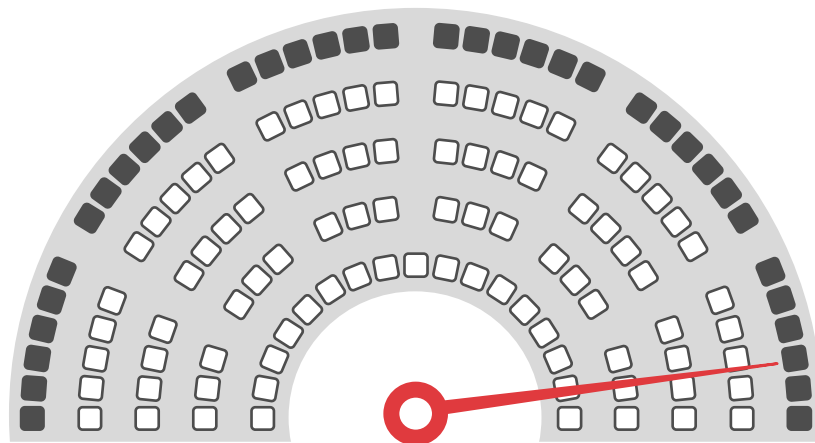




# *Optimized Democracy*

Spring 2024 | Lecture 15

Redistricting as Cake Cutting

Ariel Procaccia | Harvard University

# THE CONSTITUTION

We the People

of the United States, in order to form a more perfect Union, establish Justice, insure domestic Tranquillity, provide for the common defence, promote the general Welfare, and secure the Blessings of Liberty to ourselves and our Posterity, do ordain and establish this Constitution for the United States of America.

## Article 1.

Section 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and when he shall be elected, shall have been at least seven Years a Citizen of the State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty thousand, but each State shall have at least one Representative; and until such Enumeration shall be made, the State of New Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode Island and Providence Plantations one, Connecticut five, New York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

When Vacancies happen in the Representation from any State, the Executive Authority thereof shall issue Writs of Election to fill such Vacancies.

“Representatives ... shall be apportioned among the several states ... according to their respective numbers. ... The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative.”

Section 3. The Senate of the United States shall be composed of two Senators from each State, chosen by the Legislature thereof, for six Years, and each Senator shall have the Qualifications requisite for Senators of the most numerous Branch of the State Legislature. Immediately after they shall be assembled in Consequence of the first Election, they shall be divided as equally as may be into three Classes. In the first Class at the Expiration of the sixth Year, so that one third may be chosen every second Year; and if Vacancies happen by Resignation, or otherwise, during the Term of any Senator, the Executive thereof may make temporary Appointments until the next Meeting of the Legislature, which shall then fill such Vacancies.

# THE CONSTITUTION

Section. 3. The Senate of the United States shall be composed of two Senators from each State, chosen by the Legislature thereof, for six Years, and each Senator shall have one Vote.

Immediately after they shall be chosen, in each State, the Electors in each State shall be divided as equally as may be into three Clases. The Seats of the Senators of the first Class shall be vacated at the Expiration of the first Year, of the second Class at the Expiration of the second Year, and of the third Class at the Expiration of the sixth Year, so that one third may be chosen every second Year; and if Vacancies happen by Resignation, or otherwise, during the recess of the Legislature of any State, the Executive thereof may make temporary Appointments until the next Meeting of the Legislature, which shall then fill such Vacancies.

No Person shall be a Senator who shall not have attained to the Age of thirty Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of that State for which he shall be chosen.

The Vice President of the United States shall be President of the Senate, but shall have no Vote, unless they be equally divided. The Senate shall choose their other Officers, and also a President, pro tempore, in the Absence of the Vice President, or when he shall exercise the Office of President of the United States.

The Senate shall have the sole Power to try all Impeachments. When sitting for that Purpose, they shall be on Oath or Affirmation. When the President of the United States is tried, the Chief Justice shall preside: And no Person shall be convicted without the Concurrence of two thirds of the Members present. Judgment in Cases of Impeachment shall not extend further than to removal from Office, and disqualification to hold and enjoy any Office of Honor, Trust, or Profit under the United States; but the Party convicted shall nevertheless be liable and subject to Indictment, Trial, Judgment, and Punishment, according to Law.

Section. 4. The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State by the Legislature thereof; but the Congress may at any time by Law make or alter such Regulations, except as to the Places of choosing Senators. The Congress shall assemble at least once in every Year, and the Meeting shall be on the first Monday in December, unless they shall by Law appoint a different Day.

Section. 5. Each House shall be the Judge of the Elections, Returns and Qualifications of its own Members, and a Majority of each shall constitute a quorum to do Business, and to which a Majority of each shall have Power to compel the Attendance of absent Members, in such Manner, and under such Penalties as each House may provide. Neither House shall adjourn for more than three Days, nor to any other Place than that in which the two Houses shall be sitting.

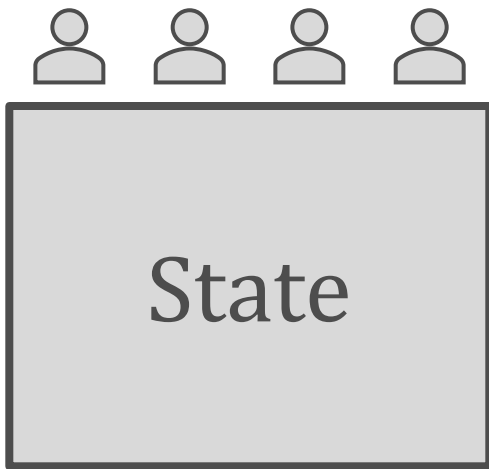
Section. 6. The Senators and Representatives shall receive a Compensation for their Services, to be ascertained by Law, and paid out of the Treasury of the United States. They shall in all Cases, except Treason, Felony and Breach of the Peace, be privileged from Arrest during their Attendance at the Session of their respective Houses, and in going to and returning from the same; and for any Speech or Debate in either House, they shall not be questioned in any other Place.

