one Vote.

Immediately after they shall be assembled in Consequence of the first Election, they shall be divided as equally as may be into three Classes. The Seats of the Senators of the first Class shall be vacated at the Expiration of the second Year, of the second Class at the Expiration of the fourth Year, and of the third Class at the Expiration of the sixth Year, so that one third may be chosen every second Year; and if Vacancies happen by Resignation, or otherwise, during the Recess of the Legislature of any State, the Executive thereof may make temporary Appointments until the next Meeting of the Legislature, which shall then fill such Vacancies.

No Person shall be a Senator who shall not have attained to the Age of thirty Years, and been nine Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of that State for which he shall be chosen.

The Vice President of the United States shall be President of the Senate, but shall have no Vote, unless they be equally divided.

The Senate shall chuse their other Officers, and also a President pro tempore, in the Absence of the Vice President, or when he shall exercise the Office of President of the United States.

The Senate shall have the sole Power to try all Impeachments. When sitting for that Purpose, they shall be on Oath or Affirmation. When the President of the United States is tried, the Chief Justice shall preside: And no Person shall be convicted without the Concurrence of two thirds of the Members present.

Judgment in Cases of Impeachment shall not extend further than to removal from Office, and disqualification to hold and enjoy any Office of Honor, Trust or Profit under the United States: but the Party convicted shall nevertheless be liable and subject to Indictment, Trial, Judgment and Punishment, according to Law.

Section. 4. The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State by the Legislature thereof; but the Congress may at any time by Law make or alter such Regulations, except as to the Places of chusing Senators.

The Congress shall assemble at least once in every Year, and such Meeting shall be on the first Monday in December, unless they shall by Law appoint a different Day.

Section. 5. Each House shall be the Judge of the Elections, Returns and Qualifications of its own Members, and a Majority of each shall constitute a Quorum to do Business; but a smaller Number may adjourn from day to day, and may be authorized to compel the Attendance of absent Members, in such Manner, and under such Penalties as each House may provide.

"The times, places and manner of holding elections for Senators and Representatives, shall be prescribed in each state by the legislature thereof..."

Each House may determine the Rules of its Proceedings, punish its Members for disorderly Behaviour, and, with the Concurrence of two thirds, expel a Member.

Each House shall keep a Journal of its Proceedings, and from time to time publish the same, excepting such Parts as may in their Judgment require Secrecy; and the Yeas and Nays of the Members of either House on any question shall, at the Desire of one fifth of those Present, be entered on the Journal.

Neither House, during the Session of Congress, shall, without the Consent of the other, adjourn for more than three days, nor to any other Place than that in which the two Houses shall be sitting.

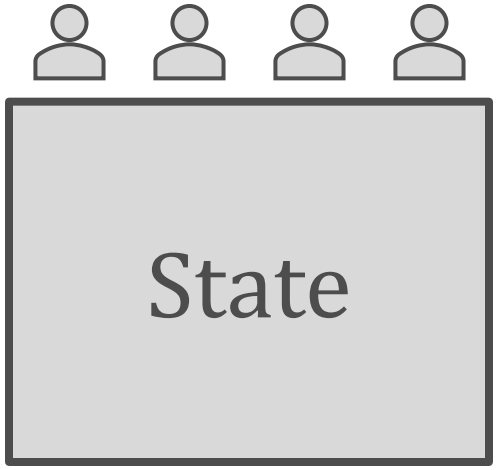
Section. 6. The Senators and Representatives shall receive a Compensation for their Services, to be ascertained by Law, and paid out of the Treasury of the United States. They shall in all Cases, except Treason, Felony and Breach of the Peace, be privileged from Arrest during their Attendance at the Session of their respective Houses, and in going to and returning from the same; and for any Speech or Debate in either House, they shall not be questioned in any other Place.

No Senator or Representative shall, during the Time for which he was elected, be appointed to any civil Office under the Authority of the United States, which shall have been created, or the Emoluments whereof shall have been encreased during such time; and no Person holding any Office under the United States, shall be a Member of either House during his Continuance in Office.

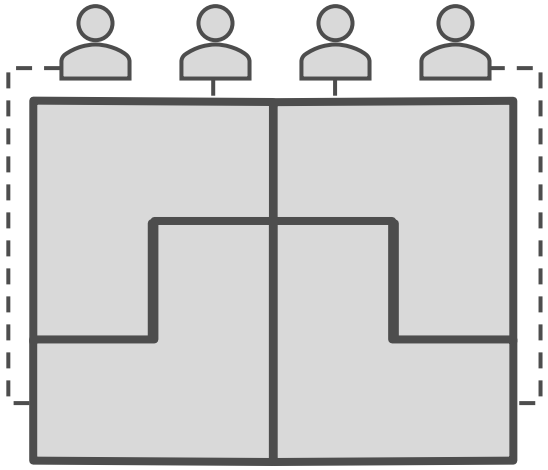
Section. 7. All Bills for raising Revenue shall originate in the House of Representatives; but the Senate may propose or concur with Amendments as on other Bills.

