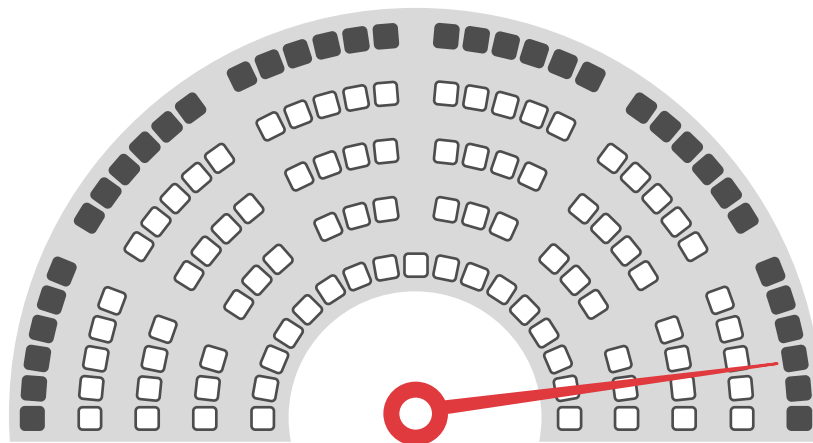




Optimized Democracy

Spring 2024 | Lecture 12

Indivisible Goods

Ariel Procaccia | Harvard University

PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.



