Optimized Democracy

Spring 2024 | Lecture 11

Rent Division

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PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

Share Rent

Split Fare

Assign Credit

Divide Goods

Distribute Tasks

Suggest an App

ONCE UPON A TIME IN JERUSALEM

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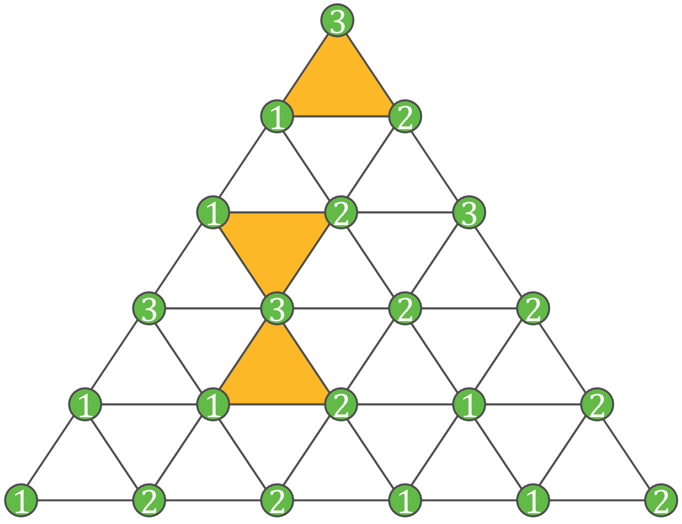
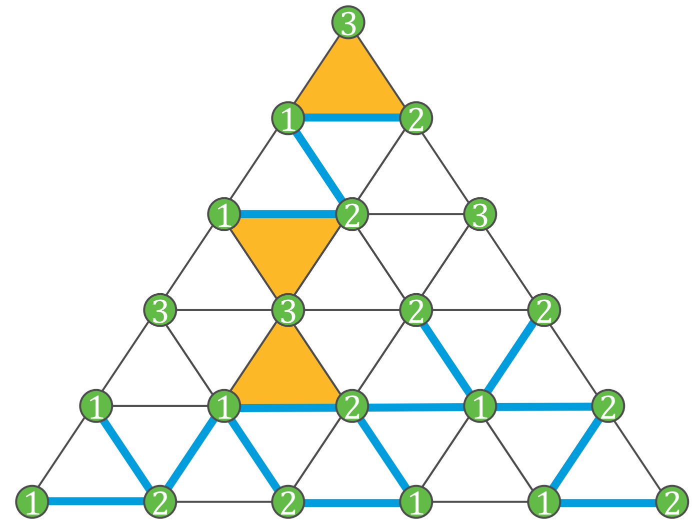
SPERNER'S LEMMA

- Triangle T partitioned into elementary triangles
- Label vertices by $\{1,2,3\}$ using Sperner labeling:
  - Main vertices are different
  - Label of vertex on an edge (i, j) of T is i or j
- Lemma: Any Sperner labeling contains at least one fully labeled elementary triangle



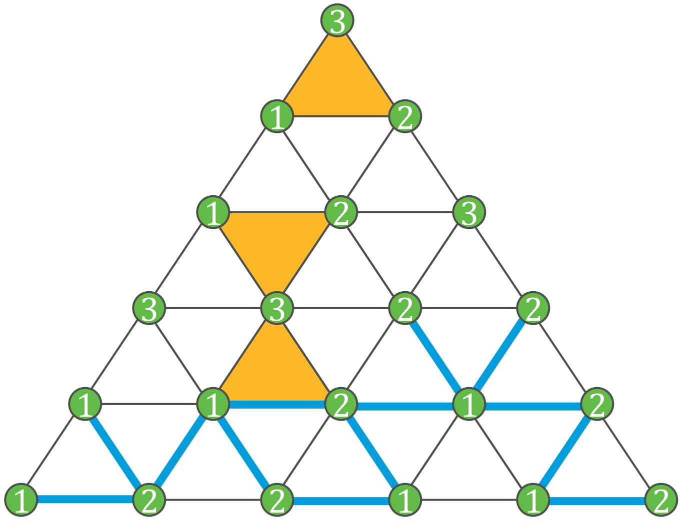
PROOF OF LEMMA

- Doors are 12 edges
- Rooms are elementary triangles
- #doors on the boundary of T is odd
- Every room has ≤ 2 doors; one door iff the room is 123



