Lecture on Participatory Budgeting

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Recap: Approval-based Committee Elections (ABC)

Last lecture you heard about approval-based committee elections.

- Setting: A set N of voters, where each $i \in N$ approves a subset α_i of the m candidates, with the task being to select k of them into a committee W.
- Proportional Approval Voting (PAV) is the rule that selects the committee W maximizing

$$\sum_{i \in N} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|\alpha_i \cap W|}.$$

- It satisfies extended justified representation (EJR) which says that for all groups $S \subseteq N$ of voters of size $|S| \geqslant \ell \cdot \frac{n}{k}$ who approve ℓ candidates in common ($|\bigcap_{i \in S} \alpha_i| \geqslant \ell$), at least one voter $i \in S$ approves at least ℓ candidates in the committee ($|\alpha_i \cap W| \geqslant \ell$).

Participatory Budgeting

Participatory Budgeting (PB) is used by many cities to let their residents vote for projects that will be funded by the city government.

— Ballot Paper —

Total available budget: € 3 000 000.

Approve up to 4 projects.

- ☒ Extension of the Public Library
  Cost: € 200 000
- ☐ Additional Public Toilets
  Cost: € 340 000
- ☐ Photovoltaic Panels on City Buildings
  Cost: € 150 000
- ☐ Digital White Boards in Classrooms
  Cost: € 250 000
- ☒ Bicycle Racks on Main Street
  Cost: € 20 000
- ☐ Improve Accessibility of Town Hall
  Cost: € 600 000
- ☐ Sports Equipment in the Park
  Cost: € 15 000
- ☒ Beautiful Night Lighting of Town Hall
  Cost: € 40 000
- ☐ Renovate Fountain in Market Square
  Cost: € 65 000
- ☐ Resurface Broad Street
  Cost: € 205 000

Participatory Budgeting around the World

see Wikipedia:List of participatory budgeting votes.

Participatory Budgeting: Model

- A set C of projects.
- Each project $c \in C$ has a cost $\text{cost}(c) \geqslant 0$.
- A set N of n voters.
- Each voter $i \in N$ approves a subset $A_i \subseteq C$; we get a profile $P = (A_i)_{i \in N}$.
- A total budget B.
- Outcome: an outcome $W \subseteq C$ with $\sum_{c \in W} \text{cost}(c) \leqslant B$.

Notes:

- If $\text{cost}(c) = 1$ for all $c \in C$ and B is an integer, this is a committee election.
- For a set $T \subseteq C$ of projects, we write $\text{cost}(T) = \sum_{c \in T} \text{cost}(c)$.

The Greedy Method

Almost all cities that use PB use the same voting method for deciding which projects win, which we call the Greedy Method.

1. $R \leftarrow C$, the remaining projects
2. $W \leftarrow \emptyset$
3. **while** $R \neq \emptyset$ **do**
    - $c^* \leftarrow$ a project in R with the most votes, i.e. $c^* \in \arg\max_{c \in R} |\{i \in N : c \in A_i\}|$
    - **if** $\sum_{c \in W} \text{cost}(c) + \text{cost}(c^*) \leqslant B$ **then**
        - $W \leftarrow W \cup \{c^*\}$
    - $R \leftarrow R \setminus \{c^*\}$
4. **return** W

Poll

What notion of social welfare does the Greedy Method approximately maximize?

- $\sum_{i \in N} |A_i \cap W|$ (utilitarian welfare where utility = number of approved funded projects)
- $\sum_{i \in N} \text{cost}(A_i \cap W)$ (utilitarian welfare where utility = funding for approved projects)
- $\sum_{c \in W} |\{i \in N : c \in A_i\}| / \text{cost}(c)$ (average efficiency of the funded projects)

The Greedy Method

Recall the knapsack problem: a list of items with payoffs $p_1, \dots, p_m$ and costs $c_1, \dots, c_m$, with an available budget B. The greedy algorithm for this problem selects projects in order of "bang-per-buck", i.e. p_i / c_i, until the budget runs out.

Let $v_c = |\{i \in N : c \in A_i\}|$ be the vote count of project c. The Greedy Method goes in order of

$$v_c = \frac{v_c \cdot \text{cost}(c)}{\text{cost}(c)}.$$

Thus, the Greedy Method approximately solves a knapsack problem with payoffs $v_c \cdot \text{cost}(c)$, and thus it approximately maximizes $\sum_{i \in N} \text{cost}(A_i \cap W)$.