Share Rent



Split Fare



Assign Credit



Divide Goods



Distribute Tasks

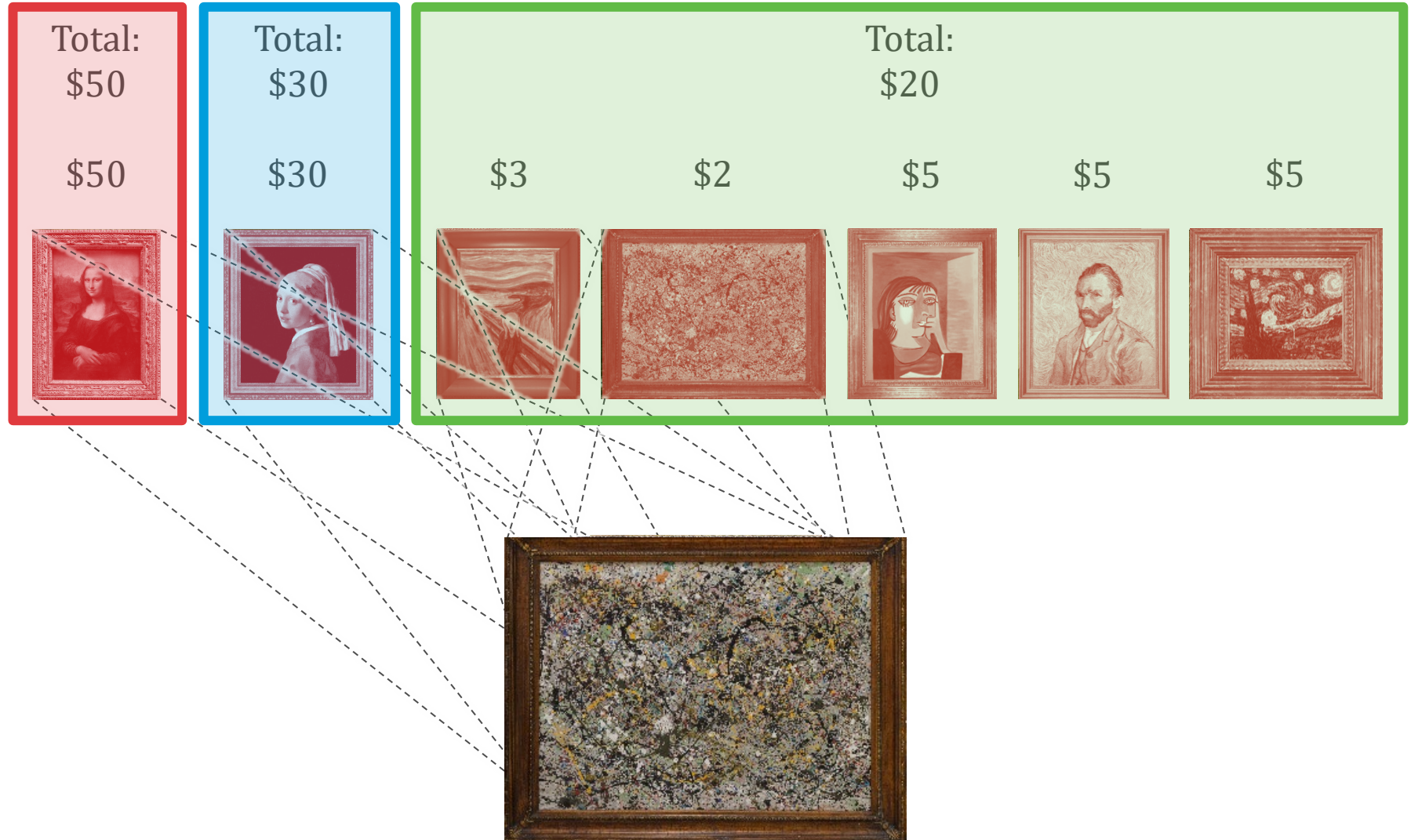


Suggest an App

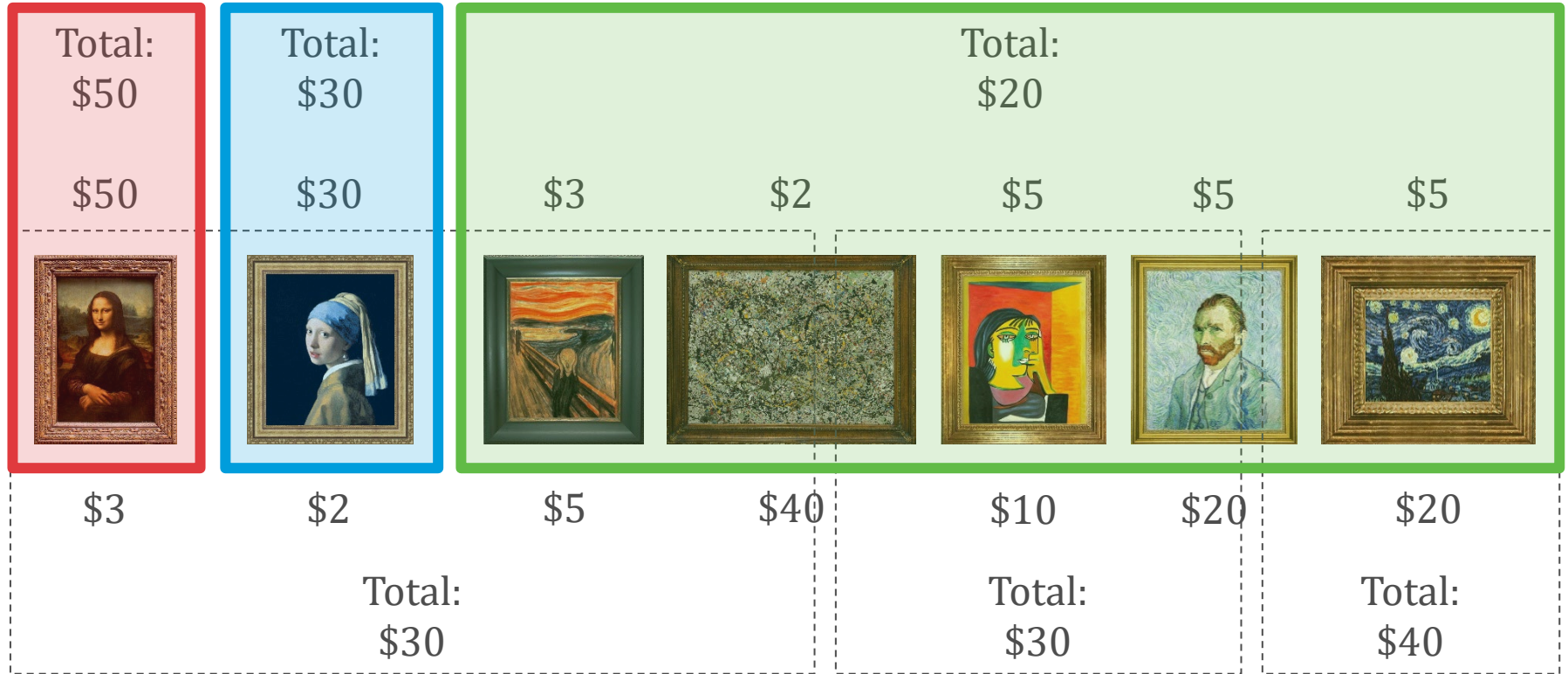
INDIVISIBLE GOODS

- Set G of m goods
- Each good is **indivisible**
- Players $N = \{1, \dots, n\}$ have valuations V_i for bundles of goods
- Valuations are **additive** if for all $S \subseteq G$ and $i \in N$, $V_i(S) = \sum_{g \in S} V_i(g)$
- Assume additivity unless noted otherwise
- An **allocation** is a partition of the goods, denoted $A = (A_1, \dots, A_n)$
- Envy-freeness and proportionality are infeasible!