Every Bill which shall have passed the House of Representatives and the Senate, shall, before it become a Law, be presented to the President of the

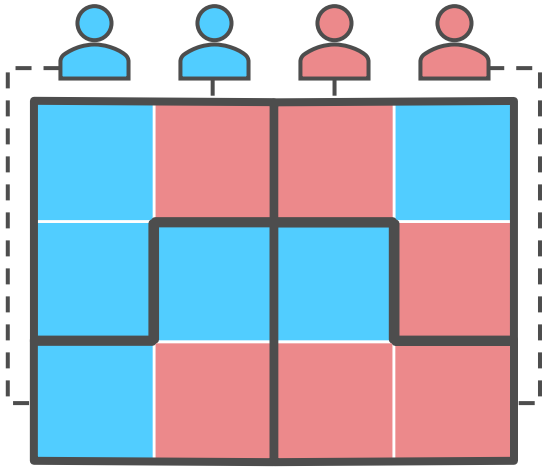
REDISTRICTING



Apportionment

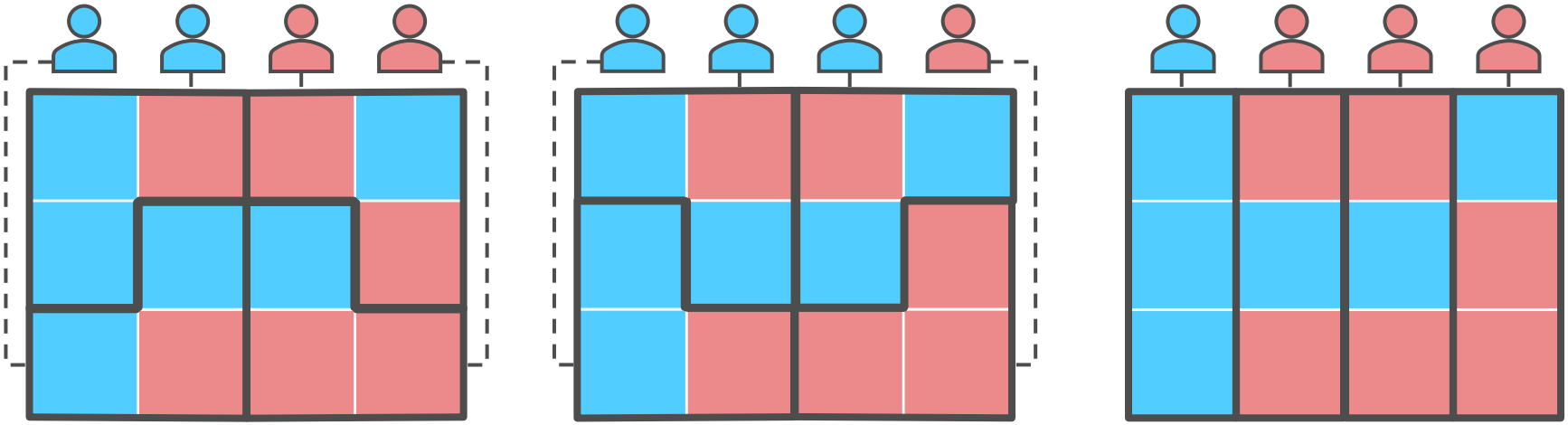


Redistricting



Elections

GERRYMANDERING



gerrymander • verb • to divide or arrange (a territorial unit) into election districts in a way that gives one political party an unfair advantage

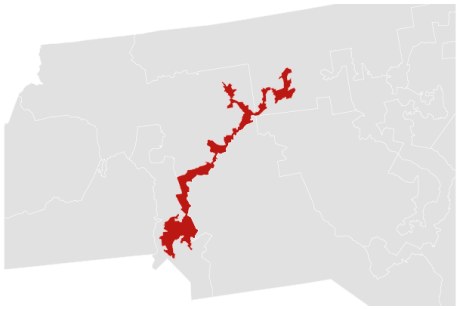


The gerrymander

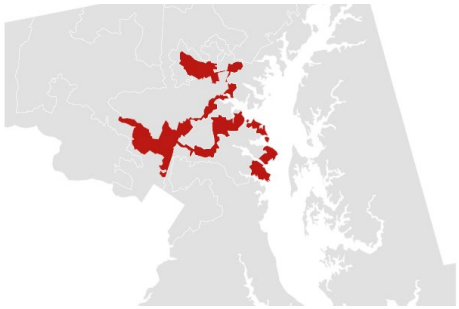
1812–present

"A new species of monster, which appeared in Essex South District in Jan. 1812."

