Optimized Democracy

Spring 2024 | Lecture 2

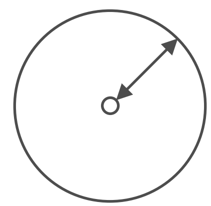
The Axiomatic Approach

Ariel Procaccia | Harvard University

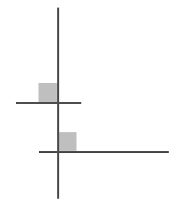
AXIOMS OF EUCLIDEAN GEOMETRY

To draw a straight line from any point to any point

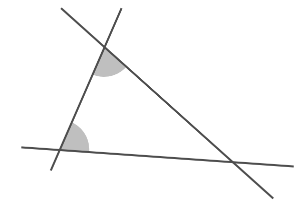
To produce a finite straight line continuously in a straight line



To describe a circle with any center and distance



That all right angles are equal to one another



That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles

Social choice theory similarly tries to analyze group decision making through an axiomatic lens. Another point of similarity is that, as we shall see, some axioms are much more intuitive than others.

THE VOTING MODEL

- Set of voters $N = \{1, \ldots, n\}$ (assume that $n \geq 2$)
- Set of alternatives A; denote $|A| = m$
- Each voter $i \in N$ has a ranking $\sigma_i \in \mathcal{L}$ over the alternatives; $x \succ_{\sigma_i} y$ means that voter i prefers x to y
- A preference profile $\boldsymbol{\sigma} \in \mathcal{L}^n$ is a collection of all voters' rankings
- A social choice function is a function $f: \mathcal{L}^n \to A$, and a social welfare function is $f: \mathcal{L}^n \to \mathcal{L}$

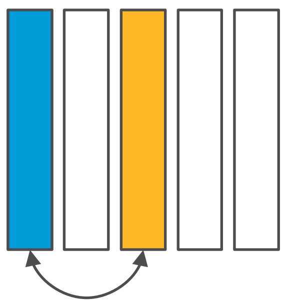
THE CASE OF TWO ALTERNATIVES

- A ubiquitous special case: (essentially) presidential elections in the US, criminal trials,…
- In this case social choice functions and social welfare functions coincide, so let’s use social choice functions

THE CASE OF TWO ALTERNATIVES

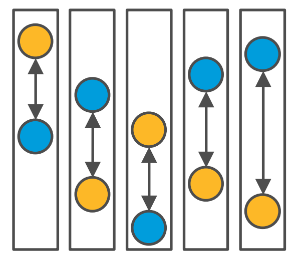
Majority seems to be the only sensible rule

AXIOMS SATISFIED BY MAJORITY



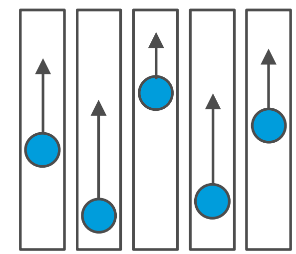
Anonymity

The rule is indifferent to voters' identities, that is, permuting the assignment of voters to rankings doesn't change the outcome



Neutrality

The rule is indifferent to alternatives' identities, that is, permuting the alternatives permutes the outcome in the same way



Monotonicity

Pushing x upwards in the votes doesn't harm x, or for $m = 2$: flipping voters from y to x can't flip the outcome from x to y

MAY'S THEOREM

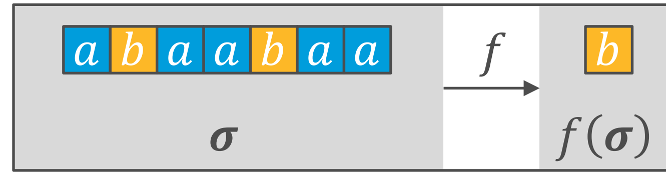
- **Theorem [May 1952]:** Assume $m = 2$ and n is odd, then $f: \mathcal{L}^n \rightarrow A$ is anonymous, neutral and monotonic if and only if it is majority
- The case of even n requires a bit more care in handling ties but is essentially the same

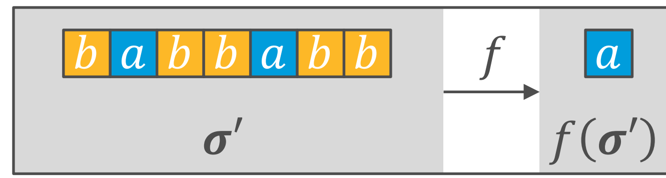
Question

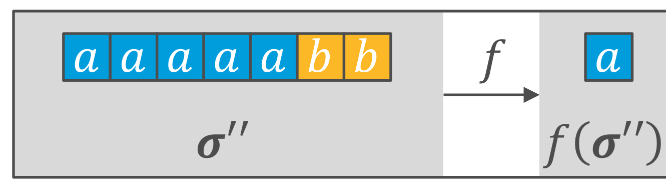
For $m = 2$, are there rules other than majority that satisfy anonymity and neutrality? Anonymity and monotonicity? Neutrality and monotonicity?

