Optimized Democracy

Spring 2024 | Lecture 5

Electoral Competition

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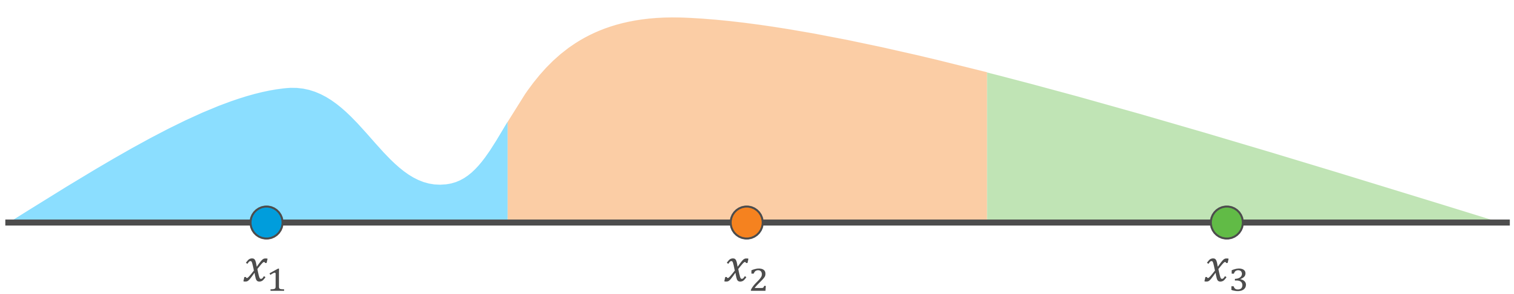
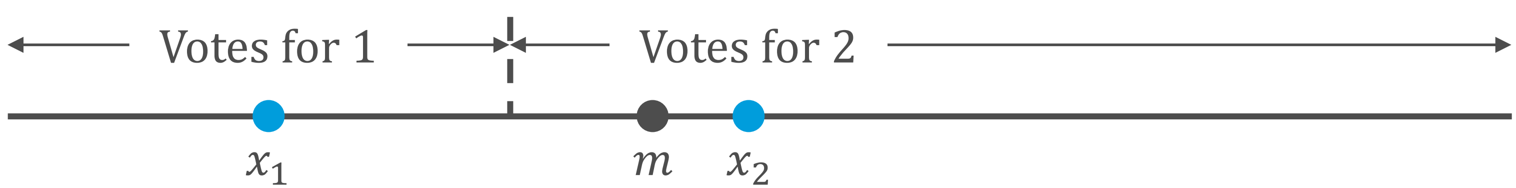
THE HOTELLING MODEL

- Political spectrum is $\mathbb{R}$
- There is a continuous distribution F of voters, each with a peak in $\mathbb{R}$
- Players are candidates, who strategically choose positions $x_1, \ldots, x_n$
- Each candidate attracts the votes of voters who are closest to them, with votes being split equally in case of a tie



THE HOTELLING MODEL

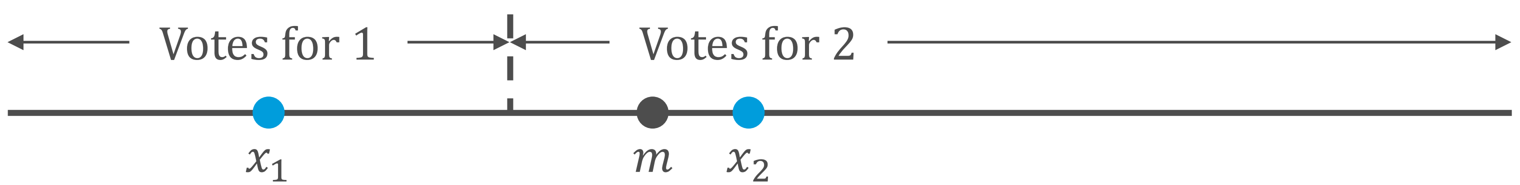
- Two candidates seek to win a plurality of votes
- The utility of each candidate is 1 if they win, 1/2 if they tie, and 0 if they lose
- Denote the median peak by m (assume for simplicity that it's unique)



Who wins?