No Senator or Representative shall, during the Time for which he was elected, be appointed to any civil Office under the Authority of the United States, which shall have been created, or the Emoluments whereof shall have been increased during such Time; and no Person holding any Office under the United States, shall be a Member of either House during his Continuance in Office.

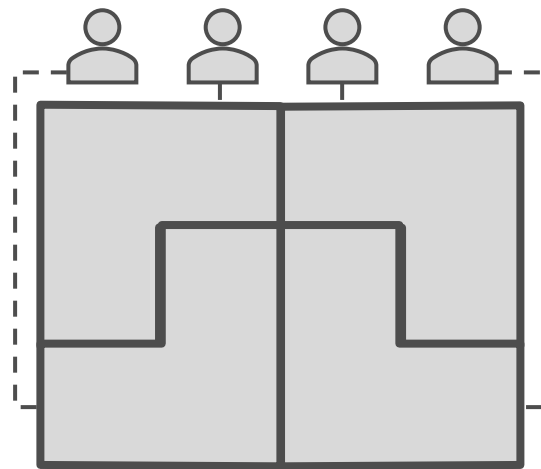
Section. 7. All Bills for raising Revenue shall originate in the House of Representatives; but the Senate may propose or concur with Amendments as on other Bills. Every Bill which shall have passed the House of Representatives and the Senate, shall, before it become a Law, be presented to the President of the

“The times, places and manner of holding elections for Senators and Representatives, shall be prescribed in each state by the legislature thereof..”

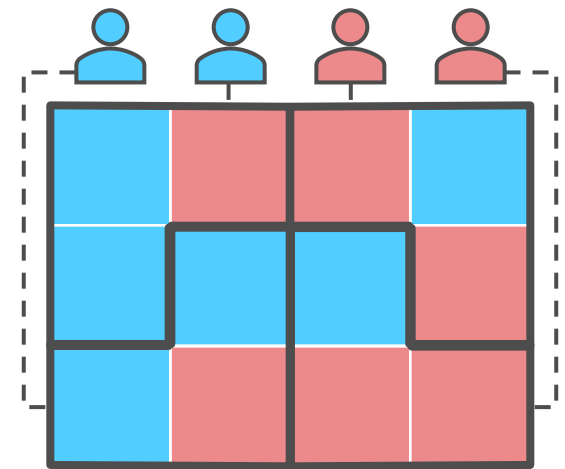
# REDISTRICTING



Apportionment

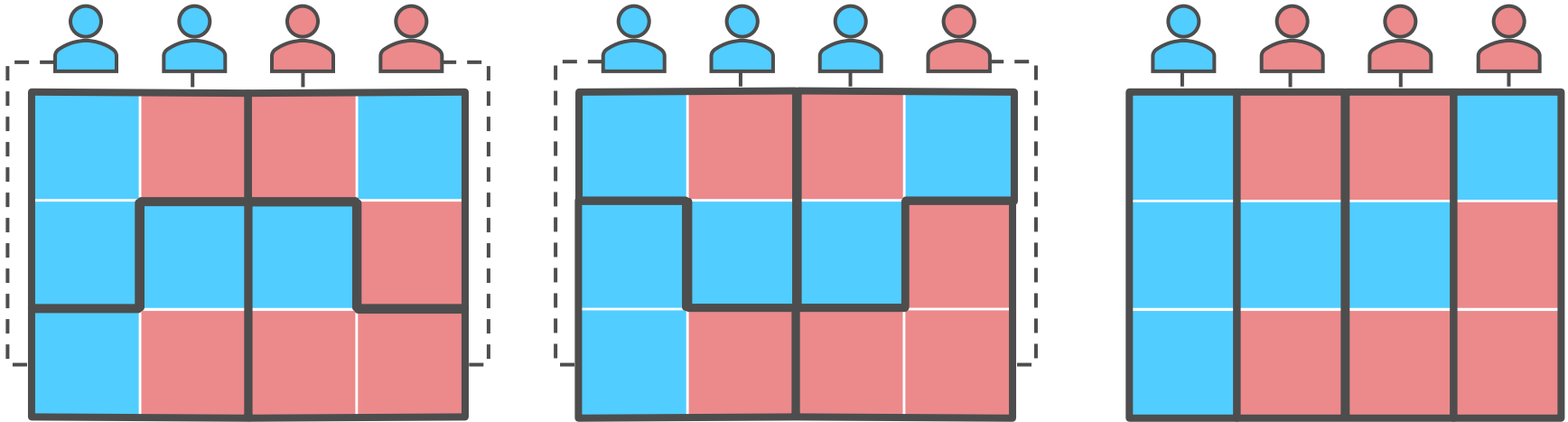


Redistricting



Elections

# GERRYMANDERING



**gerrymander** • verb • to divide or arrange (a territorial unit) into election districts in a way that gives one political party an unfair advantage