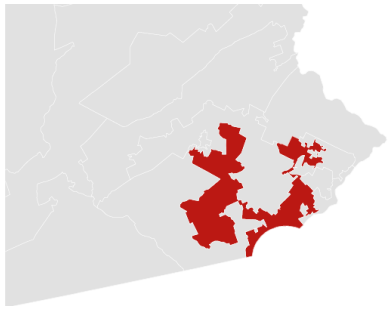
AMERICA'S MOST GERRYMANDERED



North Carolina
12th District



Maryland
3rd District

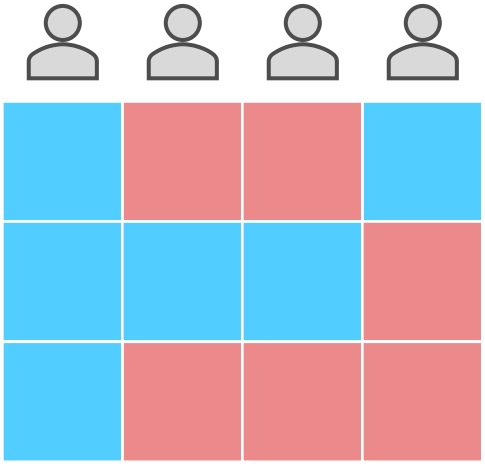


Pennsylvania
7th District

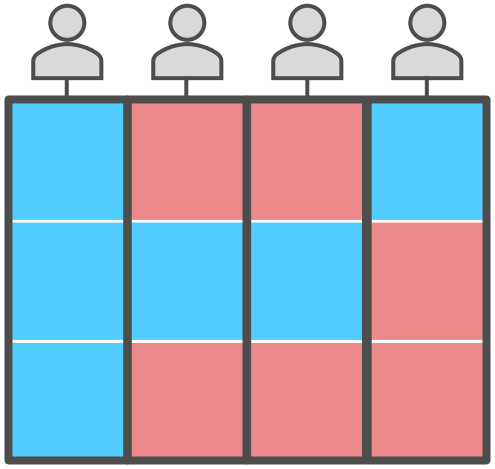
Source: The Washington Post, 2014

I CUT, YOU FREEZE

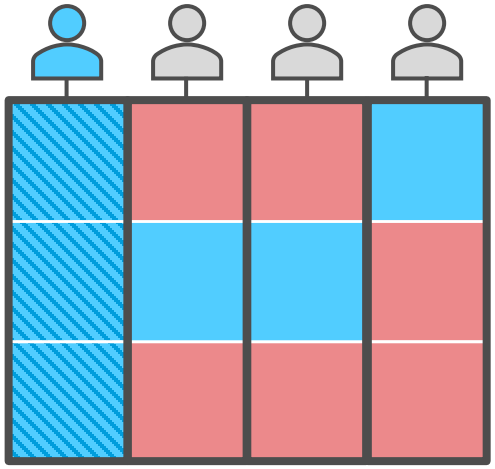
Players take turns freezing a district drawn by the other player, and redrawing the unfrozen districts, until all districts are frozen