PROOF OF LEMMA

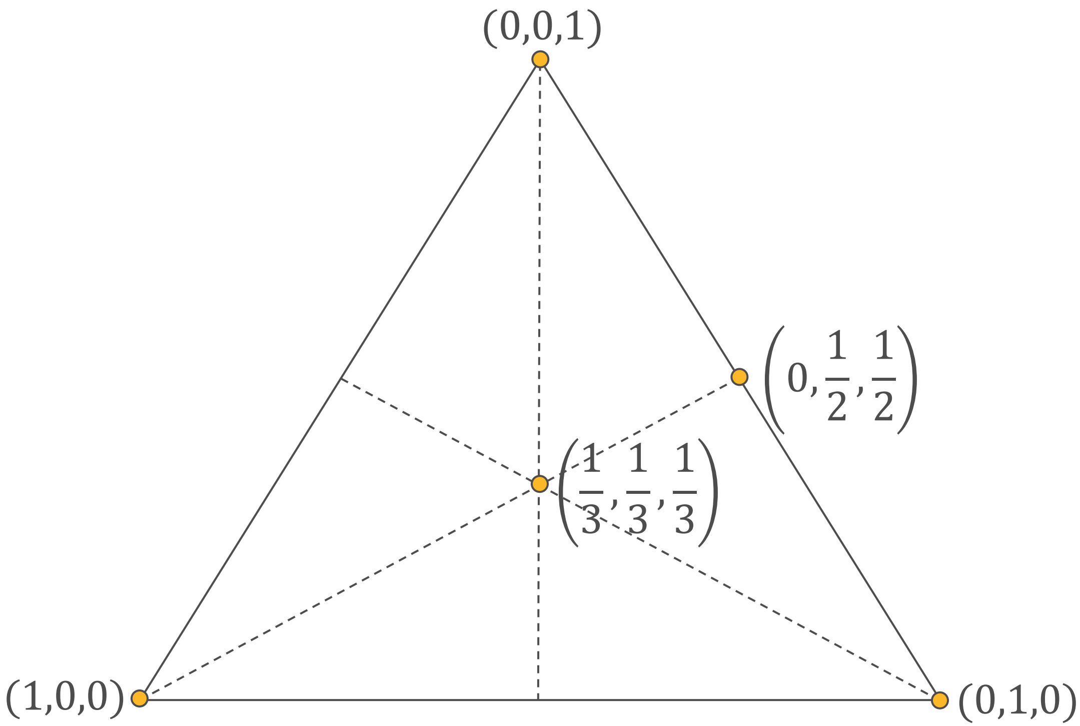
- Start at door on boundary and walk through it
- Room is fully labeled or it has another door...
- No room visited twice
- Eventually walk into fully labeled room or back to boundary
- But #doors on boundary is odd ■



THE MODEL

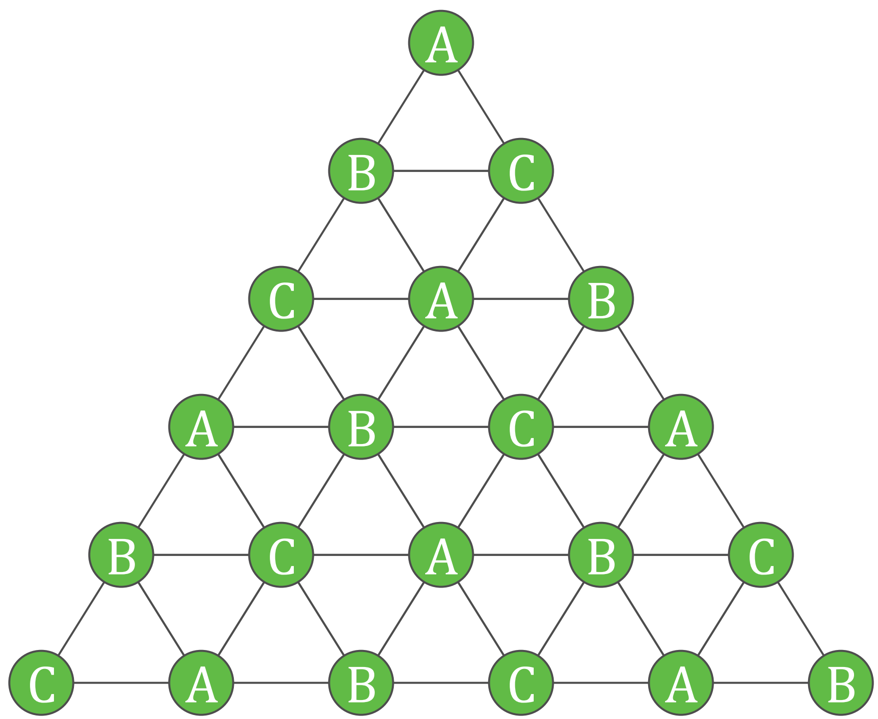
- Assume there are three players A, B, C
- Goal is to assign the rooms and divide the rent in a way that is envy free: each player prefers their own room at the given prices
- Sum of prices for three rooms is 1
- Theorem: An envy-free solution always exists under some assumptions

