Optimized Democracy (Fall 2025)
Problem Set #1

Due: 9/22/2025 11:59pm ET

Instructions:

- You may discuss the problems with classmates but please write down solutions completely on your own.
- The solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You must not use AI in any way.
- Please *type up* your solution and submit to Gradescope.

Problems:

1. We showed in class that the proportional veto core is nonempty and an alternative in the proportional veto core can be efficiently computed (e.g., via veto by consumption). But is it possible to efficiently compute the *entire* proportional veto core? Our goal is to give a positive answer. Specifically:

   [**40 points**] Design an algorithm that, given a preference profile $\boldsymbol{\sigma}$ and an alternative $x \in A$, determines whether x is in the proportional veto core of $\boldsymbol{\sigma}$ in polynomial time in the number of alternatives m and the number of voters n.

   Note: The challenge is that there is an exponential number of coalitions that could potentially veto x.

   Guidance: We will design a polynomial-time algorithm that determines whether there is a coalition S that vetoes x, i.e., denoting $|S| = k$, whether there is $B \subseteq A$ of size at least $m - \lceil mk/n \rceil + 1$ such that all voters in S prefer each $y \in B$ to x.

   The algorithm works by reducing the above problem to the *biclique* problem: Given a bipartite graph and an integer ℓ, determine whether there are a vertices on the left and b vertices on the right such that $a + b \geq \ell$ and these vertices form a biclique (there is an edge between every pair of vertices on opposite sides). It is known that the biclique problem can be solved in polynomial time and you may rely on this fact.

   You may also rely on the following fact: there are r and t, both polynomial in n and m, such that for all $k \in \{1, \ldots, n\}$, $\lceil mk/n \rceil - 1 = \lfloor rk/t \rfloor$.

Construct a bipartite graph with rn vertices on the left and $t(m-1)$ vertices on the right. Each voter $i \in N$ is associated with r vertices on the left, and each alternative $y \in A \setminus \{x\}$ is associated with t vertices on the right. There is an edge between a vertex corresponding to voter i and a vertex corresponding to alternative y if $y \succ_{\sigma_i} x$.

The analysis of the algorithm now boils down to proving the following claim: In this bipartite graph, there is a biclique with a vertices on the left and b on the right such that $a + b \geq tm$ if and only if x is vetoed by some coalition.

2. In class we discussed the following notion of monotonicity: If $f(\boldsymbol{\sigma}) = a$ and $\boldsymbol{\sigma'}$ is a profile such that (i) $[a \succ_{\sigma_i} x \;\Rightarrow\; a \succ_{\sigma'_i} x]$ for all $x \in A$ and $i \in N$, and (ii) $[x \succ_{\sigma_i} y \;\Leftrightarrow\; x \succ_{\sigma'_i} y]$ for all $x, y \in A \setminus \{a\}$ and $i \in N$, then $f(\boldsymbol{\sigma'}) = a$. Informally, if you push a upwards and everything else remains the same, a stays the winner. It can be verified that rules like plurality, Borda count, and Llull are monotonic; by contrast, we saw that STV is nonmonotonic.

   [30 points] Show that Dodgson's Rule is nonmonotonic by giving a counterexample.

   Hint: There is an example with four alternatives and four groups of voters.

3. Recall ("distortion" lecture, slide 7) that $\text{dist}_u(\text{plurality}) = \Theta(m^2)$, where m is the number of alternatives.

   [30 points] Prove that $\text{dist}_m(\text{plurality}) = O(m)$, that is, the *metric* distortion of plurality is *upper bounded* by $O(m)$.

   Hint: Like in the utilitarian distortion bound, it is helpful to observe that a plurality winner must be ranked first by at least n/m voters. Needless to say, using the triangle inequality is crucial — otherwise we know that there's a lower bound of $\Omega(m^2)$.