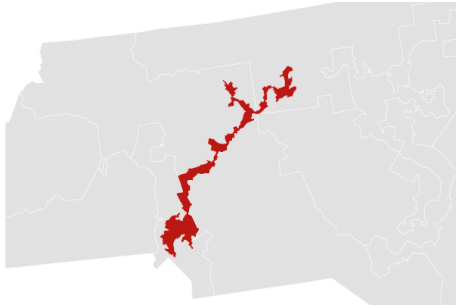
# The gerrymander

1812–present

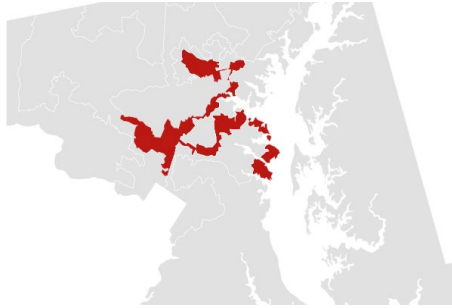
“A new species of monster, which appeared  
in Essex South District in Jan. 1812.”



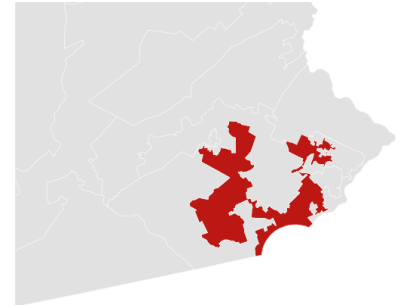
# AMERICA'S MOST GERRYMANDERED



**North Carolina**  
12<sup>th</sup> District



**Maryland**  
3<sup>rd</sup> District

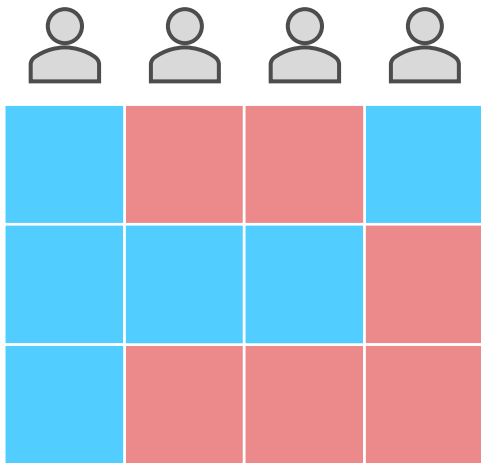


**Pennsylvania**  
7<sup>th</sup> District

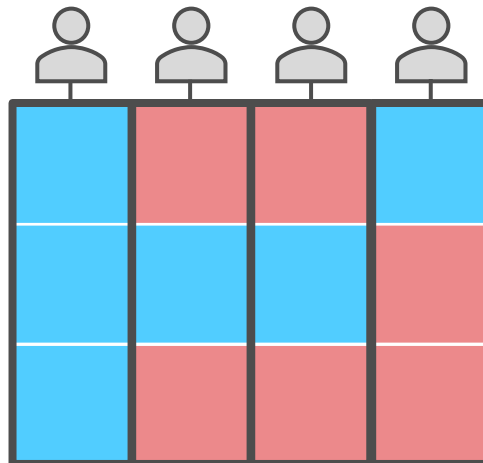
Source: The Washington Post, 2014

# I CUT, YOU FREEZE

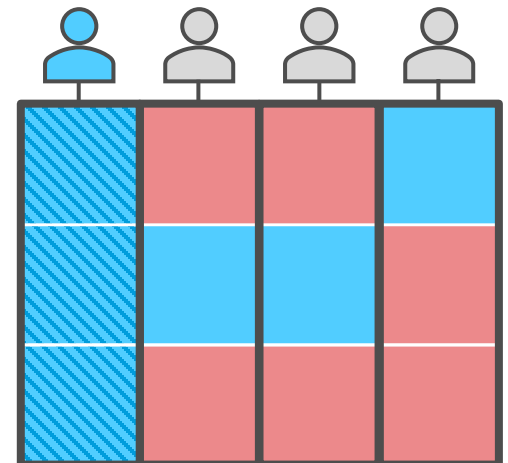
Players take turns freezing a district drawn by the other player, and redrawing the unfrozen districts, until all districts are frozen