PROOF OF THEOREM



PROOF OF THEOREM

- "Triangulate" and assign "ownership" of each vertex to each of A, B, and C, in a way that each elementary triangle is an ABC triangle

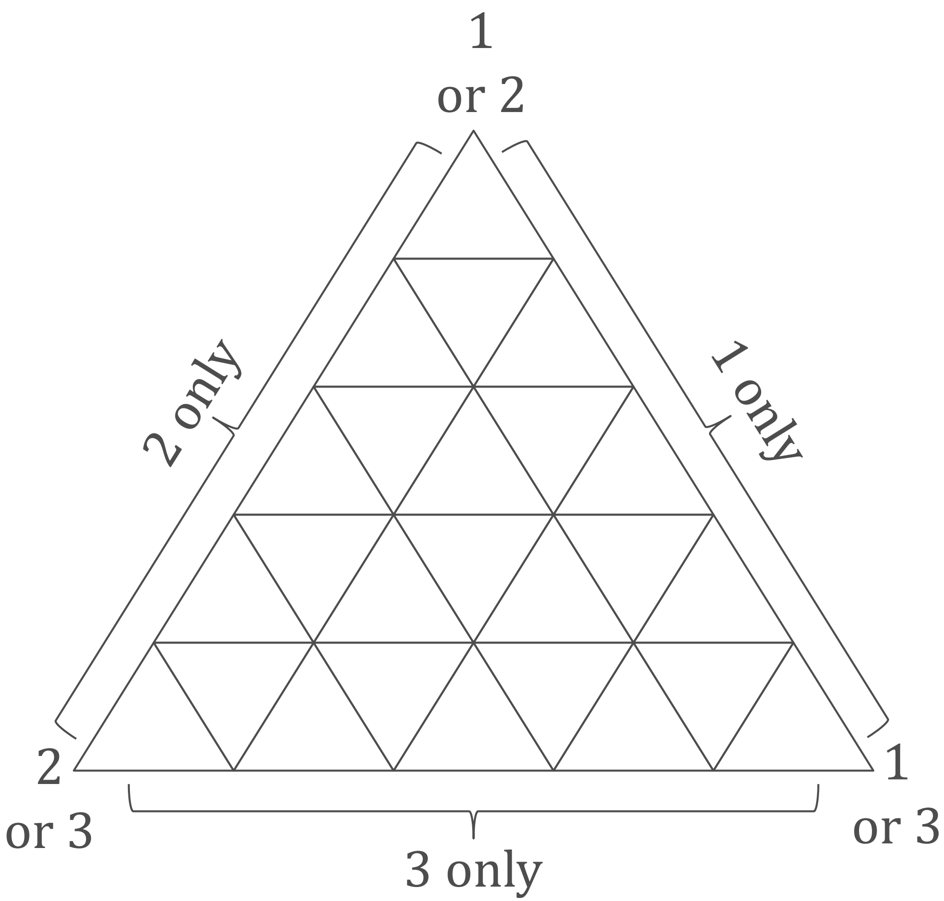


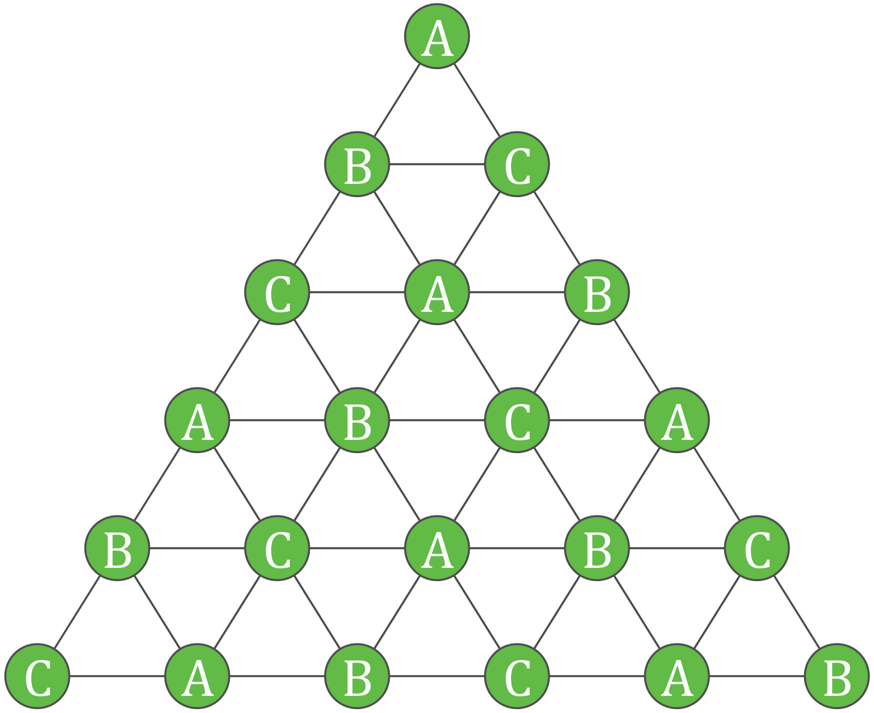
PROOF OF THEOREM