MAXIMIN SHARE GUARANTEE



MAXIMIN SHARE GUARANTEE



MAXIMIN SHARE GUARANTEE

- **Maximin share (MMS) guarantee** of player i :

$$\max_{X_1, \dots, X_n} \min_j V_i(X_j)$$

- An **MMS allocation** is such that $V_i(A_i)$ is at least i 's MMS guarantee for all $i \in N$
- For $n = 2$ an MMS allocation always exists
- **Theorem:** $\forall n \geq 3$ there exist additive valuation functions that do not admit an MMS allocation

COUNTEREXAMPLE FOR $n = 3$

17	25	12	1
2	22	3	28
11	0	21	23

3 ways of dividing these numbers into 3 subsets of 4 numbers such that each subset adds up to 55

COUNTEREXAMPLE FOR $n = 3$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \times 10^6 + \begin{array}{|c|c|c|c|} \hline 17 & 25 & 12 & 1 \\ \hline 2 & 22 & 3 & 28 \\ \hline 11 & 0 & 21 & 23 \\ \hline \end{array} \times 10^3 +$$

3	-1	-1	-1
0	0	0	0
0	0	0	0

Player 1

3	-1	0	0
-1	0	0	0
-1	0	0	0

Player 2

3	0	-1	0
0	0	-1	0
0	0	0	-1

Player 3

APPROXIMATE ENVY-FREENESS

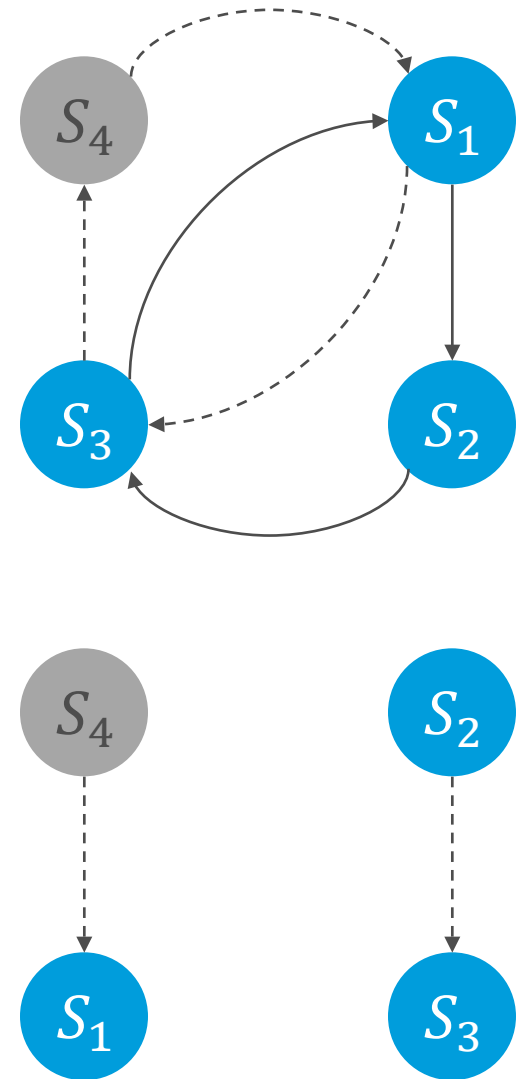
- Assume general **monotonic** valuations, i.e.,
for all $S \subseteq T \subseteq G$, $V_i(S) \leq V_i(T)$
- An allocation $A_1, \dots, A_n$ is **envy free up to one good (EF1)** if and only if
$$\forall i, j \in N, \exists g \in A_j \text{ s.t. } v_i(A_i) \geq v_i(A_j \setminus \{g\})$$
- **Theorem:** An EF1 allocation exists and can be found in polynomial time

PROOF OF THEOREM

- A **partial** allocation is an allocation of a subset of the goods
- Given a partial allocation A , we have an edge (i, j) in its **envy graph** if i envies j
- **Lemma:** An EF1 partial allocation A can be transformed in polynomial time into an EF1 partial allocation B of the same goods with an **acyclic** envy graph