PROOF OF MAY'S THEOREM

- Suppose for contradiction that there is a profile $\boldsymbol{\sigma}$ where b is selected with $t < n/2$ votes
- Obtain $\boldsymbol{\sigma}'$ by letting all voters flip their votes; by neutrality $f(\boldsymbol{\sigma}') = a$
- Flip b votes in $\boldsymbol{\sigma}'$ to obtain $\boldsymbol{\sigma}''$ where a has $n - t$ votes; by monotonicity $f(\boldsymbol{\sigma}'') = a$
- But by anonymity it holds that $f(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma}'')$ ■

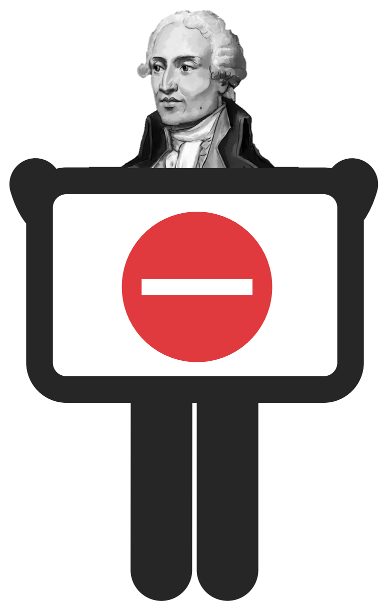






THE GENERAL CASE

We would like to design a great social welfare function for $m > 2$. We know that majority is the only reasonable way to aggregate preferences over pairs of alternatives, so why not simply determine the ordering of each pair through majority?

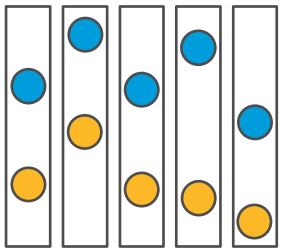


Kenneth Arrow

1921–2017

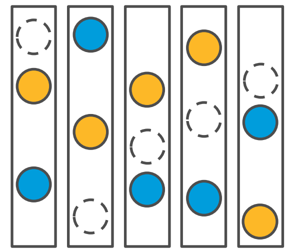
Professor at Harvard and Stanford, 1972 Nobel laureate in economics. Also remembered for his lengthy career as a grad student and for poor weather forecasts.

ARROW'S AXIOMS



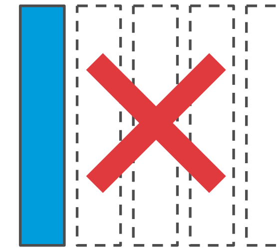
Unanimity

If all voters rank x above y then so does the social welfare function



Independence of irrelevant alternatives

The social ranking over x and y only depends on each voter's ranking restricted to x and y



Nondictatorship

There is no voter that can unilaterally determine the social ranking

ARROW'S THEOREM

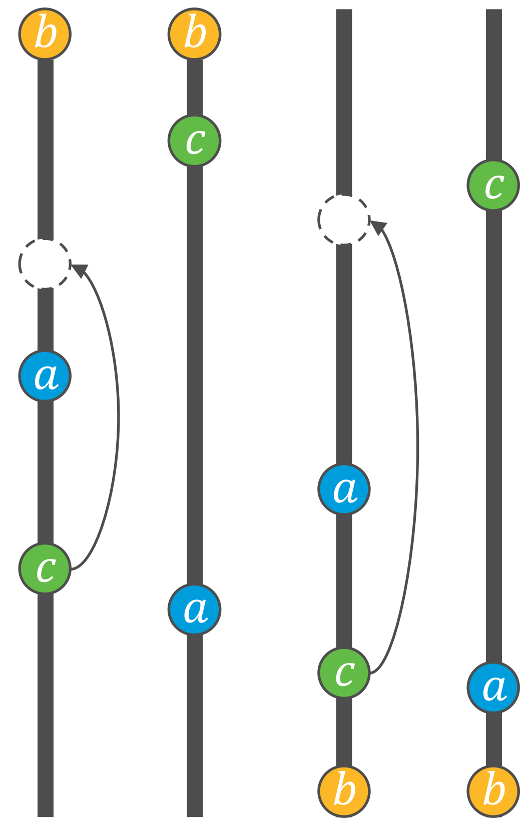
- **Theorem [Arrow, 1951]:** Assume that $m \geq 3$, then there does not exist $f: \mathcal{L}^n \to \mathcal{L}$ that satisfies unanimity, IIA, and nondictatorship
- Dictatorship satisfies unanimity and IIA, so the theorem can be seen as a characterization of dictatorship

Question

For $m \geq 3$, are there rules that satisfy nondictatorship and unanimity? Nondictatorship and IIA?