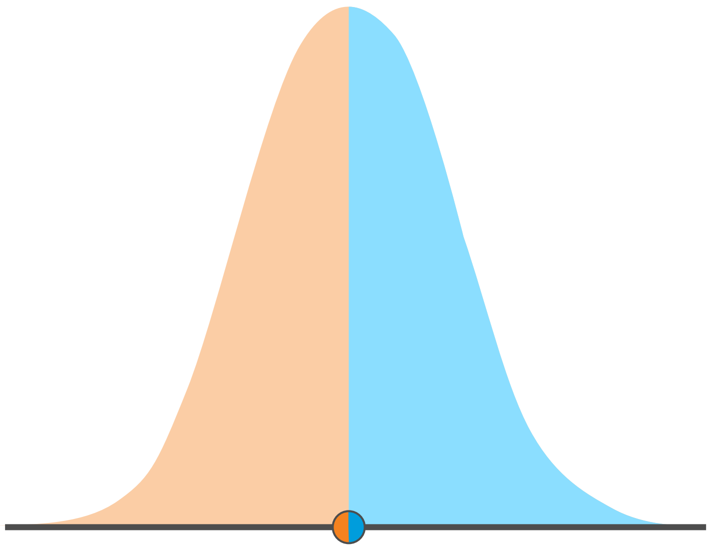
NASH EQUILIBRIUM

A Nash equilibrium is a profile $\boldsymbol{x} \in \mathbb{R}^n$ such that each player is best responding to the others, i.e., for each player i and alternative strategy $x_i' \in \mathbb{R}$, $u_i(\boldsymbol{x}) \geq u_i(x_i', \boldsymbol{x}_{-i})$



Is this a Nash equilibrium?

THE MEDIAN VOTER THEOREM



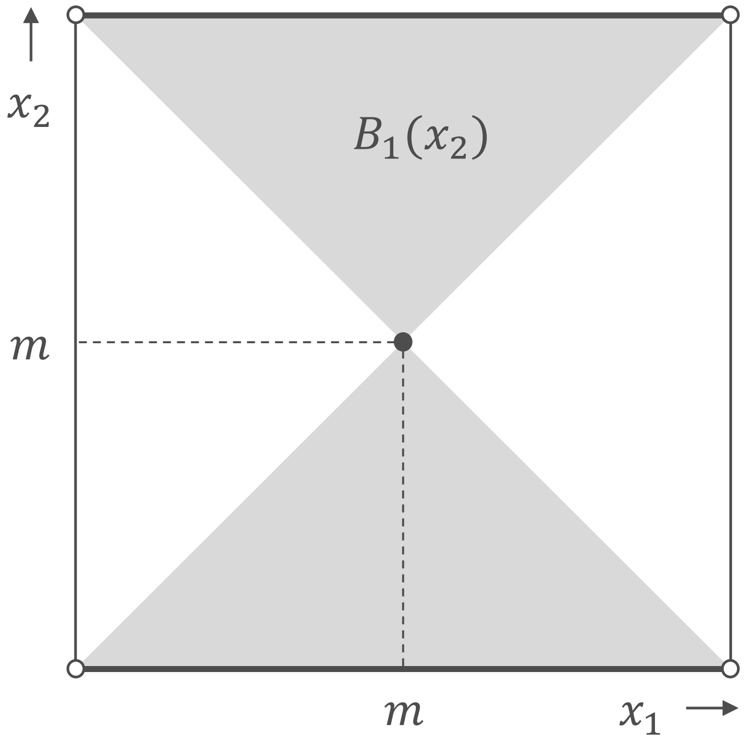
Theorem: In the Hotelling Model with two candidates, there is always a unique Nash equilibrium at (m, m)