Election results

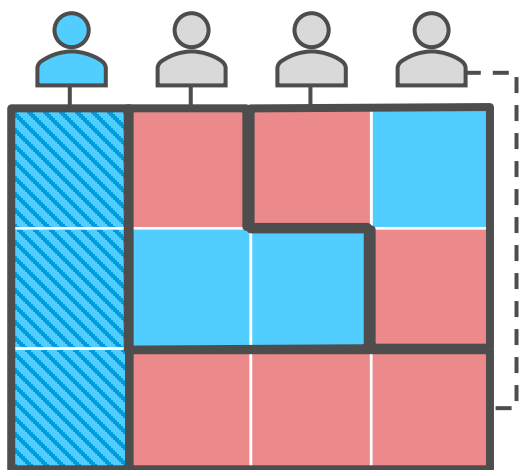


Red cuts

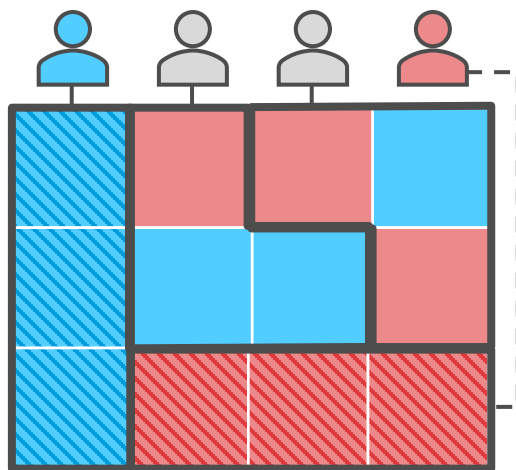


Blue freezes

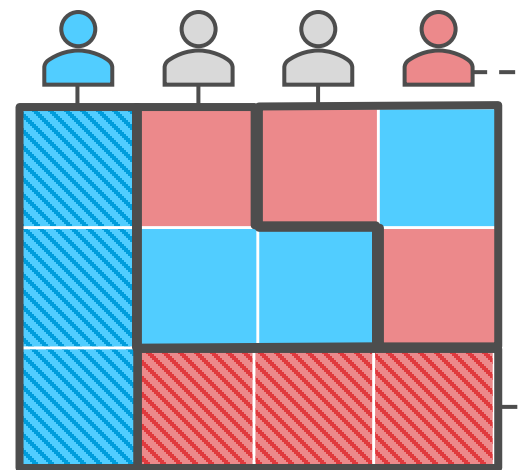
# I CUT, YOU FREEZE



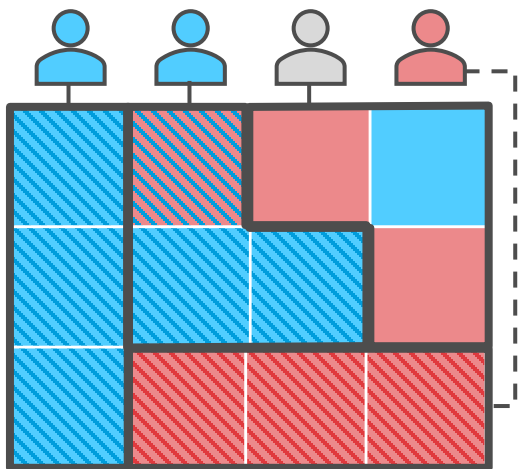
Blue cuts



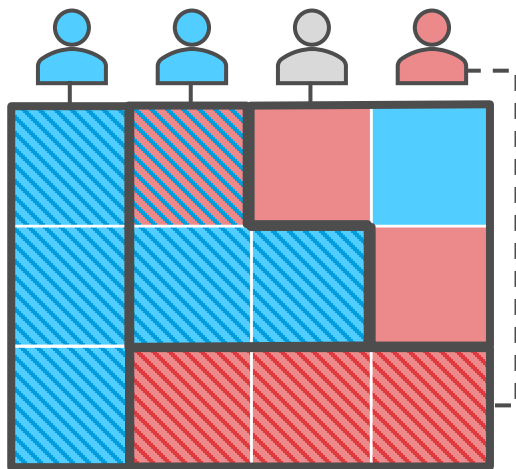
Red freezes



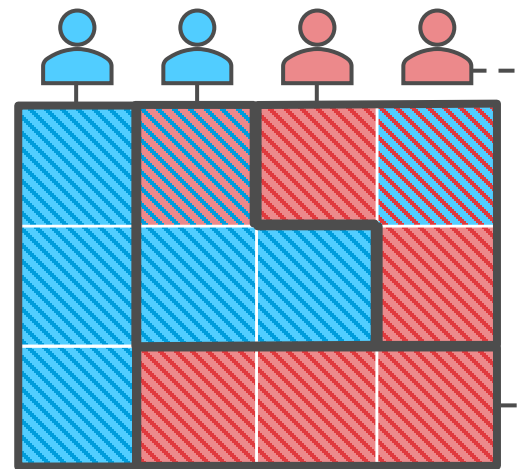
Red cuts



Blue freezes



Blue cuts



Red freezes

# The Washington Post

## Opinion | There's another way to solve gerrymandering. It's as simple as cake.

By Wesley Pegden and Ariel D. Procaccia

February 15, 2018 at 8:10 p.m. EST



# I CUT, YOU FREEZE: ANALYSIS

- A non-geometric model:
  - Each of the two players has  $V_i$  units of voters such that  $V_1 + V_2 = k$
  - A **partition** consists of  $k$  pairs  $(v_{1d}, v_{2d})$  such that  $\sum_{d=1}^k v_{id} = V_i$  for  $i = 1, 2$ , and for all districts  $d$ ,  $v_{1d} + v_{2d} = 1$
  - Player  $i$  wins district  $d$  if  $v_{id} > v_{3-i,d}$

## Poll 1

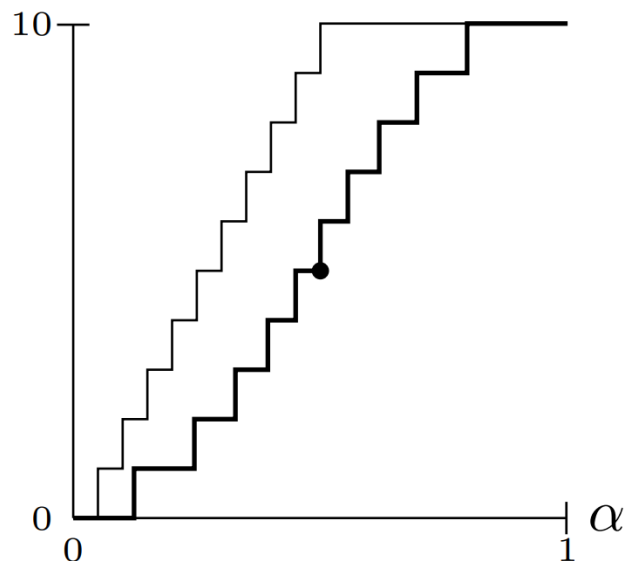
Suppose that  $V_1 = \alpha k$  for  $\alpha \leq 1/2$  and that player 1 redistricts unilaterally. What fraction of the districts can they guarantee to win (roughly)?