- Ask the owner of each vertex to tell us which room they prefer
- This gives a new labeling by 1, 2, 3
- Assume that a player wants a free room if one is offered to them

PROOF OF THEOREM

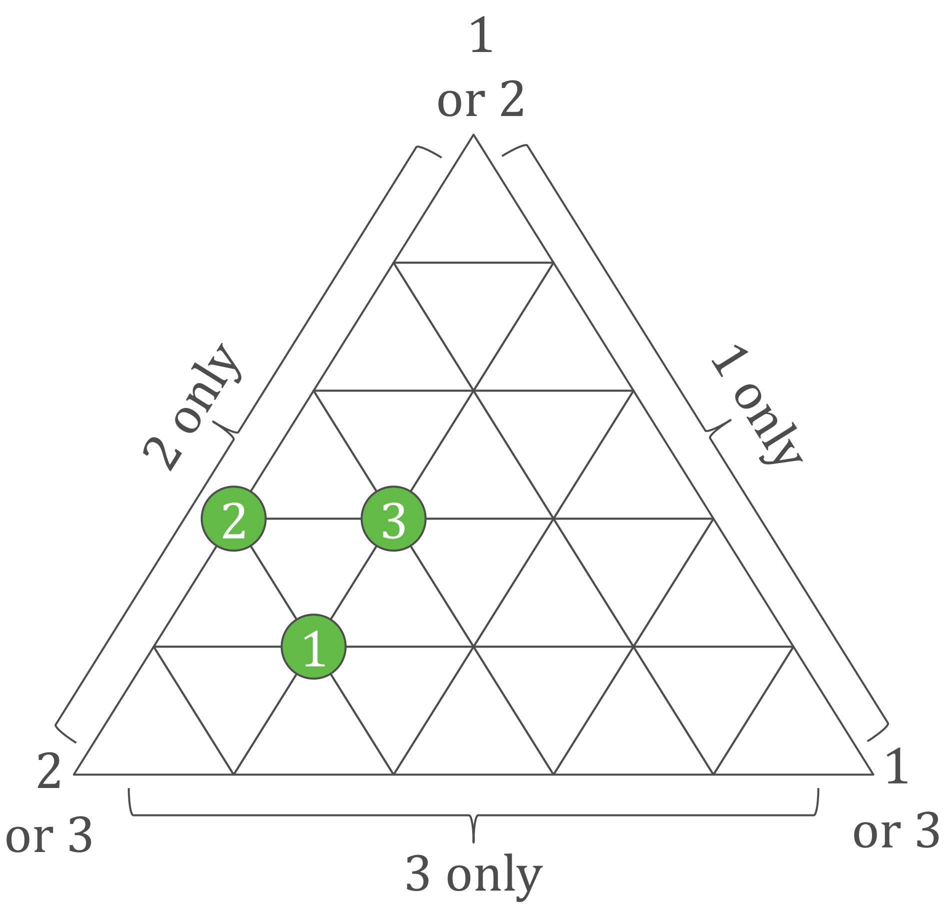
- Choice of rooms on edges is constrained by free room assumption