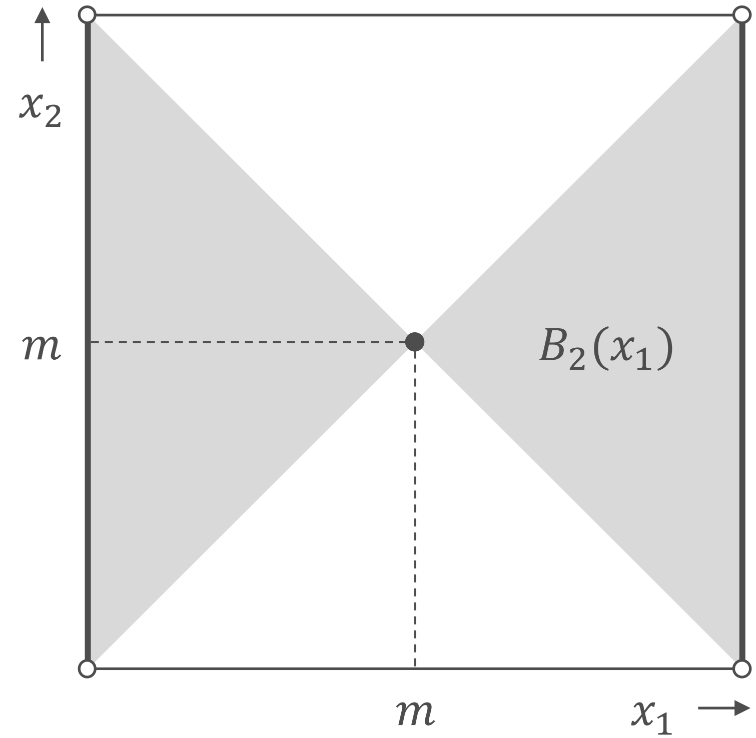
PROOF OF THEOREM

- If $x_2 < m$, the best response for 1 is all positions x_1 such that
$$x_1 > x_2 \quad \text{and} \quad \frac{x_1 + x_2}{2} < m$$
- A symmetric argument holds if $x_2 > m$
- If $x_2 = m$, the best response for 1 is m
- Therefore, it holds that

$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x_1 < 2m - x_2\} & x_2 < m \\ \{m\} & x_2 = m \\ \{x_1 : 2m - x_2 < x_1 < x_2\} & x_2 > m \end{cases}$$

PROOF OF THEOREM





The unique Nash equilibrium is at (m, m)

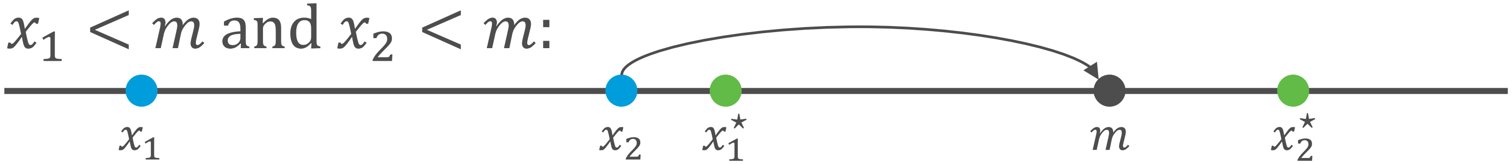
POLICY-MOTIVATED CANDIDATES

- What if candidates care about policy and not just about winning?
- Suppose i has a preferred position $x_i^\star$, and their utility depends on the distance between $x_i^\star$ and the position of the winner
- If there's a tie then candidates evaluate the induced lottery over winning positions
- Theorem: If $x_1^\star < m < x_2^\star$ then (m, m) is the unique Nash equilibrium

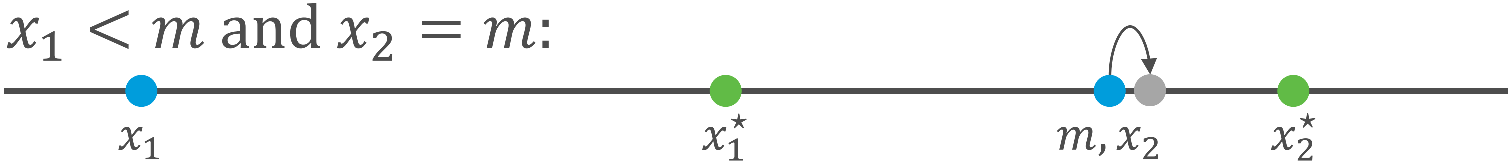
PROOF SKETCH

Rule out cases for which $(x_1, x_2) \neq (m, m)$:

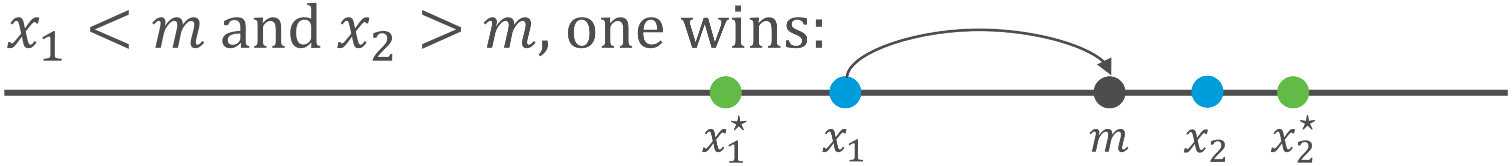
$x_1 < m$ and $x_2 < m$:



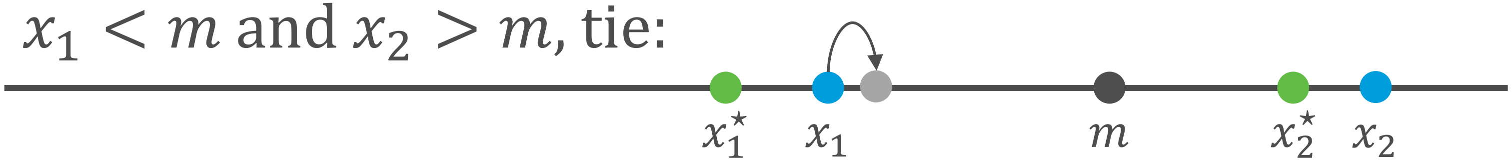
$x_1 < m$ and $x_2 = m$:



$x_1 < m$ and $x_2 > m$, one wins:



$x_1 < m$ and $x_2 > m$, tie:



Harold Hotelling

1895–1973

“The competition for votes between the Republican and Democratic parties does not lead to a clear drawing of issues, an adoption of two strongly contrasted positions between which the voter may choose. Instead, each party strives to make its platform as much like the other's as possible.”

(m, m) USED TO MAKE SENSE

TOWARD A MORE RESPONSIBLE
TWO-PARTY SYSTEM

A Report of the Committee on Political Parties
American Political Science Association

Vol. XLIV September, 1950 Number 3, Part 2

INTRODUCING UNCERTAINTY

- Both candidates believe that the median peak m is distributed according to a distribution μ with strictly positive density over an interval I
- For $x_1 < x_2$, the probability that 1 wins is

$$\pi_1(x_1, x_2) = \Pr_{m \sim \mu}\left[m < \frac{x_1 + x_2}{2}\right]$$

- Candidate i's utility for (x_1, x_2) is

$$\pi_1(x_1, x_2)U_i(x_1) + \pi_2(x_1, x_2)U_i(x_2)$$

where the maximizer of U_i is $x_i^\star$

INTRODUCING UNCERTAINTY

- Theorem: Assume $x_1^\star, x_2^\star \in I$ and $x_1^\star \neq x_2^\star$, then in any Nash equilibrium (x_1, x_2) it holds that $x_1 \neq x_2$
- Proof:
  - If $x_1 = x_2 = x^\star$ then $x^\star$ is enacted with probability 1
  - Without loss of generality $x^\star < x_1^\star$
  - If 1 moves to $x_1' \in (x^\star, x_1^\star)$, then $\pi_1(x_1', x_2) > 0$ and they are better off ■

