

Optimized Democracy

Spring 2024 | Lecture 13

Random Assignment

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ASSIGNMENT PROBLEMS



School choice

Assign students to schools



Housing allocation

Assign applicants to
public housing

Common thread:

Each player requires exactly one good
EF1 is trivial

THE MODEL

- Set of players $N = \{1, \dots, n\}$
- Set G of n goods (we're assuming $|N| = |G|$ for convenience)
- Each player has a ranking $\sigma_i \in \mathcal{L}$ over G
- An **assignment** is a perfect matching π between players and goods, where $\pi(i)$ is the good assigned to i
- We are interested in rules f that take $\sigma \in \mathcal{L}^n$ and output π

SERIAL DICTATORSHIP

- Players select their favorite goods according to a predetermined order τ
- Example for the order $1 \succ_{\tau} 2 \succ_{\tau} 3 \succ_{\tau} 4$:

1	2	3	4
<i>a</i>	<i>a</i>	<i>d</i>	<i>a</i>
<i>b</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>c</i>
<i>d</i>	<i>d</i>	<i>a</i>	<i>b</i>

SERIAL DICTATORSHIP: PROPERTIES

- An assignment π is **Pareto efficient** if there is no assignment π' such that $\pi'(i) \succsim_{\sigma_i} \pi(i)$ for all $i \in N$ and $\pi'(j) \succ_{\sigma_j} \pi(j)$ for some $j \in N$
- A rule f is **strategyproof (SP)** if for all $\sigma \in \mathcal{L}^n$, for all $i \in N$ and for all $\sigma'_i \in \mathcal{L}$,
$$f(\sigma)(i) \succsim_{\sigma_i} f(\sigma'_i, \sigma_{-i})(i)$$

Poll 1

Which of the following properties is satisfied by serial dictatorship?

- Pareto efficiency
- Both
- Strategyproofness
- Neither



RANDOM SERIAL DICTATORSHIP

(Serial dictatorship with the order τ chosen uniformly at random.)

1	2	3
a	a	b
b	b	c
c	c	a
$1 \succ_{\tau} 2 \succ_{\tau} 3$		

1	3	2
a	b	a
b	c	b
c	a	c
$1 \succ_{\tau} 3 \succ_{\tau} 2$		

2	1	3
a	a	b
b	b	c
c	c	a
$2 \succ_{\tau} 1 \succ_{\tau} 3$		

2	3	1
a	b	a
b	c	b
c	a	c
$2 \succ_{\tau} 3 \succ_{\tau} 1$		

3	1	2
b	a	a
c	b	b
a	c	c
$3 \succ_{\tau} 1 \succ_{\tau} 2$		

3	2	1
b	a	a
c	b	b
a	c	c
$3 \succ_{\tau} 2 \succ_{\tau} 1$		

A distribution over assignments is called a
lottery

LOTTERY TO RANDOM ASSIGNMENT

1	2	3
a	a	b
b	b	c
c	c	a
$1 \succ_{\tau} 2 \succ_{\tau} 3$		

1	3	2
a	b	a
b	c	b
c	a	c
$1 \succ_{\tau} 3 \succ_{\tau} 2$		

2	1	3
a	a	b
b	b	c
c	c	a
$2 \succ_{\tau} 1 \succ_{\tau} 3$		

2	3	1
a	b	a
b	c	b
c	a	c
$2 \succ_{\tau} 3 \succ_{\tau} 1$		

3	1	2
b	a	a
c	b	b
a	c	c
$3 \succ_{\tau} 1 \succ_{\tau} 2$		

3	2	1
b	a	a
c	b	b
a	c	c
$3 \succ_{\tau} 2 \succ_{\tau} 1$		

A **random assignment** is a **bi-stochastic** matrix $P = [p_{ix}]$ where p_{ix} is the probability player i is assigned to x

	a	b	c
1	1/2	1/6	1/3
2	1/2	1/6	1/3
3	0	2/3	1/3

RSD: PROPERTIES

- RSD is ex post strategyproof: Players cannot gain from lying regardless of the random coin flips
- In contrast to SD, RSD satisfies **equal treatment of equals**: For $i, j \in N$ such that $\sigma_i = \sigma_j$ it holds that $p_{ix} = p_{jx}$ for all $x \in G$
- RSD is ex post Pareto efficient: every assignment in its support is Pareto efficient
- Is this a satisfying notion of efficiency for lotteries?

ORDINAL EFFICIENCY

- Random assignment P' **stochastically dominates** P iff for all $i \in N$ and $x \in G$,
 $\sum_{y \succsim_{\sigma_i} x} p'_{iy} \geq \sum_{y \succsim_{\sigma_i} x} p_{iy}$, with at least one strict inequality
- A random assignment is **ordinally efficient** if it isn't stochastically dominated by any other assignment

Poll 2

What is the relation between ex post efficiency and ordinal efficiency?