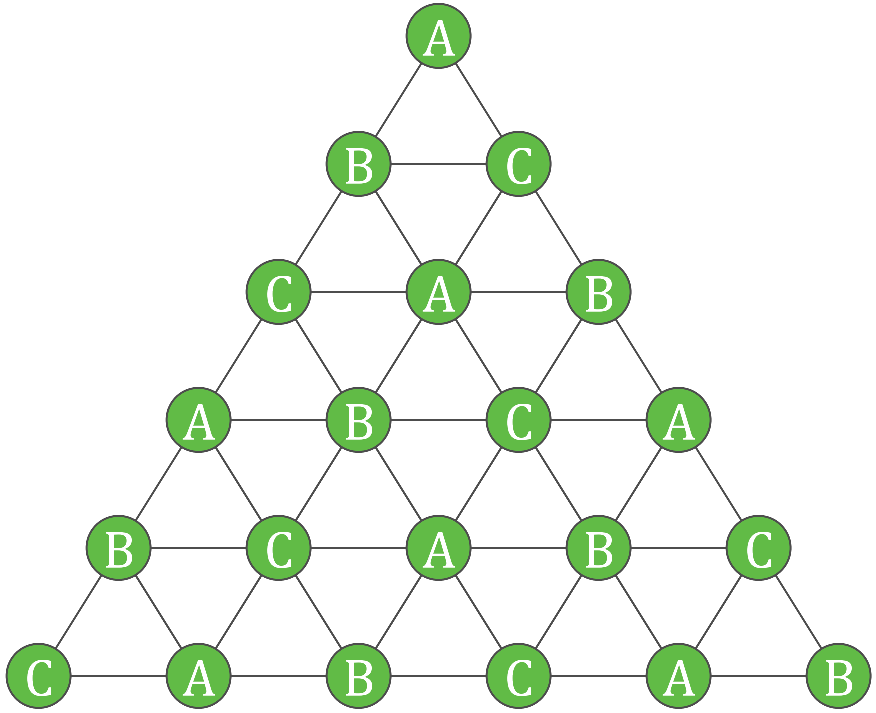




PROOF OF THEOREM

- Sperner's lemma (variant): such a labeling must have a 123 triangle





PROOF OF THEOREM

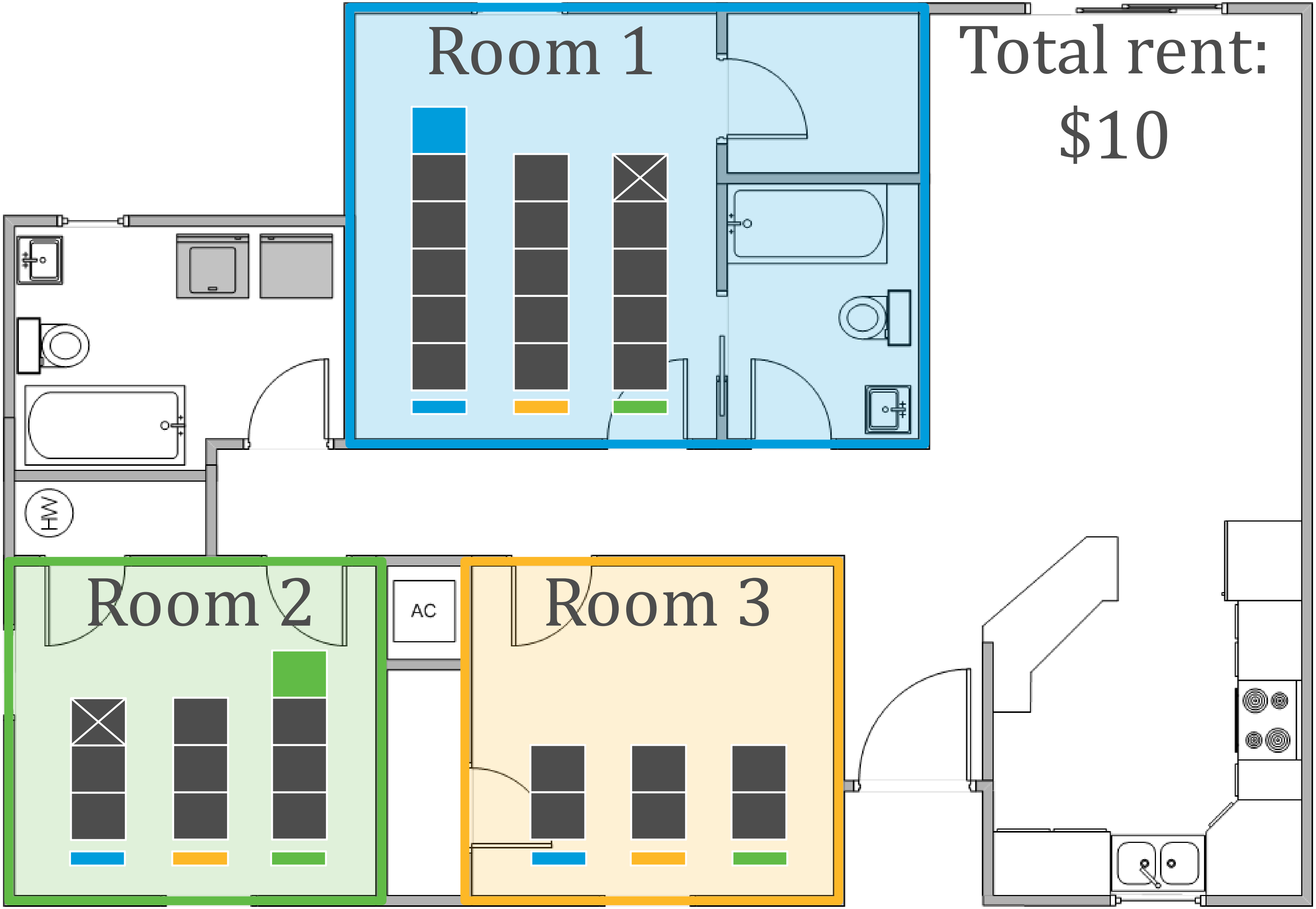
- Such a triangle is nothing but an approximately EF solution!
- By making the triangulation finer, we can approach envy-freeness
- Under additional closedness assumption, leads to existence of an EF solution ■