CITIZEN CANDIDATES

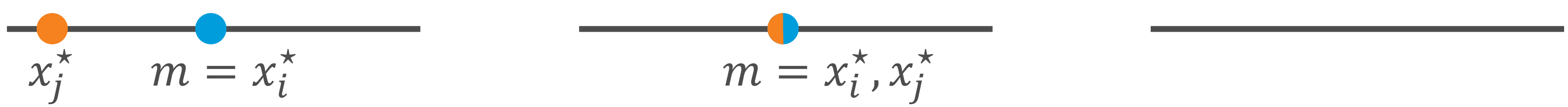
- What if citizens can run in the election?
- Continuum of voters as before, which are now the players
- The position of player i is $x_i^\star$
- Each player chooses whether to run or not
- Players who run are “honest” about their position
- The cost of running is c, and the benefit of winning is w

CITIZEN CANDIDATES

- The utility of player i is
  - $-|x_i^\star - x_j^\star|$ if i doesn't run and j wins
  - $-|x_i^\star - x_j^\star| - c$ if i runs and j wins
  - $w - c$ if i runs and wins
  - $-\infty$ if nobody runs (for simplicity)
- In case of a tie, each player gets their expected utility
- **Theorem:** There is a one-candidate equilibrium if and only if $w \leq 2c$, where if $c \leq w \leq 2c$ then the candidate's position is m, and if $w < c$ then it is in $\left[m - \frac{c-w}{2}, m + \frac{c-w}{2}\right]$

PROOF OF THEOREM

- If $w \leq 2c$, there is an equilibrium where a single player with position m runs:



If another candidate j with a different position runs, they lose

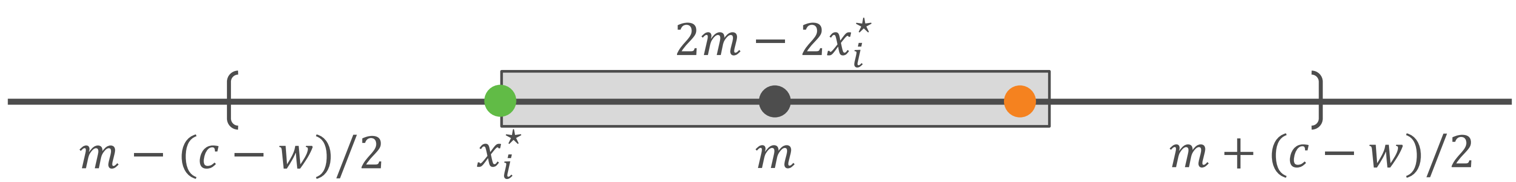
If another candidate j at m runs, they get $w/2 - c \leq 0$ instead of 0

If the single candidate i drops out, they get $-\infty$

- If $w > 2c$, for any profile where one player enters, another player with the same position would wish to enter

PROOF OF THEOREM

- Consider a single candidate with w.l.o.g. $x_i^\star < m$
- Any candidate j in the interval $(x_i^\star, 2m - x_i^\star)$ can run, win, and "pay" $c - w$ instead of $x_j^\star - x_i^\star$
- Since $x_j^\star - x_i^\star$ can be arbitrarily close to $2m - 2x_i^\star$, for this to be an equilibrium it must hold that $c - w \geq 2m - 2x_i^\star$(and hence $c > w$)
- These conditions are also sufficient ■