- $\alpha/2$
- $\alpha$
- $2\alpha$
- $1/2$

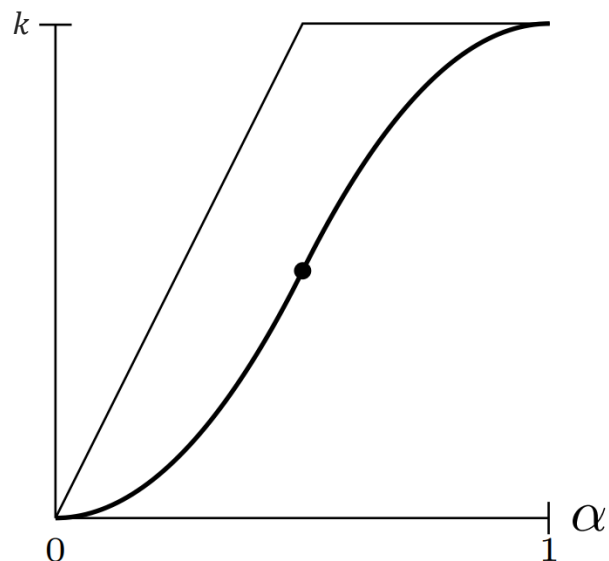


# I CUT, YOU FREEZE: ANALYSIS

**Theorem:** Let  $V_i = \alpha k$ . As  $k \rightarrow \infty$ , under the ICYF protocol and optimal play, the fraction of districts won by player  $i$  is  $2\alpha^2$  for  $\alpha \leq 1/2$  and  $1 - 2(1 - \alpha)^2$  for  $\alpha > 1/2$



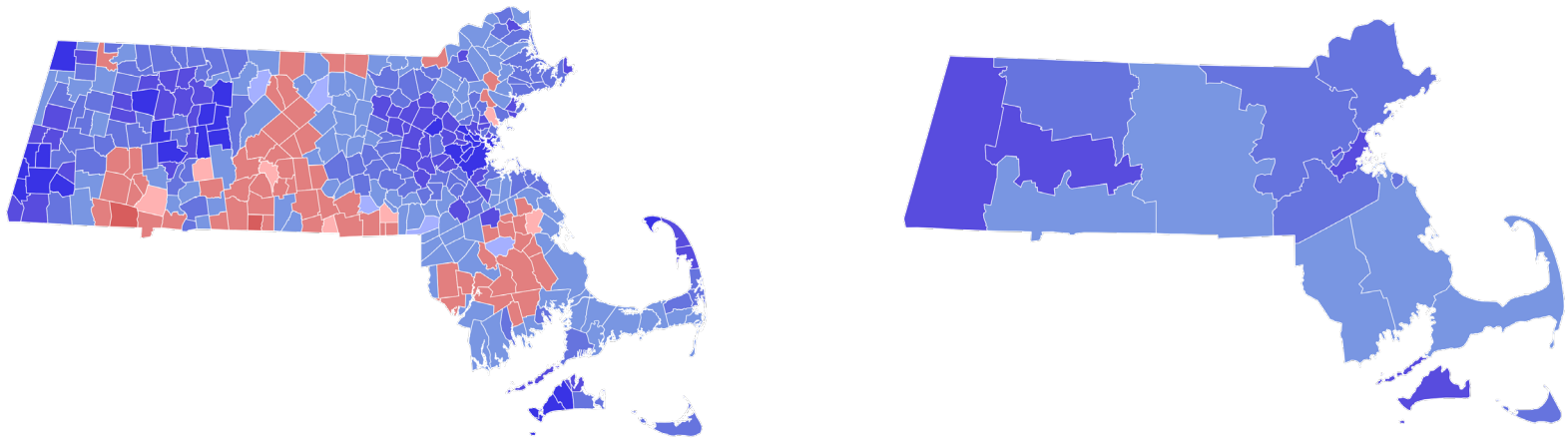
Case of  $k = 10$



Case of  $k \rightarrow \infty$

# PROPORTIONALITY

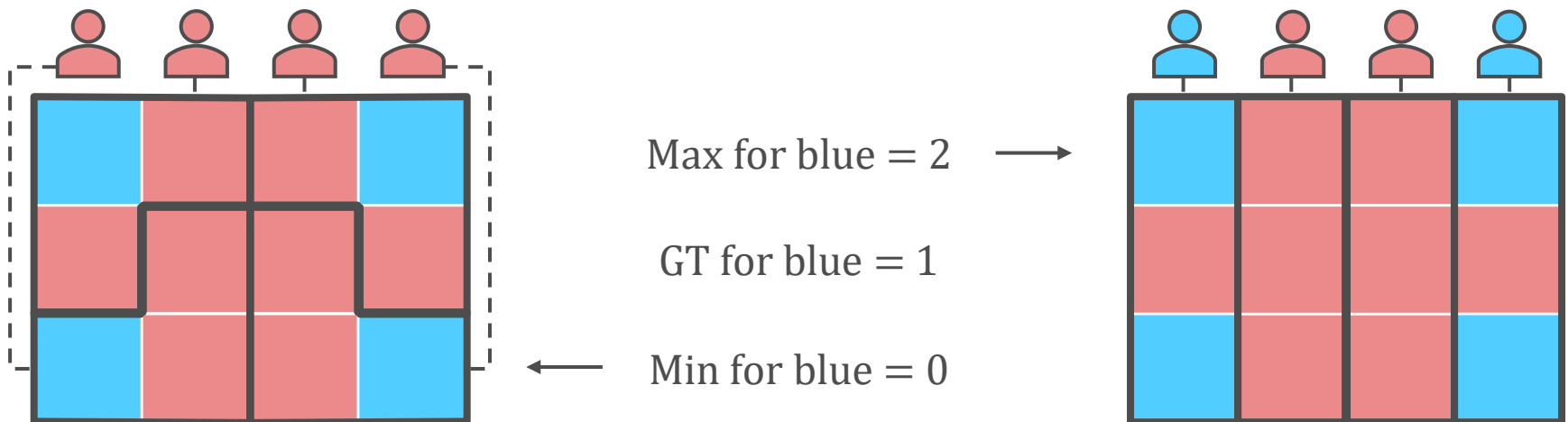
- A **proportional** partition is such that if  $V_i = \alpha k$ ,  $i$  wins an  $\alpha$  fraction of the districts
- But when geometric constraints are taken into account, this is not a feasible standard



