

Optimized Democracy (Fall 2025)

Problem Set #3

Due: 10/29/2025 11:59pm ET

Instructions:

- You may discuss the problems with classmates but please write down solutions completely on your own.
- The solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You must not use AI in any way.
- Please *type up* your solution and submit to Gradescope.

Problems:

1. Recall the epistemic liquid democracy model, which was introduced and analyzed in the “liquid democracy” lecture, slides 5–12. We saw that local delegation mechanisms cannot satisfy do no harm (DNH) and positive gain (PG).

Now consider the following non-local delegation mechanism. For each voter $i = 1, \dots, n$, if $A_G(i) \neq \emptyset$ (i approves other voters), the mechanism determines the lowest j such that i approves j , that is, $j = \min A_G(i)$. Then i delegates to j if and only if j has not already delegated their vote and there is no other voter who has already delegated to i or j . Intuitively, under this mechanism, some delegations could take place but no voter would ever have a weight of more than 2.

For simplicity, let us fix $\alpha = 0.1$ for this problem, that is, i approves j if and only if $(i, j) \in E$ and $p_j > p_i + 0.1$.

- (a) [5 points] Show that the above mechanism satisfies PG.
- (b) [15 points] Show that the above mechanism does *not* satisfy DNH.

Note and hint: This is surprising because the mechanism prevents the problem of voters amassing large weight. It turns out that the mechanism does satisfy DNH with an additional assumption: for all $i \in N$, $p_i \in [\beta, 1 - \beta]$ for $\beta > 0$. Here you are asked to give a family of counterexamples that would necessarily have to violate this assumption.

2. In class we discussed notions of proportionality for approval-based elections like extended justified representation (EJR), which only guarantees that one voter in each “deserving” coalition is satisfied. In this problem our goal is to provide guarantees that hold *on average*.

Recall that in the approval-based committee elections settings we have a set N of n voters and a target committee size k , where each voter $i \in N$ approves a set of alternatives $\alpha_i \subseteq A$. Let $q := \frac{n}{k}$. We say that a set of $S \subseteq N$ of voters is ℓ -cohesive if $|S| \geq \ell \cdot q$ and $|\bigcap_{i \in S} \alpha_i| \geq \ell$. As in class, we write $u_i(W) = |W \cap \alpha_i|$.

- (a) [10 points] Assume that q is an integer. Suppose that a committee $W \subseteq A$, $|W| = k$, satisfies extended justified representation (EJR), so for each $1 \leq \ell \leq k$ and every ℓ -cohesive group S , there exists $i \in S$ with $u_i(W) \geq \ell$. Now let S be an ℓ -cohesive group with $|S| = \ell \cdot q$. Prove that

$$\sum_{i \in S} \frac{1}{|S|} u_i(W) \geq \frac{\ell - 1}{2},$$

that is, S obtains high average utility.

Note: With more work it can be shown that it is possible to achieve average utility at least $\ell - 1$ for ℓ -cohesive groups.

- (b) [15 pt] Prove that for all $\varepsilon > 0$, there exists an election such that, no matter which committee is chosen, there is a 1-cohesive group S that has average utility at most ε . More formally, there is a set N of n voters, a set A of m alternatives, target committee size k , and approval set $\alpha_i \subseteq A$ for each $i \in N$, such that the following holds. For all committees $W \subseteq A$ with $|W| = k$, there is a set of voters S with $|S| \geq n/k$ and $|\bigcap_{i \in S} \alpha_i| \geq 1$ such that,

$$\sum_{i \in S} \frac{1}{|S|} u_i(W) \leq \varepsilon.$$

Note: With more work it can be shown that there exist elections such that no matter which committee is chosen, there is an ℓ -cohesive group with average utility at most $\ell - 1 + \varepsilon$. This means the lower bound mentioned in the note in part (a) is tight.

Hint: Construct a family of instances with $m = k + 1$ alternatives where each voter approves either one or two alternatives.

3. We proved (“committee elections,” slide 17) that a committee satisfying extended justified representation (EJR) can be found through the PAV rule. But computing PAV is hard.

To efficiently compute a committee satisfying EJR, then, we’ll actually define a stronger notion and show that it can be easily achieved:

A committee W of size k satisfies *EJR+* if for all alternatives $x \in A \setminus W$, if $S \subseteq N$ is such that $|S| \geq \ell n/k$ and $x \in \bigcap_{i \in S} \alpha_i$, then there is $i \in S$ such that $u_i(W) \geq \ell$.

- (a) [5 points] Show that *EJR+* implies EJR.
(b) [10 points] Show that EJR does not imply *EJR+*.
(c) [15 points] Consider the following greedy algorithm. The algorithm starts from $W = \emptyset$, and for $\ell = k, \dots, 1$, it iteratively finds and adds all alternatives $x \in A \setminus W$ such that the number of voters i with $x \in \alpha_i$ and $u_i(W) < \ell$ is at least $\ell n/k$. Intuitively, the algorithm

finds and adds all alternatives causing an EJR+ violation with respect to a given value of ℓ , which starts from $\ell = k$ and is iteratively decremented.

Prove that this algorithm outputs an EJR+ committee of size (at most) k .

Hint: The committee clearly satisfies EJR+, so the challenge is to show that at most k alternatives are added.

4. In the context of sortition we discussed *allocation rules*, which receive a set of volunteers N , a panel size k , a set of features F , set of values V_f for each $f \in F$, and quotas $u_{f,v}, \ell_{f,v}$ for all $f \in F$ and $v \in V_f$; they output a distribution over panels that satisfy the given quotas if one exists. In particular, we saw that allocation rules like Leximin and Maximum Nash Welfare lead to seemingly fair selection probabilities. But do these rules satisfy appealing axiomatic properties? On a high level the answer is negative, but that is mostly due to strong, general impossibility results that hold in this domain; below you are asked to establish those negative results.

- (a) **[15 points]** An allocation rule guarantees *population monotonicity* if, when additional volunteers are added to an instance — that is, there are two sets of volunteers N and N' such that $N \subseteq N'$, and the panel size, features, values and quotas remain unchanged — the selection probabilities of all previously existing volunteers weakly decrease.

Prove that no allocation rule satisfies population monotonicity.

- (b) **[10 points]** An allocation rule guarantees *committee monotonicity* if, when an instance is modified by increasing the size of the panel — that is, there are two panel sizes k and k' such that $k' \geq k$, and the volunteers, features, values and quotas remain unchanged (and the instance remains feasible) — the selection probabilities of all volunteers weakly increase.

Prove that no allocation rule satisfies committee monotonicity.