Based on this, one can argue that cities implicitly assume that voters have what are called cost utilities, i.e., their payoff is $u_i(W) = \text{cost}(A_i \cap W)$.

Could also use cardinality utilities with payoff $u_i(W) = |A_i \cap W|$, corresponding to selecting projects in order of $v_c / \text{cost}(c)$. In practice this leads to only the cheapest projects winning.

Discussion

What utility model makes more sense? Are there other ones?

The Greedy Method: Advantages and Disadvantages

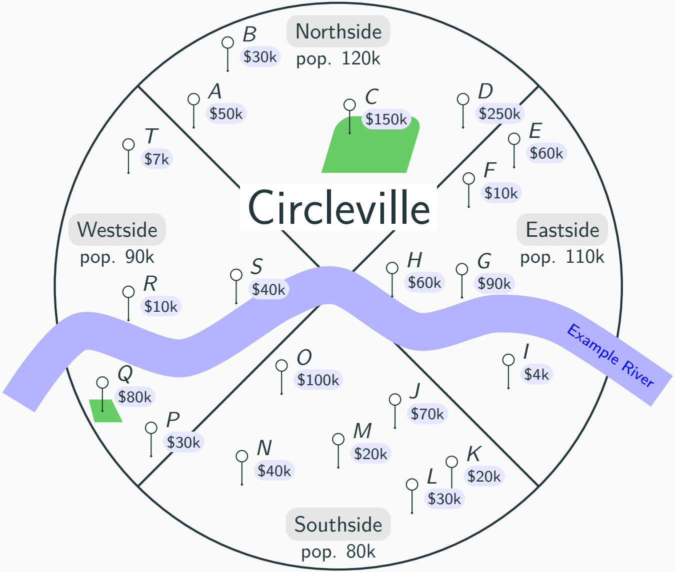
Advantages

- Easy to understand for citizens and for government officials
- Approximately maximizes the utilitarian social welfare ($\sum_{i \in N} u_i(W)$)

Disadvantages

- Many voters may be unrepresented ($W \cap A_i = \emptyset$)
- A large coordinated group of voters can get a disproportionately large share of the budget
- In practice, need to run separate elections for separate districts
- Small projects have only a small chance of winning
- The method almost ignores the cost of projects
- There are incentives for project proposer to "bundle" their projects: if we merge two projects and thereby get more votes, we are more likely to win

Circleville



Aiming for Proportional Representation

Many of the disadvantages of the Greedy Method would be fixed by a rule that computes an outcome providing proportional representation, in the sense of satisfying an EJR-style property.

Definition (EJR)

An outcome W satisfies Extended Justified Representation if for all groups $S \subseteq N$ of voters and every proposal $T \subseteq C$ consisting of a set of projects that S can afford with its proportional share of the budget ($\text{cost}(T) \leqslant \frac{|S|}{|N|} \cdot B$) and that they all approve ($T \subseteq \bigcap_{i \in S} A_i$), there exists at least one voter $i \in S$ with $\text{cost}(A_i \cap W) \geqslant \text{cost}(T)$.

EJR will imply that cohesive groups of voters will not be unrepresented ($W \cap A_i = \emptyset$), and cannot be overridden by a larger cohesive group.

Consequence: Projects are guaranteed to win if they have enough single-minded supporters – if project c costs 20% of the budget B and 20% of voters only approve c, then c must win.

What rules satisfy EJR?

Satisfying EJR

Natural first try: PAV! It is not obvious how to define it for cost utilities, and all natural definitions fail EJR.

There exist some other known ways to get EJR in the committee context, and many are polynomial time. But when projects can have different costs, the following holds:

Theorem

Unless $P = NP$, there is no polynomial time algorithm that produces an EJR outcome.

This is because for the $n = 1$ voter case, such an algorithm would need to solve SUBSET SUM.

Existence proof: there exists a method (computationally intractable) that always satisfies EJR.

But we can get "almost" EJR using natural polynomial time methods.

The Method of Equal Shares

In 2019, we proposed a different method that is fairer because it provides proportional representation: the Method of Equal Shares (MES).