MA gubernatorial election, 2022 (Healey vs. Diehl)

# GEOMETRIC TARGET

- Consider a set  $\mathcal{D}$  of possible partitions (possibly obeying geometric constraints)
- The **geometric target** of player  $i$  is the average of the maximum number of districts they can win (across partitions in  $\mathcal{D}$ ) and the minimum number, rounded down



# GEOMETRIC TARGET

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## Poll 2

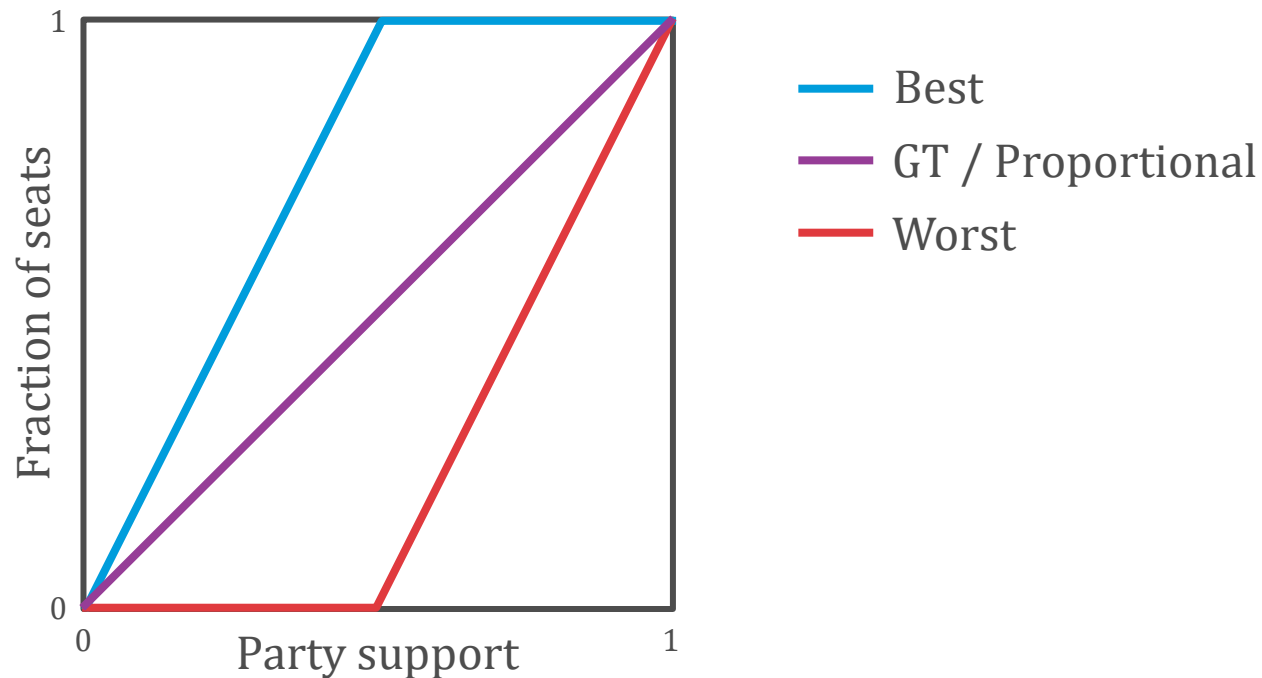
In the geometry-free model, what is the relation between geometric target and proportionality?

- $GT \Rightarrow \text{prop}$  •  $\text{Prop} \Rightarrow GT$  • Equivalent • Incomparable



# GEOMETRIC TARGET

**Theorem:** In the geometry-free model, a partition is proportional if and only if it satisfies the geometric target of both parties (up to ties)

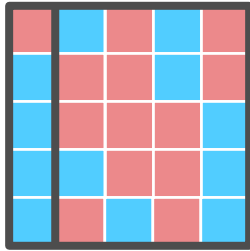


# THE LRY PROTOCOL

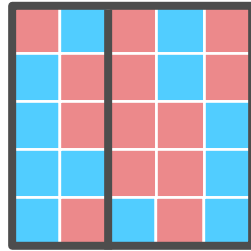
A protocol proposed by Landau, Reid and Yershov (2009):