- Ex post \Rightarrow ordinal
- Ordinal \Rightarrow ex post
- Ex post \Leftrightarrow ordinal
- Incomparable



RSD IS NOT ORDINALLY EFFICIENT

1	2	3	4
<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>b</i>	<i>a</i>	<i>a</i>
<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>d</i>	<i>d</i>	<i>d</i>	<i>d</i>



	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1	5/12	1/12	1/4	1/4
2	5/12	1/12	1/4	1/4
3	1/12	5/12	1/4	1/4
4	1/12	5/12	1/4	1/4

Random serial dictatorship

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1	1/2	0	1/4	1/4
2	1/2	0	1/4	1/4
3	0	1/2	1/4	1/4
4	0	1/2	1/4	1/4

Stochastically dominating assignment

PROBABILISTIC SERIAL RULE

- The **probabilistic serial rule** is directly defined by a random assignment (more on this later)
- Each good is a “divisible” good consisting of “probability shares”
- At every point in time, all players “eat” their favorite remaining goods at the same rate
- When all goods are eaten, each player has probability shares adding up to 1

PROBABILISTIC SERIAL RULE

1	2	3	4
<i>a</i>	<i>b</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>c</i>	<i>c</i>	<i>d</i>
<i>c</i>	<i>d</i>	<i>d</i>	<i>c</i>
<i>d</i>	<i>a</i>	<i>a</i>	<i>a</i>



Good *a*



Good *c*



Good *b*

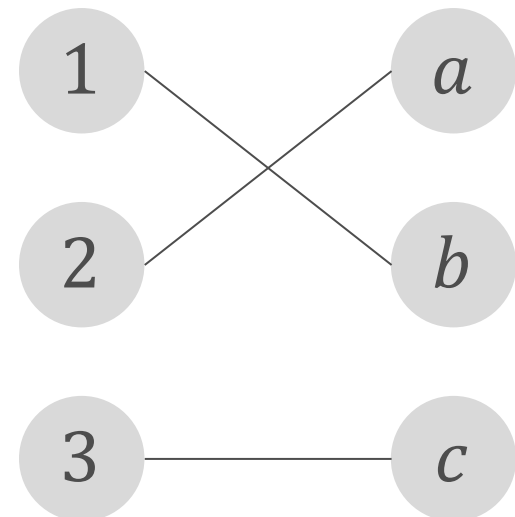


Good *d*

RANDOM ASSIGNMENT TO LOTTERY

- We saw that every lottery induces a random assignment, is the converse also true?
- A **permutation matrix** is a bistochastic matrix consisting only of zeros and ones
- A permutation matrix represents an assignment

	<i>a</i>	<i>b</i>	<i>c</i>
1	0	1	0
2	1	0	0
3	0	0	1



RANDOM ASSIGNMENT TO LOTTERY

	<i>a</i>	<i>b</i>	<i>c</i>
1	1/2	1/6	1/3
2	1/2	1/2	0
3	0	1/3	2/3

Theorem [Birkhoff-von Neumann]: Any bistochastic matrix can be obtained as a convex combination of permutation matrices

	<i>a</i>	<i>b</i>	<i>c</i>
1	0	1	0
2	1	0	0
3	0	0	1

$\times 1/6$

	<i>a</i>	<i>b</i>	<i>c</i>
1	1	0	0
2	0	1	0
3	0	0	1

$\times 1/2$

	<i>a</i>	<i>b</i>	<i>c</i>
1	0	0	1
2	1	0	0
3	0	1	0

$\times 1/3$

PS: PROPERTIES

- Probabilistic serial obviously satisfies equal treatment of equals
- **Theorem:** Probabilistic serial is ordinally efficient
- Given a random assignment P and a profile σ , define a graph $\Gamma_{P,\sigma} = (G, E)$ where $(x, y) \in E$ iff $\exists i \in N$ such that $x \succ_{\sigma_i} y$ and $p_{iy} > 0$
- **Lemma:** If $\Gamma_{P,\sigma}$ is acyclic then P is ordinally efficient

PROOF OF THEOREM

- If P is the output of PS, we claim that $\Gamma_{P,\sigma}$ is acyclic, and conclude by the lemma
- Suppose for contradiction that $\Gamma_{P,\sigma}$ has a cycle
- Let x be the first good in the cycle to be fully eaten at time t
- There is an edge (y, x) in $\Gamma_{P,\sigma}$ so there is $i \in N$ such that $y \succ_{\sigma_i} x$ and $p_{ix} > 0$
- But at any point up to t , player i should have been eating y or a more preferred good, which contradicts the fact that $p_{ix} > 0$ ■

PS IS NOT STRATEGYPROOF

1	2	3	4
<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>
<i>b</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>c</i>	<i>d</i>	<i>d</i>	<i>d</i>
<i>d</i>	<i>b</i>	<i>a</i>	<i>a</i>



	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1	1/2	0	1/4	1/4
2	1/2	0	1/4	1/4
3	0	1/2	1/4	1/4
4	0	1/2	1/4	1/4

1	2	3	4
<i>b</i>	<i>a</i>	<i>b</i>	<i>b</i>
<i>a</i>	<i>c</i>	<i>c</i>	<i>c</i>
<i>c</i>	<i>d</i>	<i>d</i>	<i>d</i>
<i>d</i>	<i>b</i>	<i>a</i>	<i>a</i>