Sketch of the definition:

1. Assign each voter $i \in N$ a virtual budget $b_i = B/n$.
2. $W \leftarrow \emptyset$
3. $R \leftarrow \{c \in C \setminus W : \sum_{i \in N : c \in A_i} b_i \geqslant \text{cost}(c)\}$, the set of affordable projects.
4. **if** $R = \emptyset$ **then return** W
5. Select $c^* \in R$ with the highest "effective vote count" (will define later, roughly: don't count voters with no money left)
6. $W \leftarrow W \cup \{c^*\}$
7. Deduct $\text{cost}(c^*)$ from the budgets of the voters of c^*, i.e. from $\{i \in N : c^* \in A_i\}$, spreading the cost as equally as possible (will define later).
8. **go to** line 3

Method of Equal Shares: Example

Budget: \$1,100

	cost	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	vote count
🚲 bike path	\$700	✓	✓	✓	✓	✓	✓					✓	7
🏋️ outdoor gym	\$400	✓	✓	✓	✓	✓	✓						6
🌳 new park	\$250		✓		✓	✓		✓			✓		5
⚽ new playground	\$200							✓	✓	✓	✓		4
📚 library for kids	\$100							✓		✓	✓		3

Greedy selects 🚲 bike path and 🏋️ outdoor gym.

Method of Equal Shares: Example

	cost	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	eff. vote c.
🚲 bike path	\$700	✓	✓	✓	✓	✓	✓					✓	7
🏋️ outdoor gym	\$400	✓	✓	✓	✓	✓	✓						6
🌳 new park	\$250		✓		✓	✓		✓			✓		5
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outdoor gym	\$400	✓	✓	✓	✓	✓	✓						6
new park	\$250		✓		✓	✓		✓			✓		5
new playground	\$200							✓	✓	✓	✓		4
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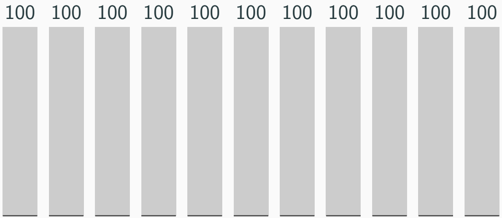

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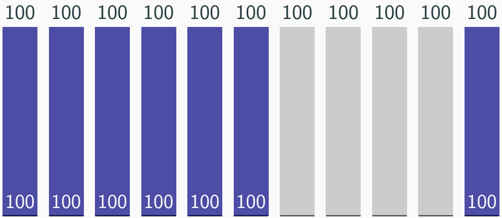
📚 library for kids	\$100							✓		✓	✓		3



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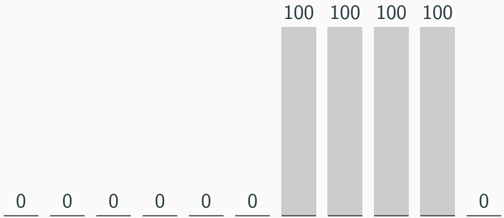
🌳 new park	\$250		✓		✓	✓		✓			✓		~~5~~ 2
⚽ new playground	\$200							✓	✓	✓	✓		4
📚 library for kids	\$100							✓		✓	✓		3



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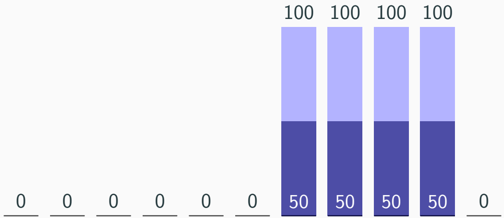

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⚽ new playground	\$200							✓	✓	✓	✓		~~4~~ ✓
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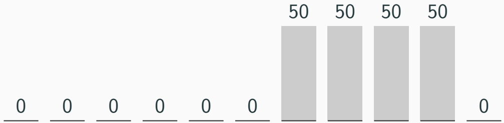

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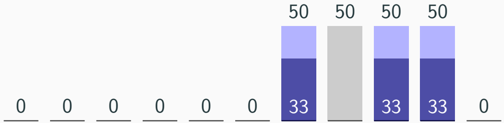

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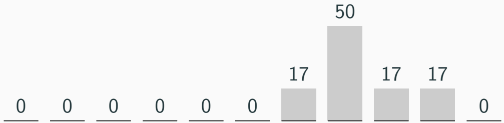
🌳 new park	$250		✓		✓	✓		✓			✓		~~5~~ ~~2~~ 0
⚽ new playground	$200							✓	✓	✓	✓		~~4~~ ✓
📚 library for kids	$100							✓		✓	✓		~~3~~ ✓