1. For  $j = 1, \dots, k - 1$ , a mediator constructs a split  $(X_j, Y_j)$  such that  $X_1 \subset X_2 \subset \dots \subset X_{k-1}$
2. For each  $j$ , each player is asked “would you rather redistrict  $X_j$ , with the other player redistricting  $Y_j$ , or vice versa?”
3. Try to find a  $j$  such that one player prefers redistricting  $X_j$  and the other  $Y_j$
4. If no such  $j$  exists, there must be  $j_0$  such that both players want to redistrict  $Y_{j_0}$  and  $X_{j_0+1}$ . Choose  $s \in \{j_0, j_0 + 1\}$  at random and let a random player redistrict  $X_s$  and the other player redistrict  $Y_s$

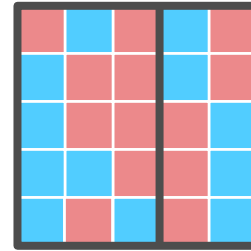
# THE LRY PROTOCOL: EXAMPLE



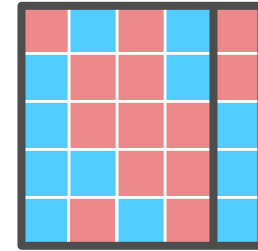
$X_1$     $Y_1$



$X_2$     $Y_2$

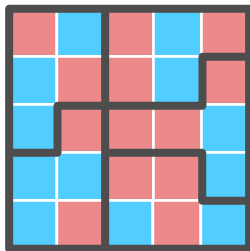


$X_3$     $Y_3$



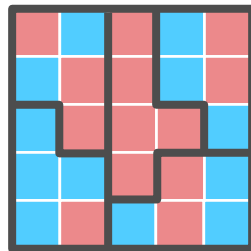
$X_4$     $Y_4$

Both players prefer redistricting  $Y_1, Y_2, X_3, X_4$ , so the crossover point is  $j_0 = 2$



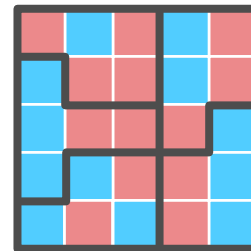
$X_2$     $Y_2$

$B$  divides  $X_2$   
 $R$  divides  $Y_2$



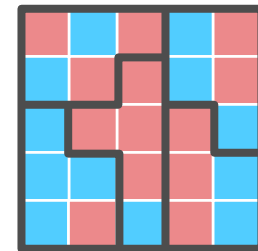
$X_2$     $Y_2$

$B$  divides  $Y_2$   
 $R$  divides  $X_2$



$X_3$     $Y_3$

$B$  divides  $X_3$   
 $R$  divides  $Y_3$

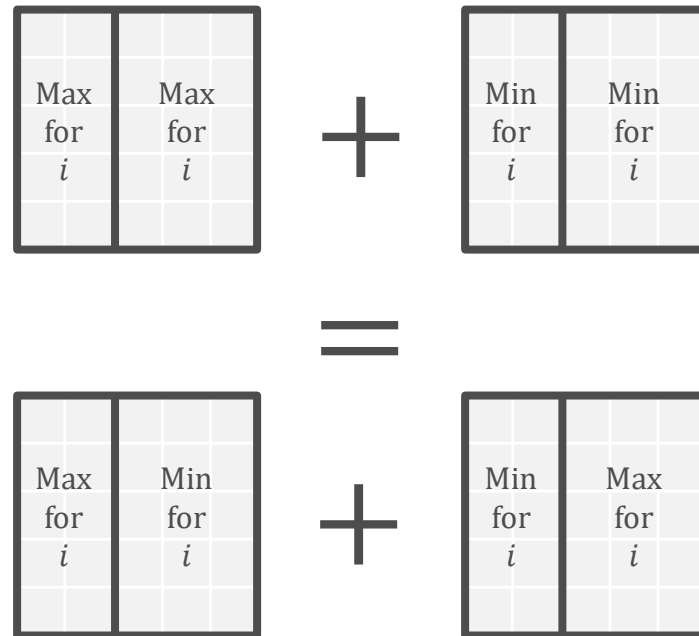


$X_3$     $Y_3$

$B$  divides  $Y_3$   
 $R$  divides  $X_3$

# THE LRY PROTOCOL: ANALYSIS

- **Theorem:** Restricting the set of partitions  $\mathcal{D}$  to satisfy a given split, a party's preferred choice satisfies their geometric target


$$\begin{array}{|c|c|} \hline \text{Max} \\ \text{for} \\ i \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Min} \\ \text{for} \\ i \\ \hline \end{array} = \begin{array}{|c|c|} \hline \text{Max} \\ \text{for} \\ i \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Min} \\ \text{for} \\ i \\ \hline \end{array}$$

# THE LRY PROTOCOL: ANALYSIS

- **Theorem:** Restricting the set of partitions to satisfy a given split, a party's preferred choice satisfies their geometric target
- Even if a party doesn't get its preferred choice in  $(X_{j_0}, Y_{j_0})$ , their preference in the very similar split  $(X_{j_0+1}, Y_{j_0+1})$  is reversed, so overall we get an “approximation” of the geometric target