CITIZEN CANDIDATES

- We next consider equilibria with two candidates

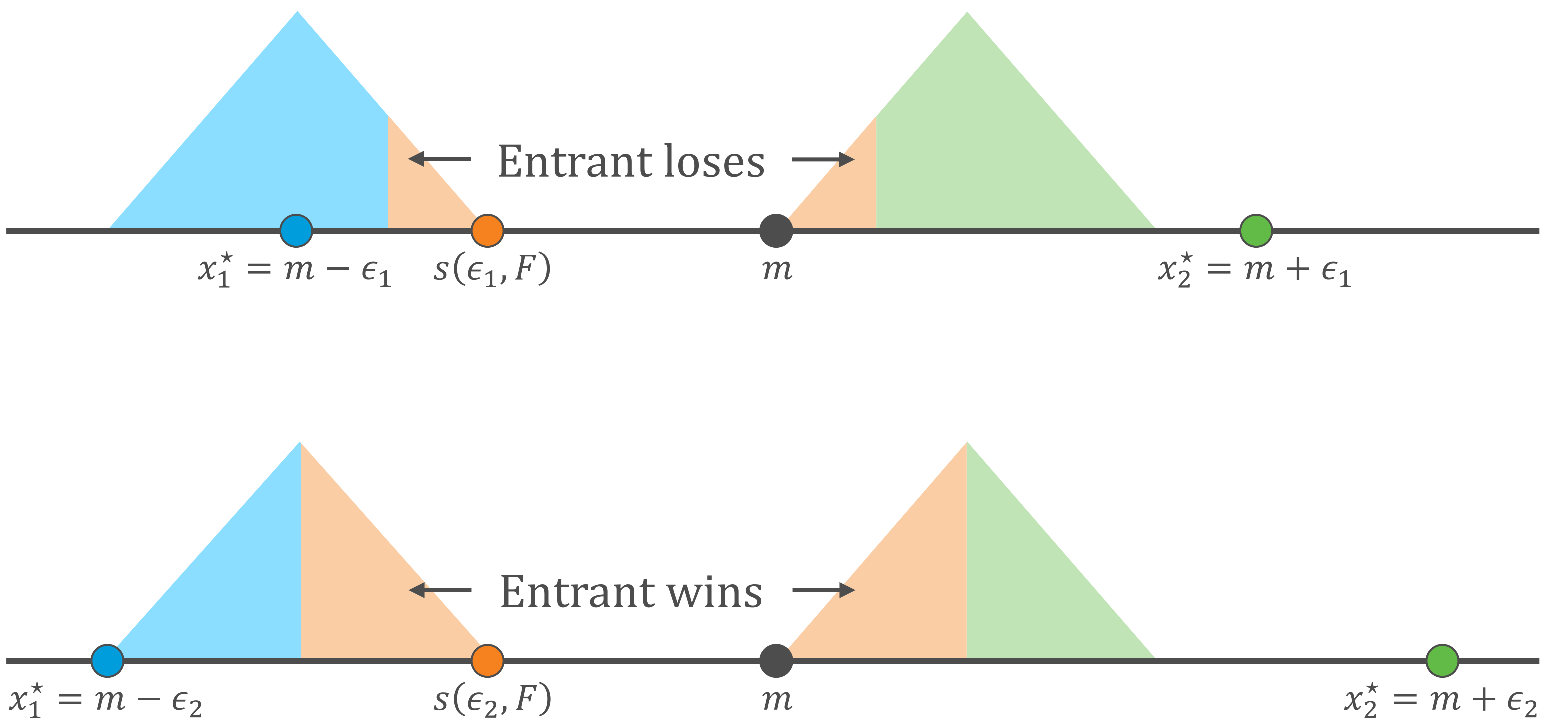
Poll 1

Is there a two-candidate equilibrium with both candidates at m?

- Always
- Sometimes
- Never

CITIZEN CANDIDATES

For candidates at $m - \epsilon$ and $m + \epsilon$, $s(\epsilon, F)$ is the position between them such that if a candidate enters at that position then the number of votes received by the original candidates remains equal



CITIZEN CANDIDATES

- Let $e(F)$ be the maximum value such that for all $\epsilon < e(F)$, entrants at $s(\epsilon, F)$ lose
- Theorem: Two-candidate equilibria exist if and only if $w \geq 2(c - e(F))$, and in any such equilibrium, the positions of the candidates are $m - \epsilon$ and $m + \epsilon$ for $0 < \epsilon \leq e(F)$
- Two-candidate equilibria are such that the positions of the candidates are neither identical nor far apart

EXTENSIONS

- We introduced policy-motivation, uncertainty, and citizen candidates into the original Hotelling model, but there are other gaps from reality

Poll 2

What are some other ingredients that are missing from the model?

BIBLIOGRAPHY

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