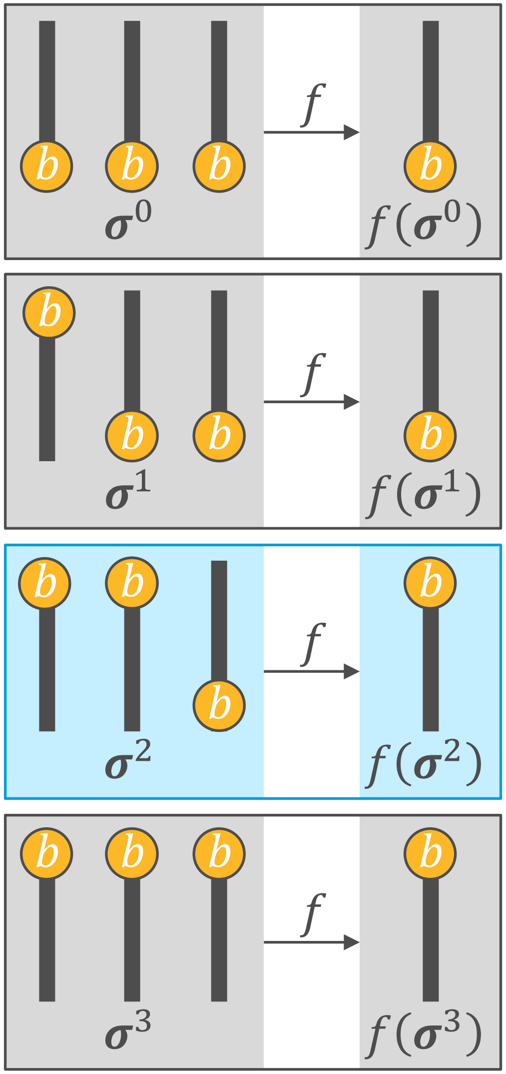
PROOF OF ARROW'S THEOREM

- Assume f is unanimous and IIA, we will show it's dictatorial
- Step 1: Let $b \in A$. If $\boldsymbol{\sigma}$ is such that b is at the top or bottom of each σ_i then b is at the top or bottom of $f(\boldsymbol{\sigma})$
- Suppose not; there are a and c such that $a \succ_{f(\boldsymbol{\sigma})} b \succ_{f(\boldsymbol{\sigma})} c$
- By IIA, if $\boldsymbol{\sigma}'$ is obtained by every voter moving c above a, then it still holds that $a \succ_{f(\boldsymbol{\sigma}')} b$ and $b \succ_{f(\boldsymbol{\sigma}')} c$, hence $a \succ_{f(\boldsymbol{\sigma}')} c$
- This is a contradiction to unanimity at $\boldsymbol{\sigma}'$