DISCUSSION

- It is possible to derive an algorithm from the proof
- Same techniques generalize to more players
- Same proof (with the original Sperner's Lemma) shows existence of EF cake division!

QUASI-LINEAR UTILITIES

- Suppose each player $i \in N$ has value v_{ir} for room r
- For all $i \in N$, $\sum_r v_{ir} = R$, where R is the total rent
- The utility of player i for getting room r at price p_r is $v_{ir} - p_r$
- A solution consists of an assignment π and a price vector $\boldsymbol{p}$, where p_r is the price of room r
- Solution $(\pi, \boldsymbol{p})$ is envy free if and only if
$$\forall i, j \in N, v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\pi(j)} - p_{\pi(j)}$$
- **Theorem:** An envy-free solution always exists under quasi-linear utilities


Room 1
Total rent:
$10
HW
Room 2
AC
Room 3

PROPERTIES OF EF SOLUTIONS

- Assignment π is welfare-maximizing if

$$\pi \in \text{argmax}_\sigma \sum_{i \in N} v_{i\sigma(i)}$$

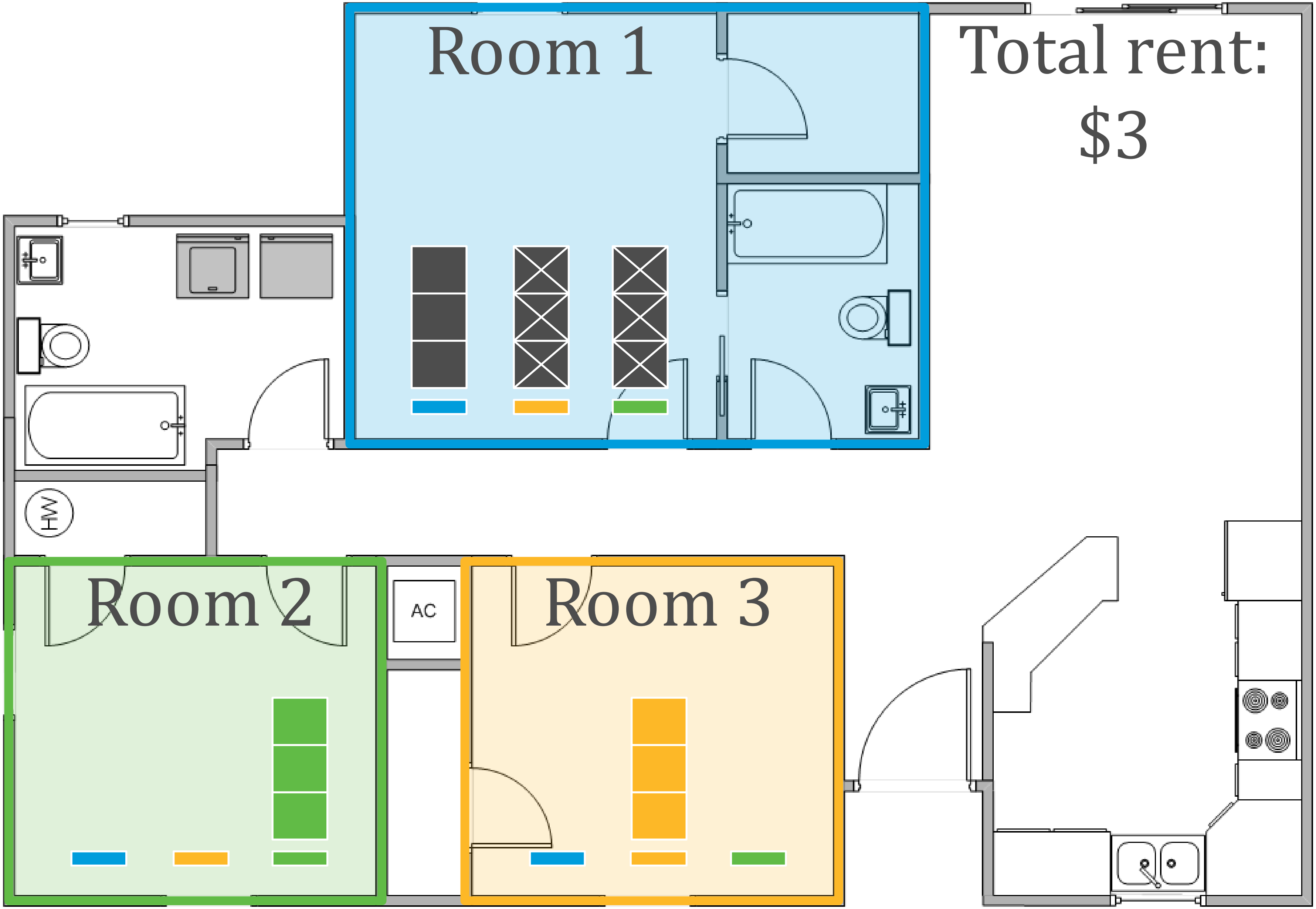
- Lemma 1: If $(\pi, \boldsymbol{p})$ is an EF solution, then π is a welfare-maximizing assignment
- Lemma 2: If $(\pi, \boldsymbol{p})$ is an EF solution and σ is a welfare-maximizing assignment, then $(\sigma, \boldsymbol{p})$ is an EF solution

PROOF OF LEMMA 1

- Let $(\pi, \boldsymbol{p})$ be an EF solution, and let σ be another assignment