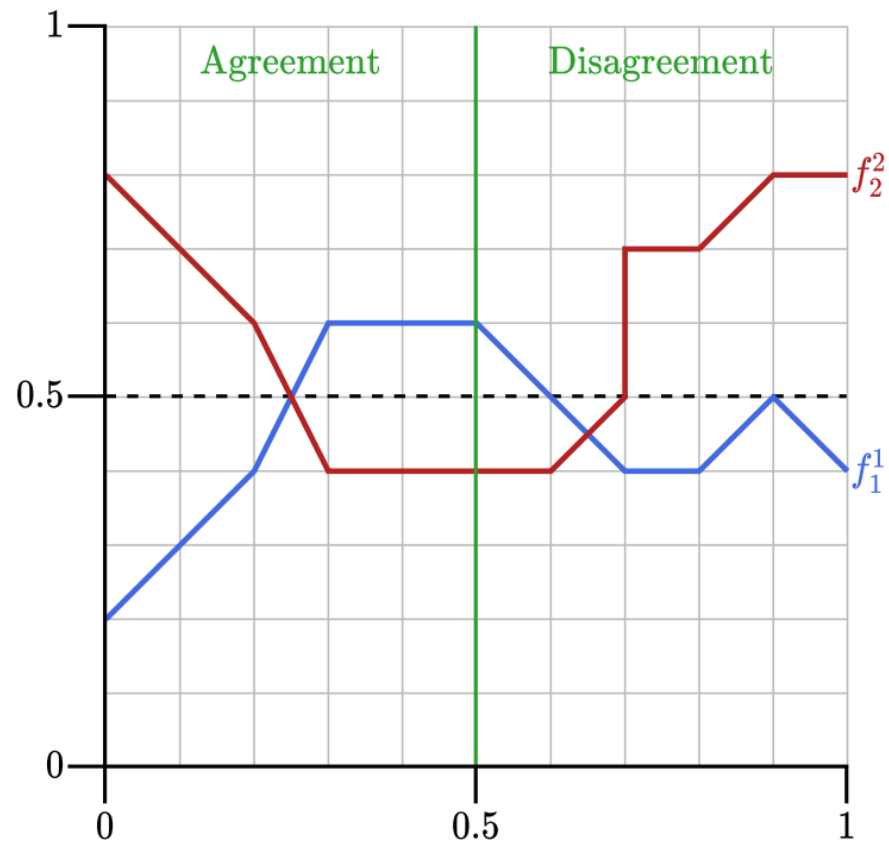
# OPTIMIZATION SUBJECT TO FAIRNESS

- Instead of an interactive protocol, optimize an objective function subject to a fairness constraint, namely geometric target
- Possible objectives:
  - Compactness
  - Number of competitive districts
  - Efficiency gap
- Two obstacles:
  - How to solve the optimization problem?
  - Is the geometric target feasible in practice?

# THE STATE CUTTING PROBLEM

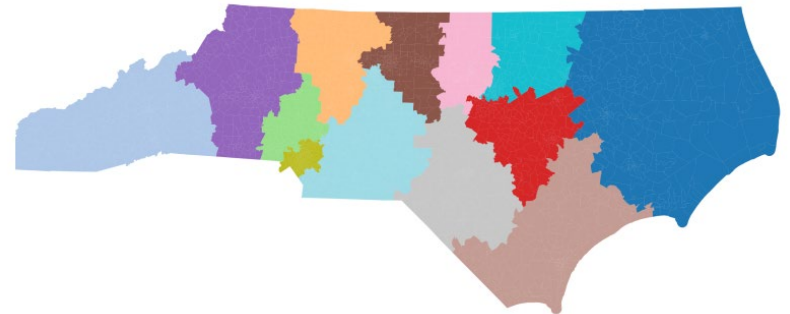
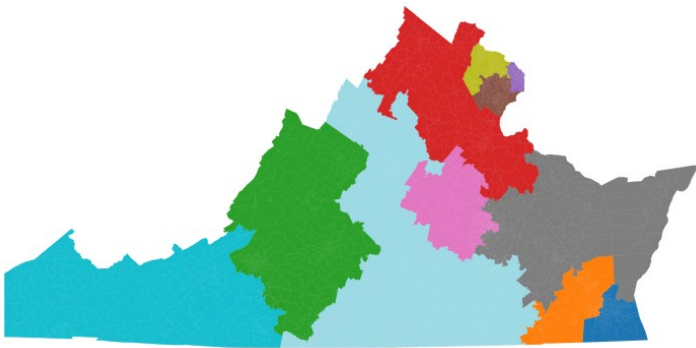
- The state  $S$  is the interval  $[0,1]$
- A district  $D$  is a finite union of subintervals
- Each player  $i \in \{1,2\}$  has a density function  $f_i^j$  such that for all  $x$ ,  $f_i^1(x) + f_i^2(x) = 1$
- Let  $u_i^j(D) = \int_D f_i^j(x) dx$
- We consider a partition into  $k$  districts, where for each,  $\mu(D) = 1/k$
- A player  $i$  believes they won a district  $D$  if  $u_i^i(D) > u_i^{3-i}(D)$
- Compared to the non-geometric model, this captures a key real-world constraint: voters can't be partitioned arbitrarily

# THE STATE CUTTING PROBLEM



# EXISTENCE OF GT PARTITIONS

**Theorem:** In any instance of the state cutting problem, a partition satisfying the geometric targets of both players with respect to their own beliefs exists



GT partitions in Virginia (left) and North Carolina (right) which outperform their implemented plans in terms of competitiveness, efficiency gap and compactness

# BIBLIOGRAPHY

W. Pegden, A. D. Procaccia and D. Yu. **A Partisan Districting Protocol with Provably Nonpartisan Guarantees.** Manuscript, 2017.

Z. Landau, O. Reid and I. Yershov. **A Fair Division Solution to the Problem of Redistricting.** Social Choice and Welfare, 2009.

G. Bendadè, A. D. Procaccia and J. Tucker-Foltz. **You Can Have Your State and Redistrict It Too.** Manuscript, 2022.