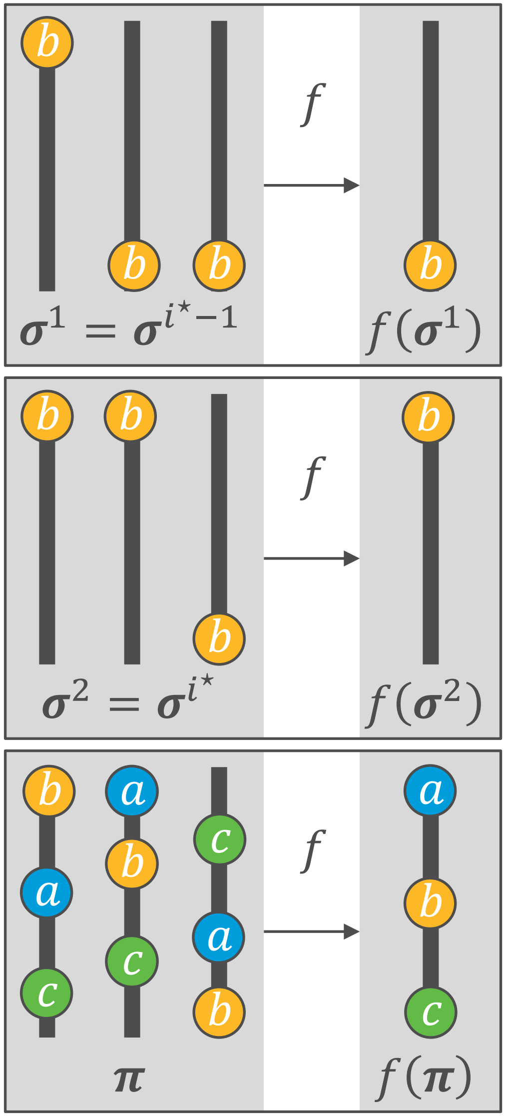
PROOF OF ARROW'S THEOREM

- Step 2: There exists a profile and a voter $i^\star$ such that $i^\star$ can move b from the bottom to the top of the social ranking by changing their vote
- Define profiles $\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n$ where all voters rank b last in $\boldsymbol{\sigma}^0$, and $\boldsymbol{\sigma}^i$ is obtained from $\boldsymbol{\sigma}^{i-1}$ by i pushing b to the top
- By unanimity, b is at the bottom of $f(\boldsymbol{\sigma}^0)$ and at the top of $f(\boldsymbol{\sigma}^n)$
- The position of b first changes in $f(\boldsymbol{\sigma}^{i^\star})$, and by Step 1 it must change from top to bottom



PROOF OF ARROW'S THEOREM

- Step 3: $i^\star$ is a dictator over any pair $\{a, c\}$ not involving b
- W.l.o.g. show $i^\star$ can force $a \succ c$
- Obtain $\boldsymbol{\pi}$ from $\boldsymbol{\sigma}^{i^\star}$ by letting $i^\star$ rank $a \succ_{\pi_{i^\star}} b \succ_{\pi_{i^\star}} c$ and letting others arbitrarily rank a and c while keeping the position of b
- The order of $\{a, b\}$ is the same in $\boldsymbol{\sigma}^{i^\star-1}$ and $\boldsymbol{\pi}$, hence $a \succ_{f(\boldsymbol{\pi})} b$ by IIA
- The order of $\{b, c\}$ is the same in $\boldsymbol{\sigma}^{i^\star}$ and $\boldsymbol{\pi}$, hence $b \succ_{f(\boldsymbol{\pi})} c$ by IIA
- It follows that $a \succ_{f(\boldsymbol{\pi})} c$
- The conclusion follows from IIA by observing that all rankings of $\{a, c\}$ are arbitrary except that of $i^\star$



PROOF OF ARROW'S THEOREM

The conclusion follows from IIA by observing that all rankings of $\{a, c\}$ are arbitrary except that of $i^\star$

1	2	3
a	a	b
b	d	c
c	c	d
d	b	a

1	2	3
b	a	d
a	b	c
c	c	a
d	d	b

Same output ranking over $\{a, c\}$ by IIA

1	2	3
c	d	c
d	b	d
b	a	b
a	c	a

1	2	3
b	a	d
c	b	c
a	c	a
d	d	b

Same output ranking over $\{a, c\}$ by IIA

PROOF OF ARROW'S THEOREM

- Step 4: $i^\star$ is a dictator
- By Step 3, there is $i^{\star\star}$ that is a dictator for every pair not involving $c \neq b$, such as $\{a, b\}$
- But we know that $i^\star$ can affect the social ordering of $\{a, b\}$, by moving from $\boldsymbol{\sigma}^{i^\star-1}$ to $\boldsymbol{\sigma}^{i^\star}$
- Hence $i^\star = i^{\star\star}$, which means that $i^\star$ dictates the social order on every pair except $\{b, c\}$
- Another application of this argument to a dictator for every pair not involving $a \notin \{b, c\}$ completes the proof ■

MONOTONICITY, REVISITED

Monotonicity seemed very natural for two alternatives, but it isn't quite so obvious for more

1	2	3
d	b	c
a	d	b
c	c	a
b	a	d

If winner is b here

1	2	3
d	b	b
b	d	c
a	c	a
c	a	d

Then winner must be b here

Poll

Which of the following rules is not monotonic?

- Plurality
- Borda
- Llull
- STV

STV IS NOT MONOTONIC

c is the winner in the following profile:

6 voters	2 voters	3 voters	4 voters	2 voters
c	b	b	a	a
a	a	c	b	c
b	c	a	c	b

But b becomes the winner if the rightmost voters push c upwards:

6 voters	2 voters	3 voters	4 voters	2 voters
c	b	b	a	c
a	a	c	b	a
b	c	a	c	b

BIBLIOGRAPHY

K. Arrow. Social Choice and Individual Values. Wiley, 1951.

K. O. May. A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions. Econometrica, 1952.

J. Geanakopolis. Three Brief Proofs of Arrow's Impossibility Theorem. Economic Theory, 2005.