Election results

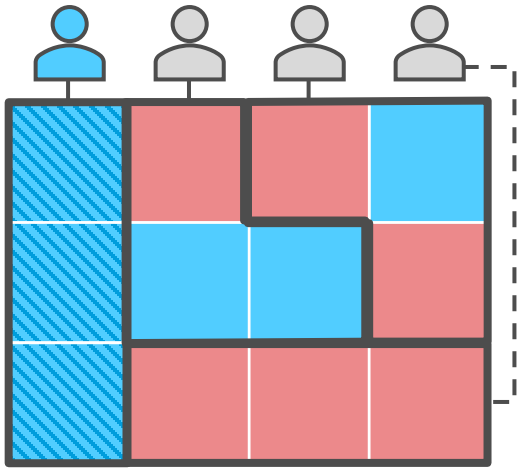


Red cuts

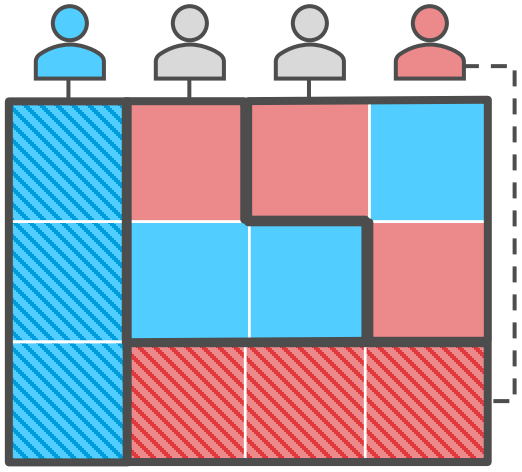


Blue freezes

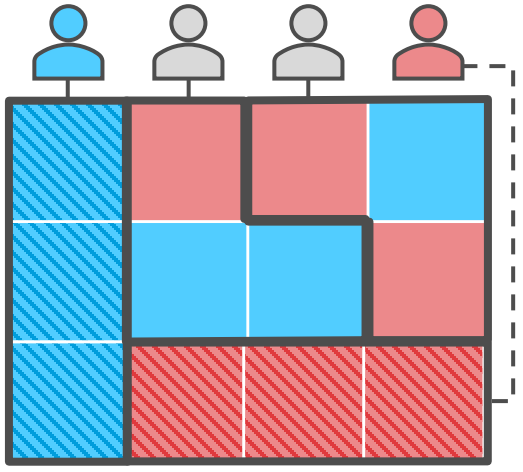
I CUT, YOU FREEZE



Blue cuts



Red freezes



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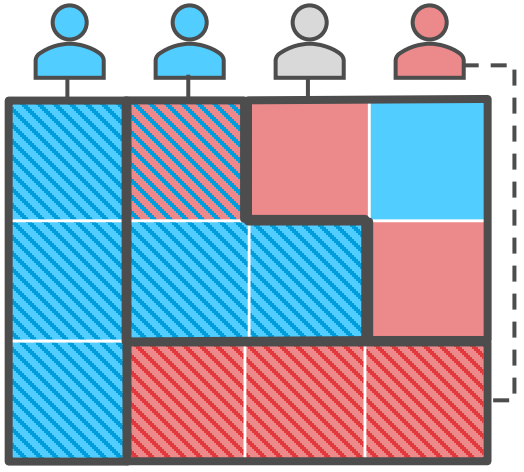
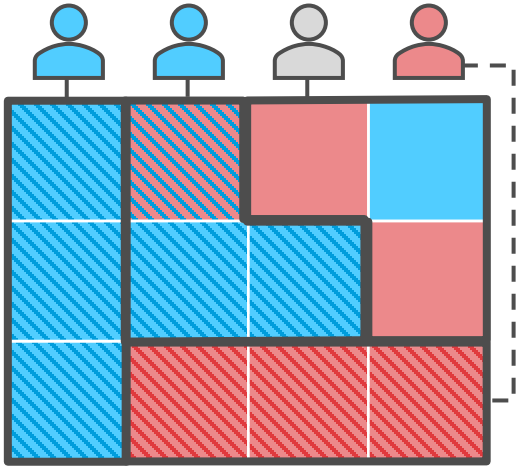
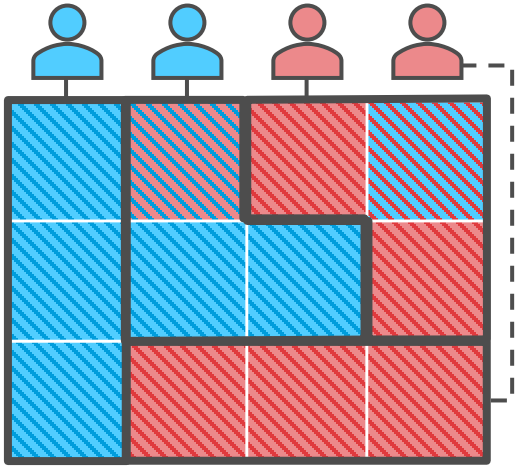


Blue freezes



Blue cuts



Red freezes

The Washington Post

Opinion | There's another way to solve gerrymandering. It's as simple as cake.

By Wesley Pegden and Ariel D. Procaccia

February 15, 2018 at 8:10 p.m. EST



I CUT, YOU FREEZE: ANALYSIS

- A non-geometric model:
  - Each of the two players has V_i units of voters such that $V_1 + V_2 = k$
  - A partition consists of k pairs (v_{1d}, v_{2d}) such that $\sum_{d=1}^{k} v_{id} = V_i$ for $i = 1,2$, and for all districts d, $v_{1d} + v_{2d} = 1$
  - Player i wins district d if $v_{id} > v_{3-i,d}$

Poll

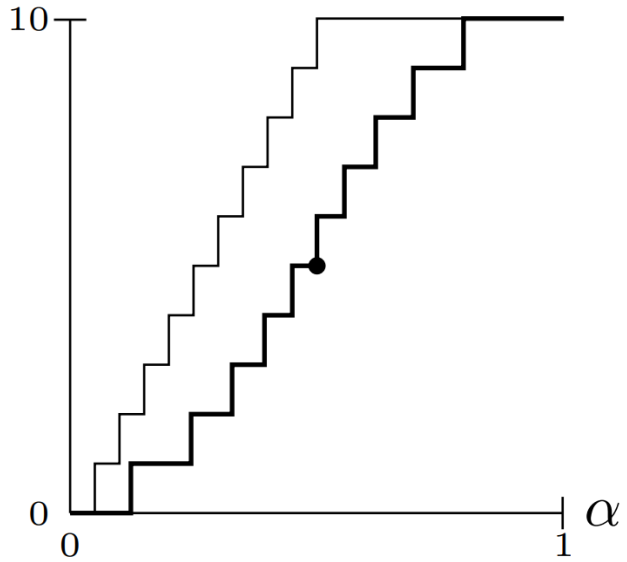
Suppose that $V_1 = \alpha k$ for $\alpha \leq 1/2$ and that player 1 redistricts unilaterally. What fraction of the districts can they guarantee to win (roughly)?

