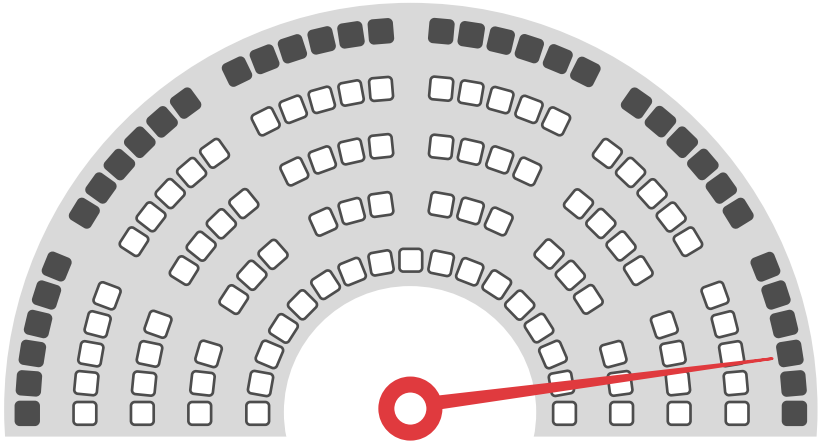


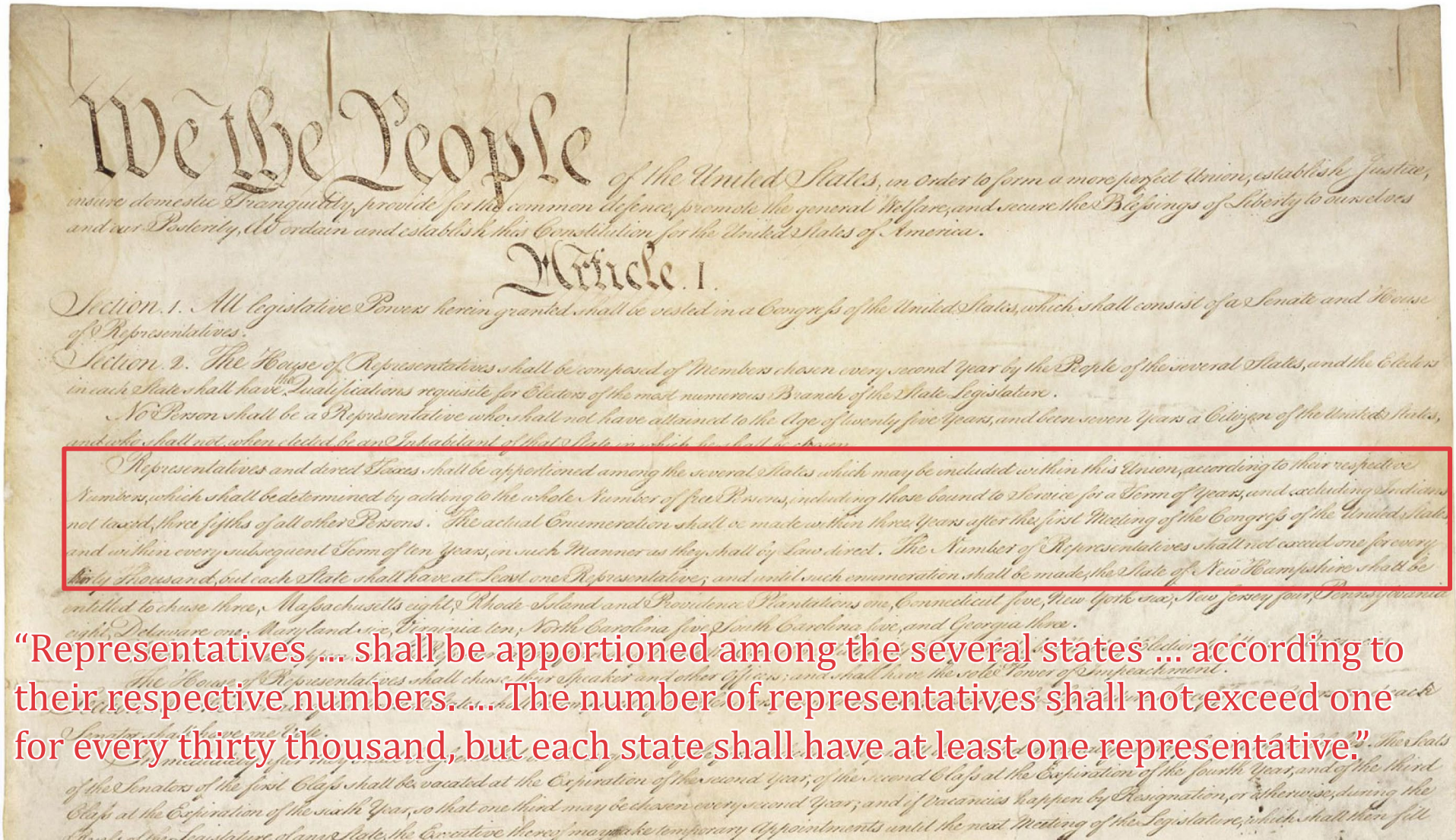
Optimized Democracy

Fall 2026 | Lecture 16

Apportionment in the 19th Century

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THE CONSTITUTION



"Representatives … shall be apportioned among the several states … according to their respective numbers. … The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative."