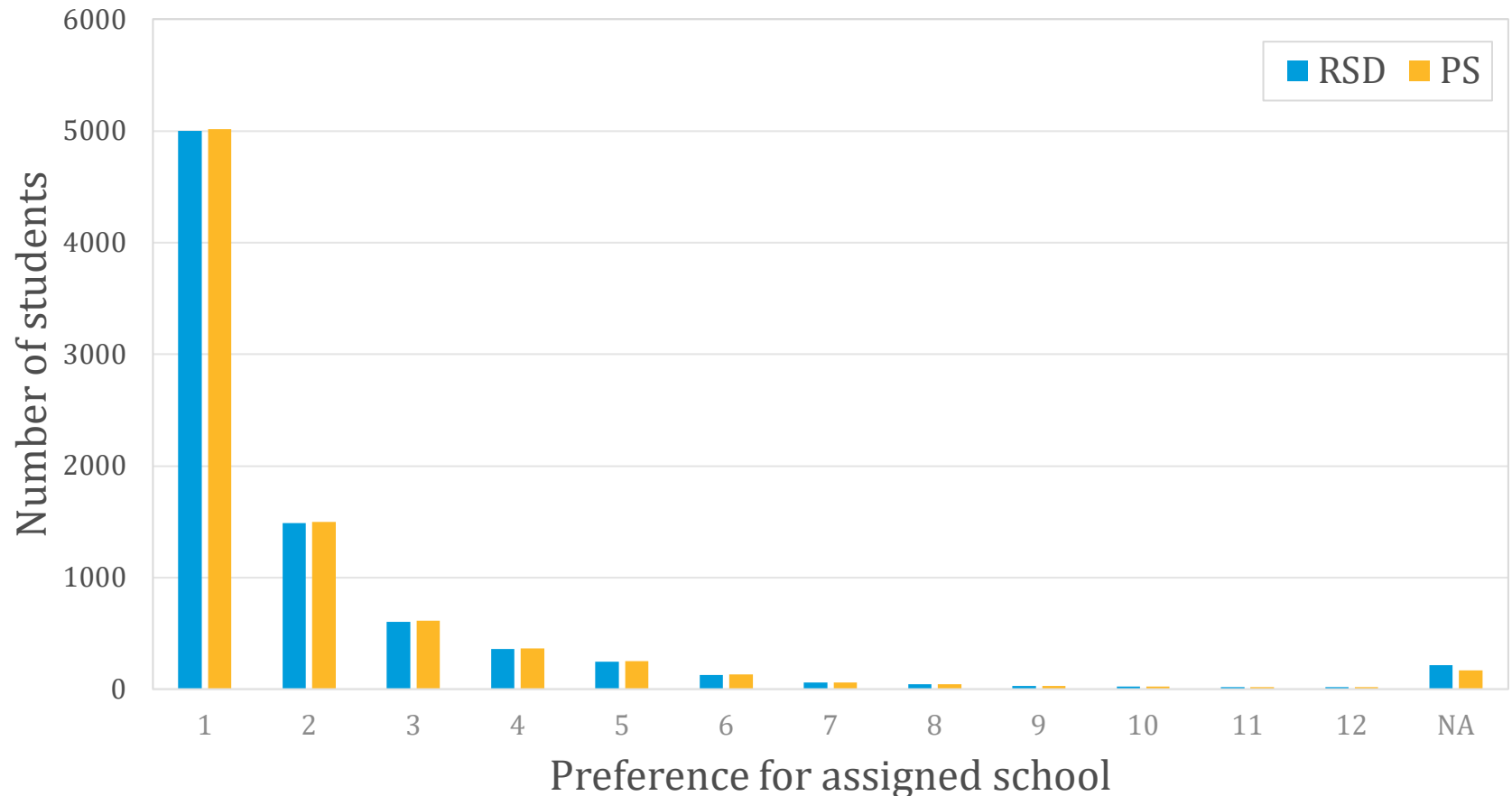
	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1	1/3	1/3	1/12	1/4
2	2/3	0	1/12	1/4
3	0	1/3	5/12	1/4
4	0	1/3	5/12	1/4

AN IMPOSSIBILITY RESULT

- **Theorem:** There is no rule that satisfies ordinal efficiency, strategyproofness and equal treatment of equals
- If we accept equal treatment of equals as non-negotiable then the tradeoff between ordinal efficiency and strategyproofness is unavoidable

PS VS. RSD ON NYC DATA

Pathak [2006] ran RSD and PS on (“truthful”) data from 8255 students in NYC



PS VS. RSD IN THEORY

- A result by Che and Kojima [2010] formalizes this “equivalence in the large” between RSD and PS: the two random assignments converge to the same limit as the instance grows larger

Poll 3

In light of these results, which rule would you use for school choice?

- Random serial dictatorship
- Probabilistic serial



BIBLIOGRAPHY

A. Bogomolnaia and H. Moulin. **A New Solution to the Random Assignment Problem.** Journal of Economic Theory, 2001.

Y.-K. Che and F. Kujima. **Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms.** Econometrica, 2010.