• $\alpha/2$ • α • 2α • $1/2$

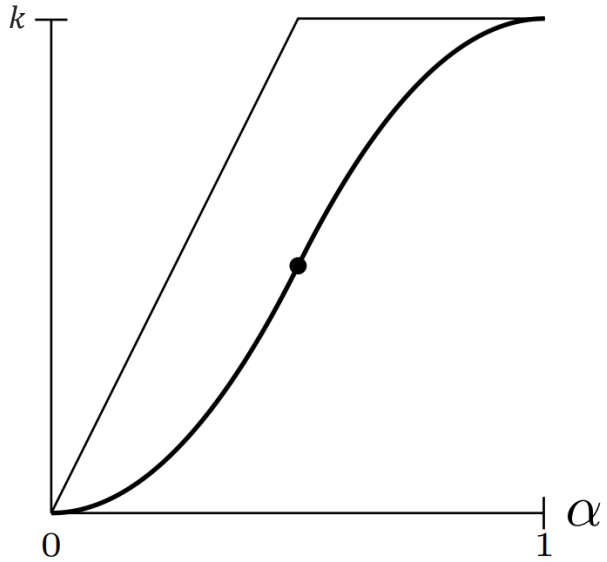


I CUT, YOU FREEZE: ANALYSIS

Theorem: Let $V_i = \alpha k$. As $k \to \infty$, under the ICYF protocol and optimal play, the fraction of districts won by player i is $2\alpha^2$ for $\alpha \leq 1/2$ and $1 - 2(1-\alpha)^2$ for $\alpha > 1/2$



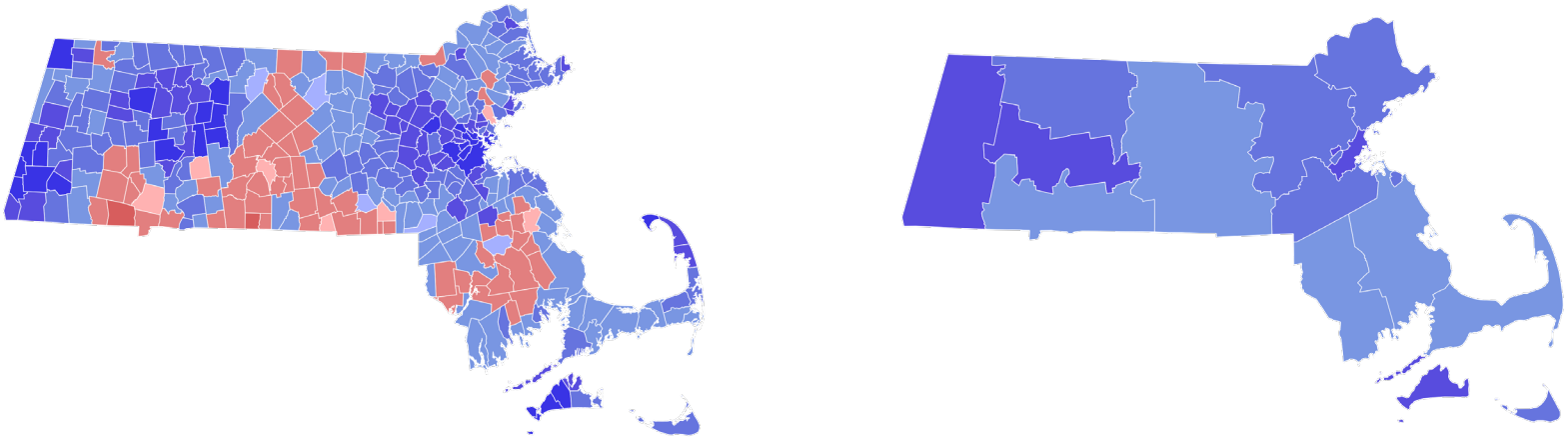
Case of $k = 10$



Case of $k \to \infty$

PROPORTIONALITY

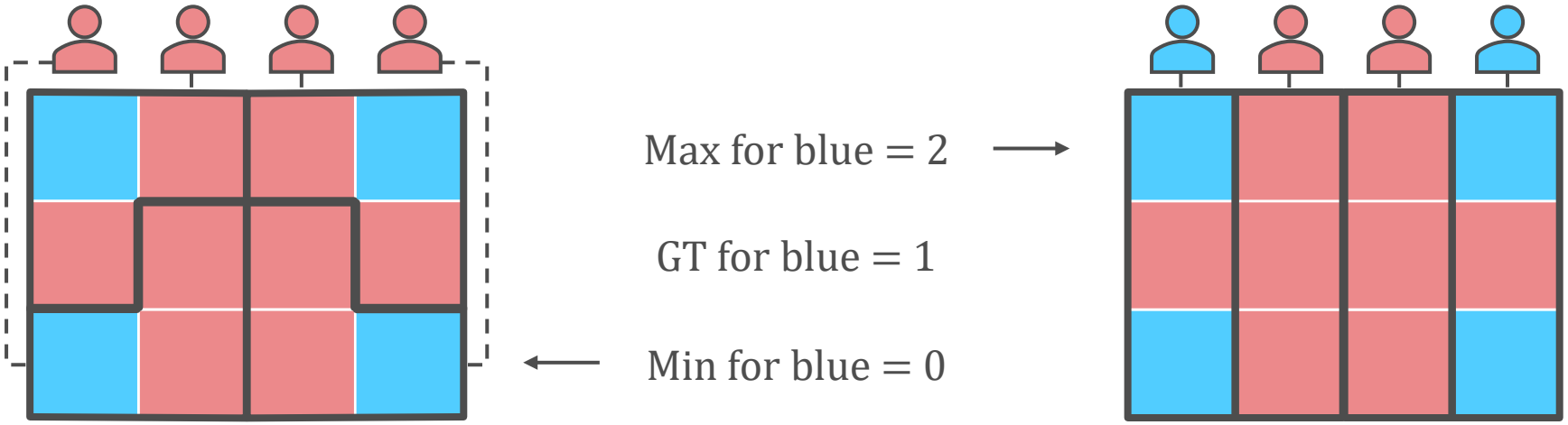
- A proportional partition is such that if $V_i = \alpha k$, i wins an α fraction of the districts
- But when geometric constraints are taken into account, this is not a feasible standard