PROOF OF LEMMA

- If graph has a cycle C , shift allocations along C to obtain A' ; clearly EF1 is maintained
- #edges in envy graph of A' decreased:
 - Same edges between $N \setminus C$
 - Edges from $N \setminus C$ to C shifted
 - Edges from C to $N \setminus C$ can only decrease
 - Edges inside C decreased
- Iteratively remove cycles ■

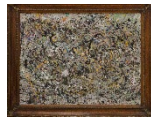


PROOF OF THEOREM

- Maintain EF1 and acyclic envy graph
- In round 1, allocate good g_1 to arbitrary player; envy graph is acyclic and EF1
- $g_1, \dots, g_{k-1}$ are allocated in acyclic and EF1 allocation $\mathbf{A}$
- Derive $\mathbf{B}$ by allocating g_k to **source** i
- $V_j(B_j) = V_j(A_j) \geq V_j(A_i) = V_j(B_i \setminus \{g_k\})$
- Use lemma to eliminate cycles ■

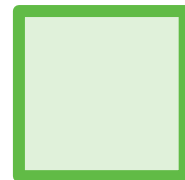
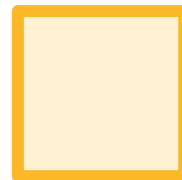
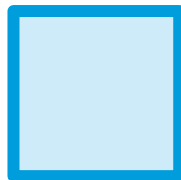
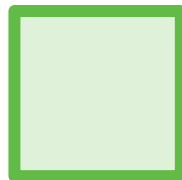
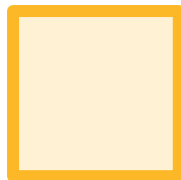
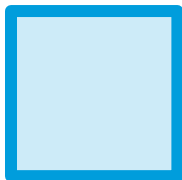
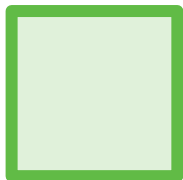
ROUND ROBIN

- Let us return to additive valuations
- Now proving the existence of an EF1 allocation is trivial
- A round-robin allocation is EF1:



Phase 1

Phase 2



EFFICIENCY AND FAIRNESS

- An allocation A is **Pareto efficient** if there is no allocation A' such that $V_i(A'_i) \geq V_i(A_i)$ for all $i \in N$, and $V_j(A'_j) > V_j(A_j)$ for some $j \in N$

Poll

Which of the following rules is Pareto efficient?

- Round Robin
- Both
- Max utilitarian social welfare
- Neither



MAXIMUM NASH WELFARE

- The **Nash welfare** of an allocation A is the product of values

$$NW(A) = \prod_{i \in N} V_i(A_i)$$

- The **maximum Nash welfare (MNW)** solution chooses an allocation that maximizes the Nash welfare
- For ease of exposition we ignore the case of $NW(A) = 0$ for all A
- **Theorem:** Assuming additive valuations, the MNW solution is EF1 and Pareto efficient