- Due to EF, for all i,
$$v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\sigma(i)} - p_{\sigma(i)}$$
- Summing over all i,
$$\sum_{i\in N} v_{i\pi(i)} - \sum_{i\in N} p_{\pi(i)} \geq \sum_{i\in N} v_{i\sigma(i)} - \sum_{i\in N} p_{\sigma(i)}$$
- We get the desired inequality because prices sum up to R ■

POLYNOMIAL-TIME ALGORITHM

- Consider the algorithm that finds a welfare-maximizing assignment π, and then finds prices $\boldsymbol{p}$ that satisfy the EF constraint
- Theorem: The algorithm always returns an EF solution, and can be implemented in polynomial time
- Proof:
    - We know that an EF solution $(\sigma, \boldsymbol{p})$ exists, by Lemma 2 $(\pi, \boldsymbol{p})$ is EF, so we would be able to find prices satisfying the EF constraints
    - The first part is max weight matching, the second part is a system of linear inequalities ■


Room 1
Total rent:
$3
HW
Room 2
AC
Room 3

OPTIMAL EF SOLUTIONS

Straw Man Solution

Max sum of utilities
Subject to envy freeness

Maximin Solution

Max min utility
Subject to envy freeness

Equitable solution

Min max difference in utils
Subject to envy freeness

OPTIMAL EF SOLUTIONS

- Theorem: The maximin and equitable solutions can be computed in polynomial time
- Theorem: The maximin solution is unique
- Theorem: The maximin solution is equitable, but not vice versa