MA gubernatorial election, 2022 (Healey vs. Diehl)

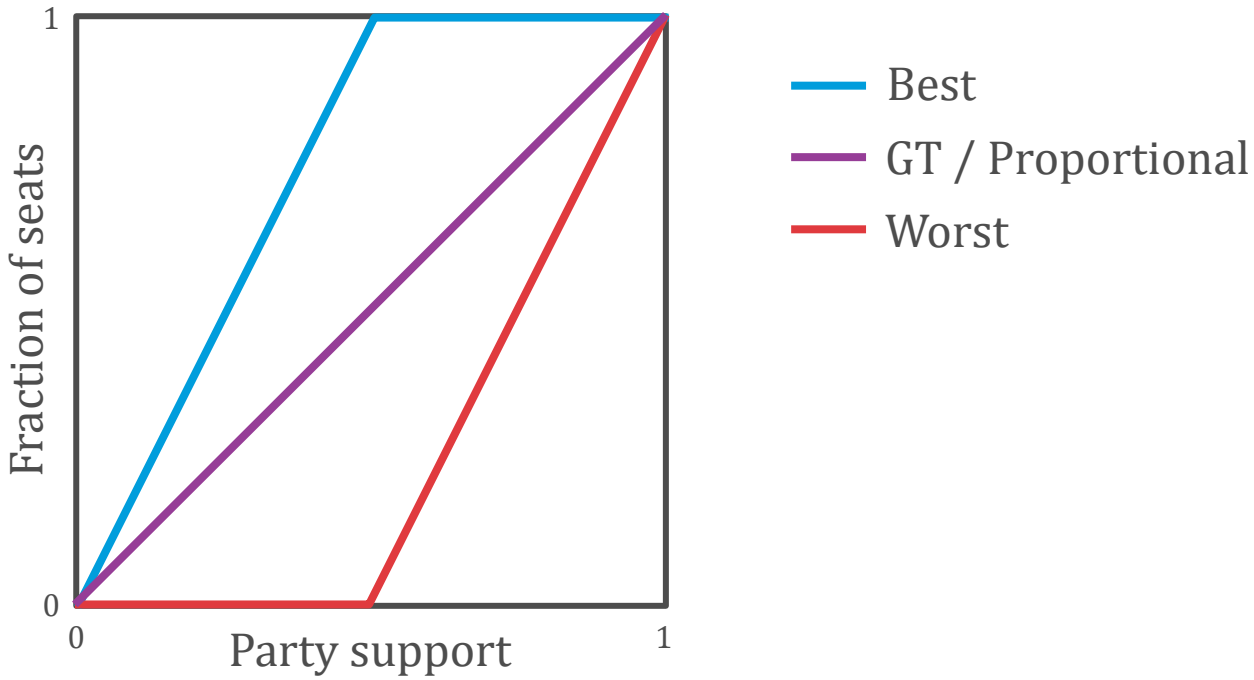
GEOMETRIC TARGET

- Consider a set $\mathcal{D}$ of possible partitions (possibly obeying geometric constraints)
- The geometric target of player i is the average of the maximum number of districts they can win (across partitions in $\mathcal{D}$) and the minimum number, rounded down



GEOMETRIC TARGET

Theorem: In the geometry-free model, a partition is proportional if and only if it satisfies the geometric target of both parties (up to ties)

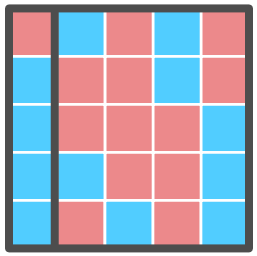


THE LRY PROTOCOL

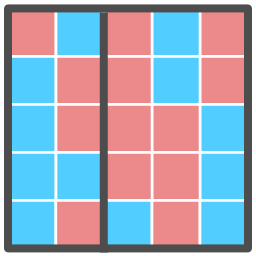
A protocol proposed by Landau, Reid and Yershov (2009):

1. For $j = 1, \ldots, k-1$, a mediator constructs a split (X_j, Y_j) such that $X_1 \subset X_2 \subset \cdots X_{k-1}$
2. For each j, each player is asked "would you rather redistrict X_j, with the other player redistricting Y_j, or vice versa?
3. Try to find a j such that both players are indifferent between redistricting X_j and Y_j
4. If no such j exists, there must be j_0 such that both players want to redistrict Y_{j_0} and X_{j_0+1}. Choose $s \in \{j_0, j_0 + 1\}$ at random and let a random player redistrict X_s and the other player redistrict Y_s