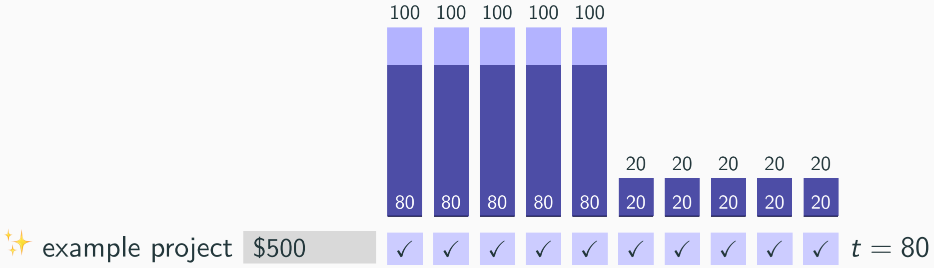


Spreading Cost as Equally as Possible

Sharing the cost of a project "as equally as possible" means that we distribute it among the project's supporters to minimize the highest amount t that any of these voters contributes.

If we choose t optimally, every voter with more than t remaining (*full* voters) will contribute t, and all other voters (*fractional* voters) contribute whatever money they have left. If we sum up these contributions, we cover precisely the cost of the project. Thus, if for every voter i we write b_i for their remaining money, we choose the (unique) value of t that satisfies

$$\underbrace{t + \cdots + t}_{\text{full voters}} + \underbrace{b_{i_1} + \cdots + b_{i_q}}_{\text{fractional voters}} = \text{cost of the project.}$$



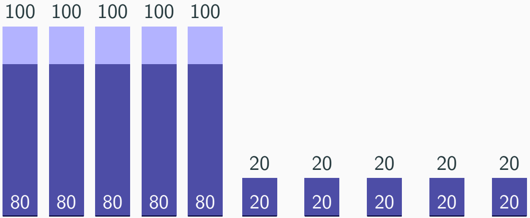
Effective Vote Count

In order to compute the effective vote count of the project we first simulate how its cost would be shared: thus, we determine the value of t and the division into full and fractional voters.

$$\underbrace{t + \cdots + t}_{\text{full voters}} + \underbrace{b_{i_1} + \cdots + b_{i_q}}_{\text{fractional voters}} = \text{cost of the project.}$$

The effective vote count of the project is then the number of full voters plus for each fractional voter i the fraction of the full contribution t that i pays, that is b_i/t. Note that voters who have no money left ($b_i = 0$) count as 0. From the above equation, we see that

$$t \cdot \text{effective vote count} = \text{cost of the project.}$$



✨ example project $500 1 1 1 1 1 0.25 0.25 0.25 0.25 0.25 eff. vote count = 6.25

Method of Equal Squares: More Difficult Example

	cost	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	eff. vote c.
🚲 bike path	\$700	✓	✓	✓	✓	✓	✓					✓	7
🏋️ outdoor gym	\$400	✓	✓	✓	✓	✓	✓						6
🌳 new park	\$250		✓		✓	✓		✓			✓		5
⚽ new playground	\$200							✓	✓	✓	✓		4
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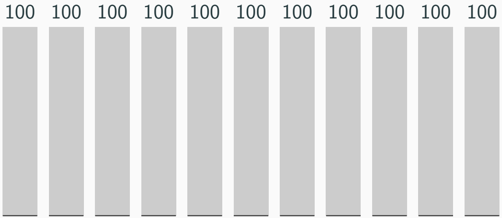

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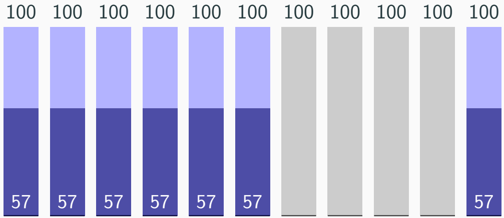

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new park	\$250		✓		✓	✓		✓			✓		~~5~~ ?
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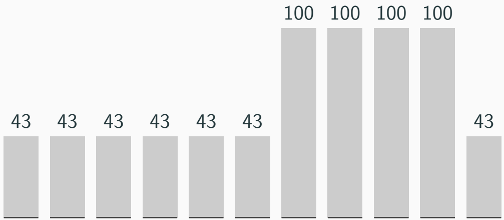
library for kids	\$100							✓		✓	✓		3