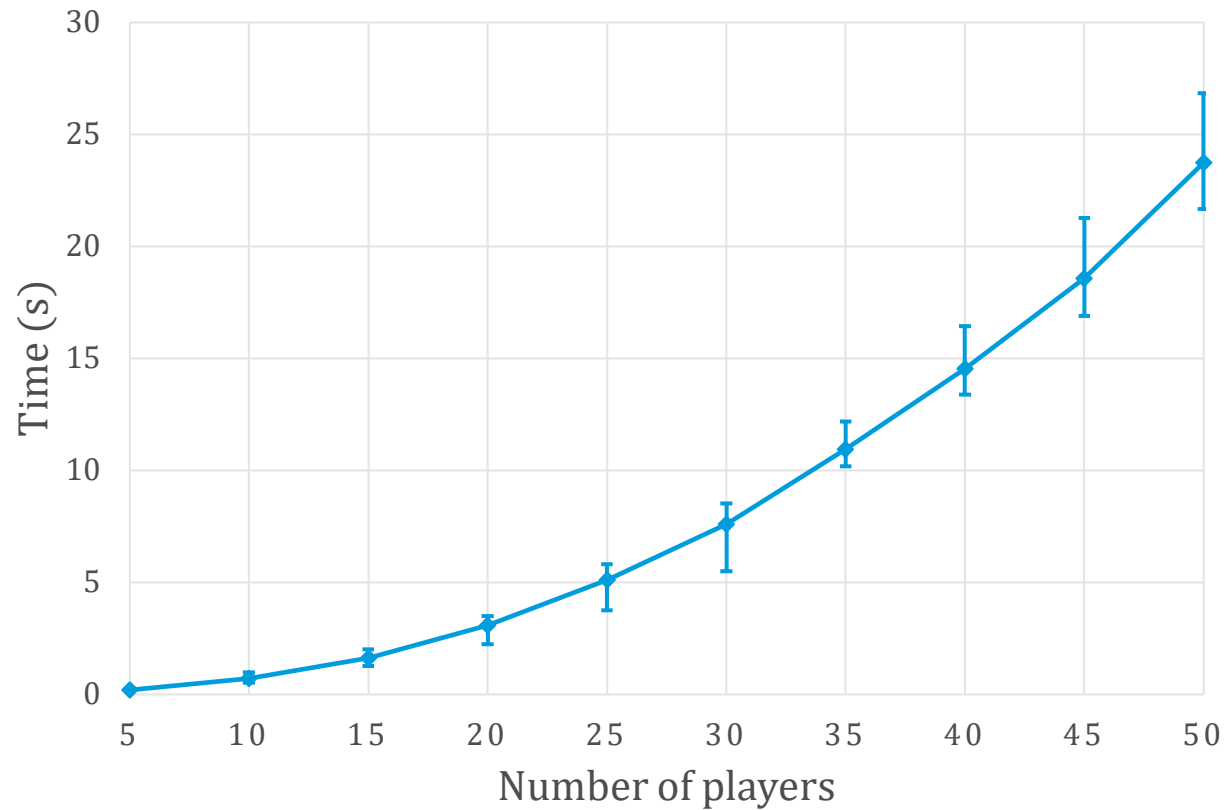
PROOF OF THEOREM

- Efficiency is obvious, so we focus on EF1
- Assume for contradiction that i envies j by more than one good
- Let $g^* \in \operatorname{argmin}_{g \in A_j} V_j(g)/V_i(g)$
- Move g^* from j to i to obtain A' , we will show that $\text{NW}(A') > \text{NW}(A)$
- It holds that $V_k(A_k) = V_k(A'_k)$ for all $k \neq i, j$,
 $V_i(A'_i) = V_i(A_i) + V_i(g^*)$, and
 $V_j(A'_j) = V_j(A_j) - V_j(g^*)$

PROOF OF THEOREM

- $\frac{NW(A')}{NW(A)} > 1 \Leftrightarrow \left[\frac{V_j(A_j) - V_j(g^*)}{V_j(A_j)} \right] \left[\frac{V_i(A_i) + V_i(g^*)}{V_i(A_i)} \right] > 1 \Leftrightarrow$
 $\frac{V_j(g^*)}{V_i(g^*)} [V_i(A_i) + V_i(g^*)] < V_j(A_j)$
- Due to our choice of g^* ,
$$\frac{V_j(g^*)}{V_i(g^*)} \leq \frac{\sum_{g \in A_j} V_j(g)}{\sum_{g \in A_j} V_i(g)} = \frac{V_j(A_j)}{V_i(A_j)}$$
- Due to EF1 violation, we have
$$V_i(A_i) + V_i(g^*) < V_i(A_j)$$
- Multiply the last two inequalities to get the first ■

TRACTABILITY OF MNW



[Caragiannis et al., 2016]

INTERFACE

THE BASICS +

ALICE'S EVALUATIONS ✓ -

Alice, use the sliders to assign values to each of the items below. All of your values must sum to 1000. You can use the *rescale* button to automatically adjust your values to add up to 1000.

Gold Ring



59

Diamond
Ring



145

Pearl
Necklace



116

Ruby
Earrings



265

Gold Watch



80

Silver
Bracelet



335

RESET

RESCALE

CONTINUE

Current Total: 1000

Target: 1000

BOB'S EVALUATIONS +

CLAIRE'S EVALUATIONS +

RESULTS +

AN OPEN PROBLEM

- An allocation $A_1, \dots, A_n$ is **envy free up to any good (EFX)** if and only if
$$\forall i, j \in N, \forall g \in A_j, v_i(A_i) \geq v_i(A_j \setminus \{g\})$$
- Strictly stronger than EF1, strictly weaker than EF
- An EFX allocation exists for two players with monotonic valuations (easy) and for three players with additive valuations (very hard)
- Existence is an open problem for $n \geq 4$ players with additive valuations

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