THE MODEL

- Set of states $N = \{1, \dots, n\}$
- K seats to be allocated
- Each state has population p_i, and the total population is $P = \sum_{i=1}^{n} p_i$
- The standard quota of state i is $q_i = \frac{p_i}{P} \cdot K$
- The upper quota of i is $\lceil q_i \rceil$, and the lower quota is $\lfloor q_i \rfloor$
- Let k_i be the number of seats allocated to i

ROUNDING STANDARD QUOTAS

- The problem is that the standard quotas are fractional
- Simply rounding the standard quotas to the nearest integers may give seat allocations that don't add up to K

State	p_i	q_i	k_i
1	506	50.6	51
2	307	30.7	31
3	187	18.7	19
Total	1,000	100	101



Alexander Hamilton

1755–1804

First secretary of the treasury, co-author of the Federalist Papers. Also known for his role in the eponymous musical.

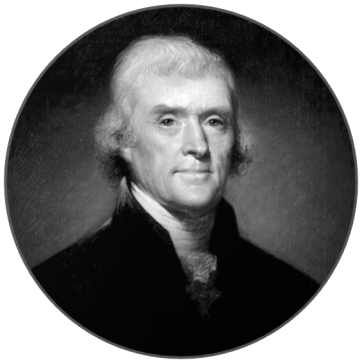
HAMILTON'S METHOD

- Hamilton's Method allocates each state its lower quota and then allocates the remaining seats one at a time to the state with the largest residue $r_i = q_i - \lfloor q_i \rfloor$
- Congress presented a bill on March 26, 1792 that would apportion seats according to Hamilton's Method

HAMILTON'S METHOD

State	p_i	$p_i/30{,}000$	k_i
Connecticut	236,841	7.895	8
Delaware	55,540	1.851	2
Georgia	70,835	2.361	2
Kentucky	68,705	2.290	2
Maryland	278,514	9.284	9
Massachusetts	475,327	15.844	16
New Hampshire	141,822	4.727	5
New Jersey	179,570	5.986	6
New York	331,589	11.053	11
North Carolina	353,523	11.784	12
Pennsylvania	432,879	14.419	14
Rhode Island	68,446	2.282	2
South Carolina	206,236	6.875	7
Vermont	85,533	2.851	3
Virginia	630,560	21.019	21
Total	3,615,920	120.821	120

Based on the census of 1790; 120 seats to be allocated.



Thomas Jefferson

1743–1826

Third president of the United States, first secretary of state. Also known for his supporting role in Hamilton.

JEFFERSON'S METHOD

- Jefferson's Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} \lfloor p_i/D \rfloor = K$, where $\hat{q}_i = p_i/D$ is the modified quota
  - Each state is allocated $k_i = \lfloor \hat{q}_i \rfloor$

- Washington was persuaded to veto the bill enacting Hamilton's Method
- Congress adopted Jefferson's Method on April 10, 1792
- It was used until 1830

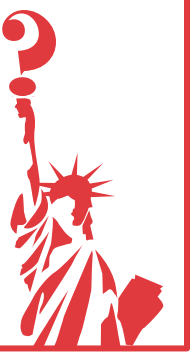
JEFFERSON'S METHOD: EXAMPLE

- Jefferson's Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} \lfloor p_i/D \rfloor = K$, where $\hat{q}_i = p_i/D$ is the modified quota
  - Each state is allocated $k_i = \lfloor \hat{q}_i \rfloor$
- Suppose there are three states with populations $p_1 = 150, p_2 = 320$, and $p_3 = 530$, and $K = 10$

Poll 1

What is the allocation given by Jefferson's Method for the above instance?

- (2,3,5)
- (1,4,5)
- (2,2,6)
- (1,3,6)



JEFFERSON IS WELL-DEFINED

- Theorem: If D and D' are two different divisors yielding Jefferson apportionments $k_1, \dots k_n$ and $k'_1, \dots, k'_n$ then $k_i = k'_i$ for all $i \in N$
- Proof:
  - Assume w.l.o.g. that $D \leq D'$, then $p_i/D \geq p_i/D'$ for all $i \in N$
  - We conclude that $k_i \geq k'_i$ for all $i \in N$
  - It also holds that $\sum_{i \in N} k_i = K = \sum_{i \in N} k'_i$
  - It can't be the case that $k_i > k'_i$ for some $i \in N$ ■

JEFFERSON'S LARGE-STATE BIAS

		$D = 100{,}000$		$D = 97{,}000$	
State	p_i	$\hat{q}_i$	k_i	$\hat{q}_i$	k_i
1	2,620,000	26.20	26	27.01	27
2	168,000	1.68	1	1.73	1
…	…	…	…		
Total	10,000,000	…	99	…	100

- State 1 gets the additional seat despite initially having the smaller residue
- When the divisor is reduced, each seat requires 3,000 fewer citizens, and state 1 gains for each of its 26 seats
- State 1 needs 97,037 citizens per seat whereas state 2 needs 168,000



John Adams

1735–1826

Second president of the United States, first vice president. Also known for being mocked by King George III.