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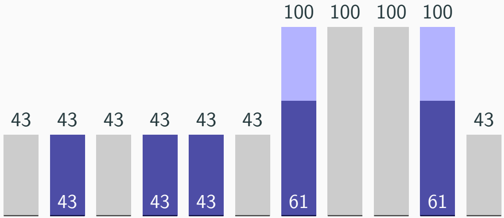
🌳 new park	\$250		✓		✓	✓		✓			✓		~~5~~ 4.13
⚽ new playground	\$200							✓	✓	✓	✓		4
📚 library for kids	\$100							✓		✓	✓		3



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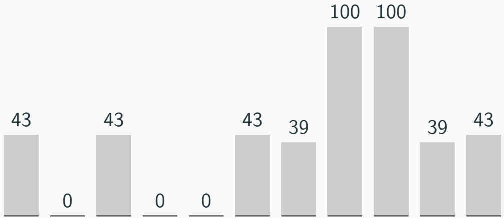

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new park	\$250		✓		✓	✓		✓			✓		~~5~~ ~~4.13~~ ✓
new playground	\$200							✓	✓	✓	✓		~~4~~ 3.31
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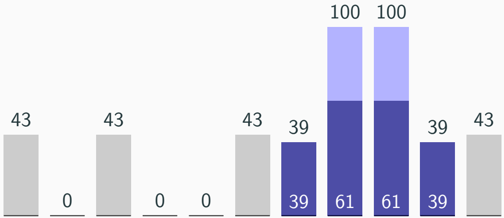

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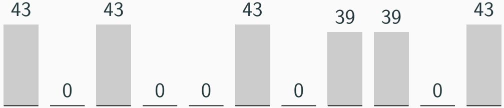
new park	\$250		✓		✓	✓		✓			✓		~~5~~ ~~4.13~~ ✓
new playground	\$200							✓	✓	✓	✓		~~4~~ ~~3.31~~ ✓
library for kids	\$100							✓		✓	✓		~~3~~ 0



MES Satisfies EJR Up To One Project

An outcome W satisfies EJR *up to one project* if for all groups $S \subseteq N$ of voters and every proposal $T \subseteq C$ consisting of a set of projects that S can afford ($\text{cost}(T) \leqslant \frac{|S|}{|N|} \cdot B$) and that they all approve ($T \subseteq \bigcap_{i \in S} A_i$), we have that either $T \subseteq W$, or there exists $i \in S$ and a project $p^* \in T \setminus W$ with $\text{cost}((A_i \cap W) \cup \{p^*\}) \geqslant \text{cost}(T)$.

Theorem

MES satisfies EJR1.

MES Satisfies EJR Up To One Project

Theorem

MES satisfies EJR1.

Consider a coalition S of $s = |S|$ voters and proposal T. If we have $T \subseteq W$, we are done. Otherwise, there is $p^* \in T \setminus W$. Thus at the end of MES, S does not have enough money left to pay for p^*. Thus their average remaining budget is less than $c(p^*)/s$.

So there must be a first time when the budget of someone in S (say i) drops below $c(p^*)/s$. Note that i has now spent more than $B/n - c(p^*)/s$. Up until this point in time, S could afford p^* by themselves with equal cost split. So the effective vote count of p^* was at least s. Since we always pick the alternative with the highest effective vote count, we've always picked alternatives with effective vote count at least s. Note that for each dollar that i spends on such a project, a total of at least s dollars was spent on it by all voters overall. Therefore

$$u_i > s \cdot \left(\frac{B}{n} - \frac{c(p^*)}{s} \right) = \frac{s}{n} \cdot B - c(p^*),$$

as desired. □

Putting MES into Practice

Since 2023, MES has been used by several European cities.

- In April 2023, Wielieczka in Poland used MES for its "Green Million" PB of ecological projects. MES selected projects covering the whole city area including outlying villages, which the Greedy Method would not have done.
- In June 2023, Aarau in Switzerland used MES. It selected projects all across the city.
- Since 2023, Świecie in Poland has been using MES. Previously, they ran separate elections for urban and for rural projects. Now they can run one joint election.
- In 2023 and 2024, Winterthur used MES as part of a deliberative PB, where half of the budget was distributed via voting and MES, and the other half based on deliberations in a committee of citizens assembled through sortition.

Additional slides on these applications: https://dominik-peters.de/scw.pdf

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