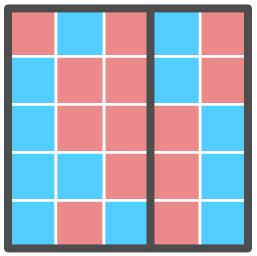
THE LRY PROTOCOL: EXAMPLE



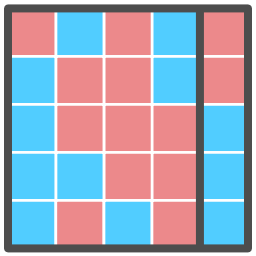
X_1 Y_1



X_2 Y_2

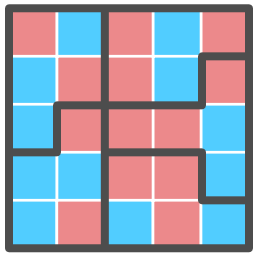


X_3 Y_3



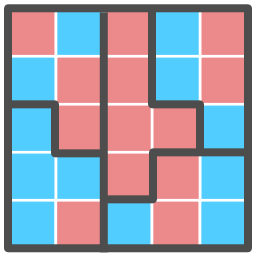
X_4 Y_4

Both players prefer redistricting Y_1, Y_2, X_3, X_4, so the crossover point is $j_0 = 2$



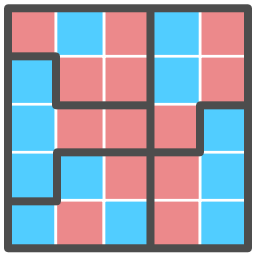
X_2 Y_2

B divides X_2
R divides Y_2



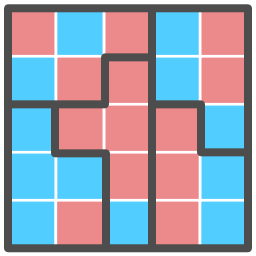
X_2 Y_2

B divides Y_2
R divides X_2



X_3 Y_3

B divides X_3
R divides Y_3

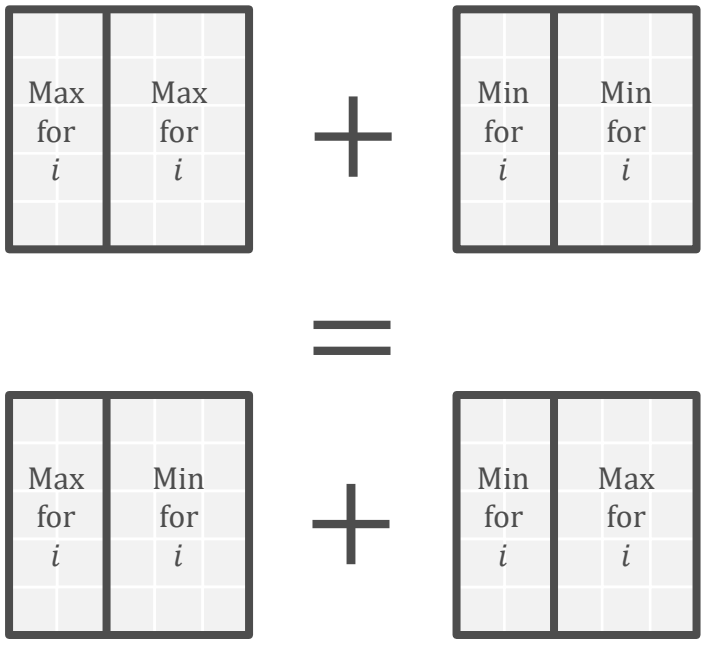


X_3 Y_3

B divides Y_3
R divides X_3

THE LRY PROTOCOL: ANALYSIS

- Theorem: Restricting the set of partitions $\mathcal{D}$ to satisfy a given split, a party's preferred choice satisfies their geometric target



THE LRY PROTOCOL: ANALYSIS

- Theorem: Restricting the set of partitions $\mathcal{D}$ to satisfy a given split, a party's preferred choice satisfies their geometric target
- Even if a party doesn't get its preferred choice in (X_{j_0}, Y_{j_0}), their preference in the very similar split (X_{j_0+1}, Y_{j_0+1}) is reversed, so overall we get an "approximation" of the geometric target