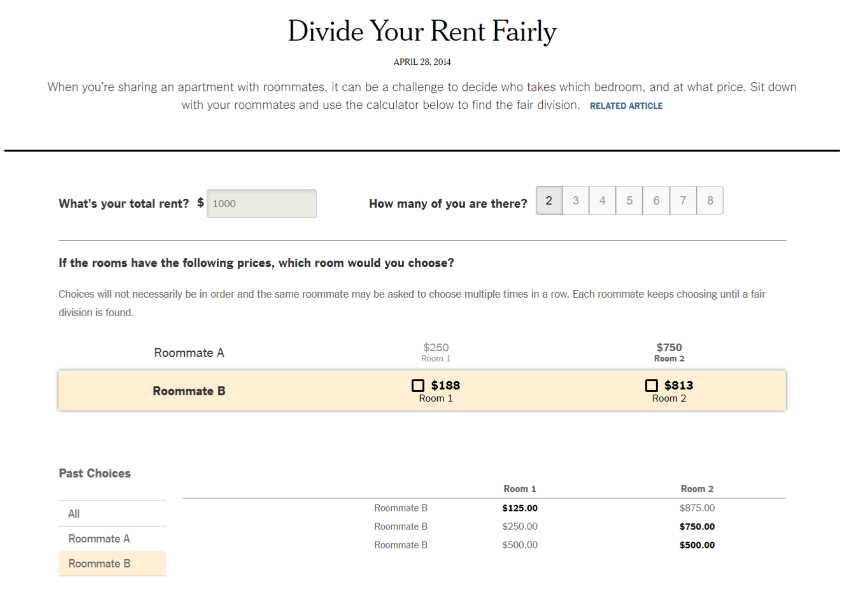
DISCUSSION

- The first model makes no assumptions on utilities other than players preferring free rooms
- The second model assumes quasi-linear utilities

Question

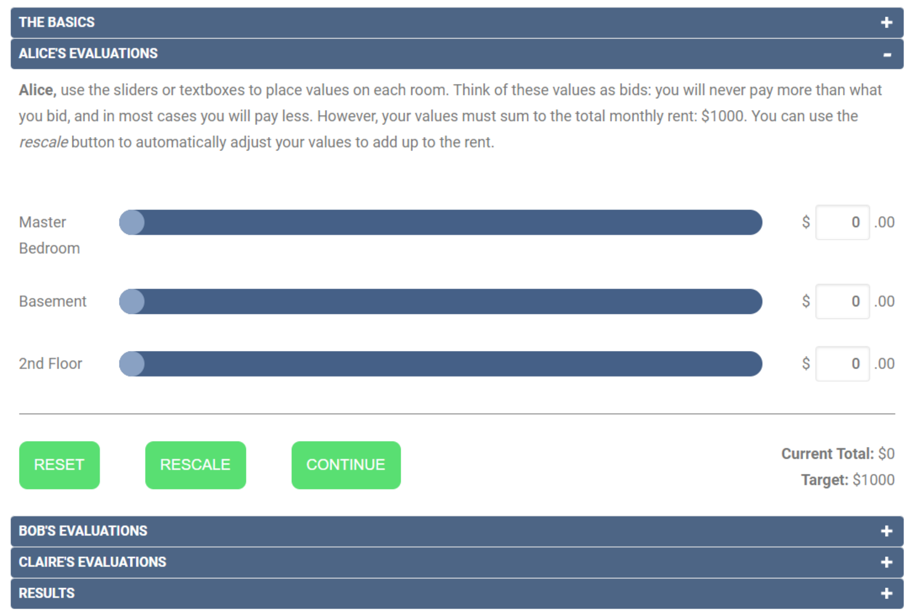
What are some advantages and disadvantages of each of the two models?

INTERFACES



NY TIMES (rental harmony)

https://www.nytimes.com/interactive/2014/science/rent-division-calculator.html



Spliddit (quasi-linear utilities)

http://www.spliddit.org/apps/rent

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A. Alkan and G. Demange and D. Gale. Allocation of Indivisible Goods and Criteria of Justice. Econometrica, 1991.

Y. Gal and M. Mash and A. D. Procaccia and Y. Zick. Which Is the Fairest (Rent Division) of Them All? Journal of the ACM, 2017.