ADAMS' METHOD

- Adams' Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n} \lceil \hat{q}_i \rceil = K$
  - Each state is allocated $k_i = \lceil \hat{q}_i \rceil$
- The large states were against the proposal
- Adams' Method was considered by Congress but never adopted

ADAMS' SMALL-STATE BIAS

		$D = 100{,}000$		$D = 104{,}000$	
State	p_i	$\hat{q}_i$	k_i	$\hat{q}_i$	k_i
1	2,668,000	26.68	27	25.65	26
2	120,000	1.20	2	1.15	2
…	…	…	…		
Total	10,000,000	…	101	…	100

- State 1 loses a seat despite initially having the larger residue
- When the divisor is increased, each seat requires 4,000 more citizens, and state 1 loses for each of its 27 seats
- State 1 needs 102,615 citizens per seat whereas state 2 needs 60,000

WEBSTER'S METHOD

- Webster's Method:
  - Takes a desired number of seats K
  - Finds a divisor D such that $\sum_{i=1}^{n}[\hat{q}_i] = K$
  - Each state is allocated $k_i = [\hat{q}_i]$
- This method isn't clearly biased towards small or large states
- Webster's Method was adopted by Congress in 1842

WEBSTER'S METHOD

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Poll 2

With exactly two states, Hamilton always produces the same allocation as:

- Jefferson
- Webster
- Adams
- None of the above



HISTORICAL INTERLUDE

- In 1850, Senator Samuel Vinton (independently?) proposed a method that is identical to Hamilton's
- Vinton's (Hamilton's) Method was finally adopted by Congress that year
- The House increased from 233 seats to 234, a size on which the allocations from Hamilton's Method and Webster's Method coincided
- The size of the House increased to 241 in 1860 and to 292 in 1870

ALABAMA PARADOX

Under Hamilton's Method, adding seats can decrease a state's allocation!

		$K = 10$		$K = 11$	
State	p_i	q_i	k_i	q_i	k_i
1	6	4.286	4	4.714	5
2	6	4.286	4	4.714	5
3	2	1.429	2	1.571	1
Total	14	10	10	11	11

A method that avoids this paradox is called house monotonic

ALABAMA PARADOX

- The Alabama Paradox was discovered in 1880 by C. W. Seaton, the chief clerk of the Census Office
- Using the 1880 census results, he calculated allocations according to Hamilton's Method for all House sizes between 275 and 350
- When he went from 299 to 300, Alabama lost a seat!
- Congress decided to go with 325 seats, on which Hamilton's Method and Webster's Method agreed
- In 1890 there were no issues, but in 1900 the Alabama Paradox reappeared with Colorado and Maine taking the place of Alabama

POPULATION PARADOX

Under Hamilton's Method, a state whose population grew can lose a seat to a state whose population shrank

	Before			After		
State	p_i	q_i	k_i	p_i	q_i	k_i
1	145	1.45	2	147	1.55	1
2	340	3.40	3	338	3.56	4
3	515	5.15	5	465	4.89	5
Total	1000	10	10	950	10	10

A method that avoids this paradox is called population monotonic

POPULATION PARADOX

- In 1900, the populations of Virginia and Maine were 1,854,184 and 694,466, respectively
- In the following year Virginia's population grew by 19,767 (+1.06%) while Maine's increased by 4,649 (+0.7%)
- Hamilton's Method would have allocated an additional seat to Maine at the expense of Virginia

OKLAHOMA PARADOX

Under Hamilton's Method, adding a state and increasing the size of the house accordingly* can change the allocation of existing states

	Before			After		
State	p_i	q_i	k_i	p_i	q_i	k_i
1	145	1.45	2	145	1.50	1
2	340	3.40	3	340	3.51	4
3	515	5.15	5	515	5.31	5
4	—	—	—	260	2.68	3
Total	1000	10	10	1260	13	13

* To calculate the new house size, divide the new population by the original standard divisor and round to the nearest integer

OKLAHOMA PARADOX

- When Oklahoma became a state in 1907, it was awarded 5 representatives and the size of the House increased by 5
- But if the allocation was recomputed according to Hamilton's method (which was used at the time) and the same 1900 census data, New York would have had to transfer a seat to Maine

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