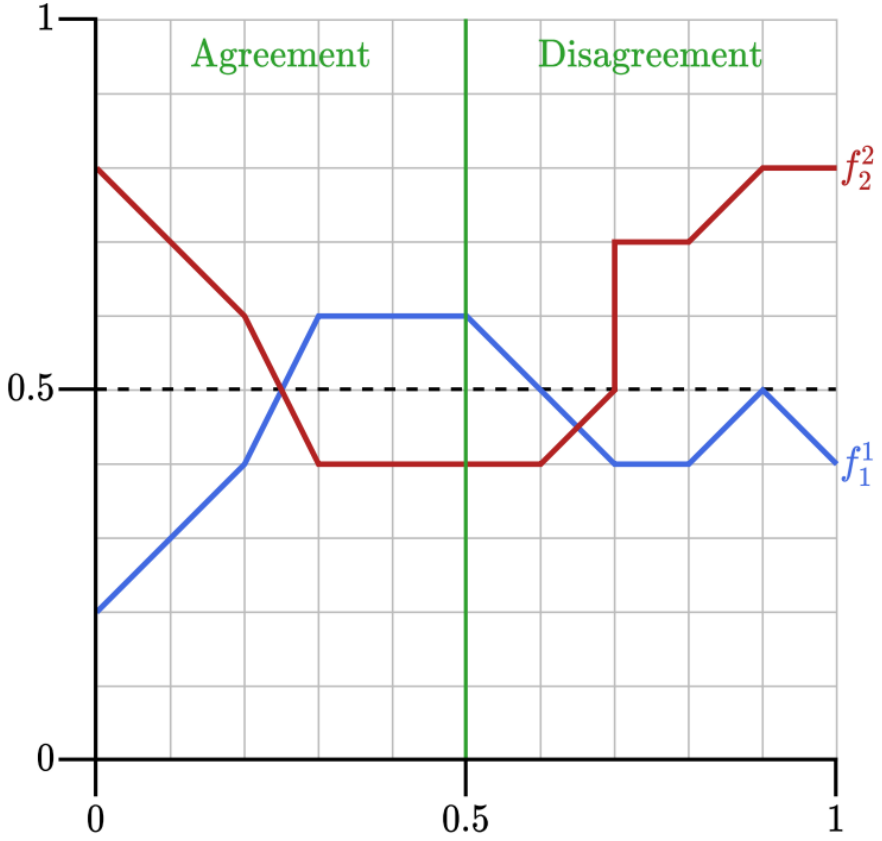
OPTIMIZATION SUBJECT TO FAIRNESS

- Instead of an interactive protocol, optimize an objective function subject to a fairness constraint, namely geometric target
- Possible objectives:
  - Compactness
  - Number of competitive districts
  - Efficiency gap
- Two obstacles:
  - How to solve the optimization problem?
  - Is the geometric target feasible in practice?

THE STATE CUTTING PROBLEM

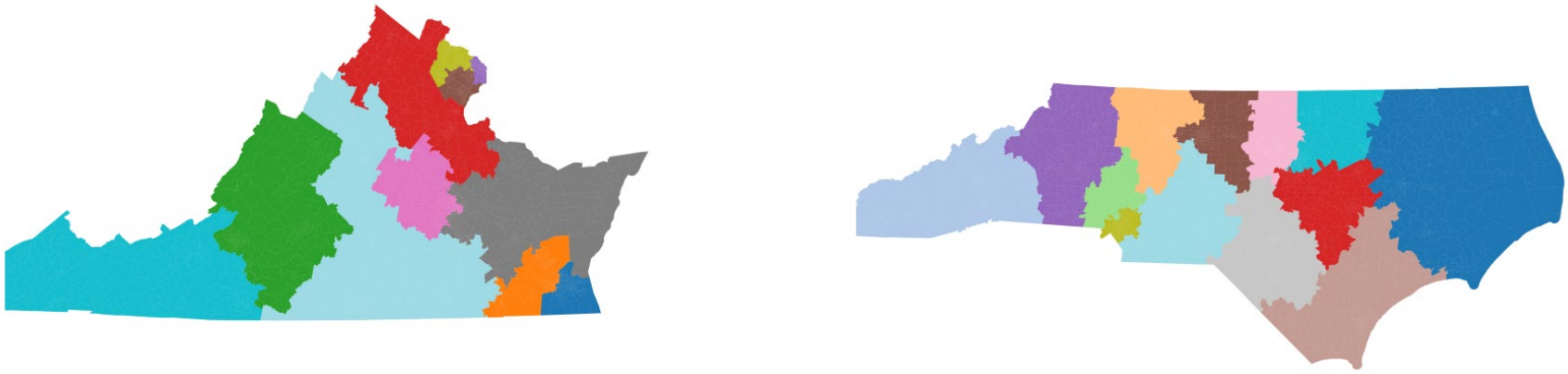
- The state S is the interval [0,1]
- A district D is a finite union of subintervals
- Each player $i \in \{1,2\}$ has a density function $f_i^{\,j}$ such that for all x, $f_i^{\,1}(x) + f_i^{\,2}(x) = 1$
- Let $u_i^j(D) = \int_D f_i^{\,j}(x)\, dx$
- We consider a partition into k districts, where for each, $\mu(D) = 1/k$
- A player i believes they won a district D if $u_i^i(D) > u_i^{3-i}(D)$
- Compared to the non-geometric model, this captures a key real-world constraint: voters can't be partitioned arbitrarily

THE STATE CUTTING PROBLEM



EXISTENCE OF GT PARTITIONS

Theorem: In any instance of the state cutting problem, a partition satisfying the geometric targets of both players with respect to their own beliefs exists



GT partitions in Virginia (left) and North Carolina (right) which outperform their implemented plans in terms of competitiveness, efficiency gap and compactness

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