Optimized Democracy (Spring 2023)
Problem Set #4

Due: 4/7/2022 11:59pm ET

Instructions:

- It is fine to look up a complicated sum or inequality, but please do not look up an entire solution. In particular, the solutions to many of the problems that we give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- You may discuss the problems with classmates but please write down solutions completely on your own.
- Please *type up* your solution and submit to Gradescope.

Problems:

1. **[20 points]** In class we stated two lemmas in the context of the rent division with quasi-linear utilities. The second lemma can actually be strengthened as follows:

   Lemma: If $(\pi, \mathbf{p})$ is an EF solution and σ is a welfare-maximizing assignment, then for all $i \in N$, $v_{i\pi(i)} - p_{\pi(i)} = v_{i\sigma(i)} - p_{\sigma(i)}$.

   Using this lemma (without proof), describe a polynomial-time algorithm for computing the Maximin solution and prove its correctness.

   Guidance: Follow the algorithm sketch on Slide 19 of Lecture 11 but replace the system of linear inequalities with an appropriate linear program. You may assume that a linear program can be solved in polynomial time. Note that a linear program can maximize the minimum of linear functions; you may search for this technique online.

2. Consider a setting with n players and a set G of m indivisible goods. Let $V_1, \ldots, V_n$ represent the additive valuation functions of the n players and assume that all players have positive valuations for all goods.

   As mentioned in class, it is an open problem whether EFX allocations exist in settings with more than three players; therefore, in this problem, we consider a relaxation of EFX. Define an allocation to be 1/2-EFX if, for any two players i and j, i's value for j's bundle minus any good is at most twice i's value for her own bundle.

Algorithm 1 1/2-EFX Allocation

Require: $n, G, (V_1, \ldots, V_n)$ ▷ Input: players, goods, and valuation functions
$P \leftarrow G$ ▷ Initialize: all goods in pool
for $i \in [n]$ **do**
  $A_i \leftarrow \emptyset$ ▷ Initialize: all players start with no goods
end for
while $P \neq \emptyset$ **do** ▷ Repeat while pool not empty
  $g^* \leftarrow \text{pop}(P)$ ▷ Remove an arbitrary good from the pool
  $j \leftarrow \text{FindUnenviedPlayer}(\boldsymbol{A})$ ▷ and give it to an unenvied player
  $A_j \leftarrow A_j \cup \{g^*\}$
  if $\exists i \in [n]$, $g \in A_j$ such that $V_i(A_i) < \frac{1}{2} V_i(A_j \setminus \{g\})$ **then** ▷ if this breaks 1/2-EFX
    $P \leftarrow P \cup A_i$ ▷ Return i's old allocation to the pool
    $A_j \leftarrow A_j \setminus \{g^*\}$
    $A_i \leftarrow \{g^*\}$ ▷ and give i $\{g^*\}$
  end if
  $A \leftarrow \text{RemoveEnvyCycles}(\boldsymbol{A})$ ▷ Ensure the envy graph is acyclic
end while

Definition: An allocation $\boldsymbol{A}$ is 1/2-EFX if, for all i and j, for all $g \in A_j$,

$$V_i(A_i) \geq (1/2) \cdot V_i(A_j \setminus \{g\}).$$

Consider Algorithm 1 for finding a 1/2-EFX allocation for n players. You may assume that FindUnenviedPlayer always returns an unenvied player (if one exists). Furthermore, given a 1/2-EFX allocation with an envy graph that contains cycles, RemoveEnvyCycles returns a 1/2-EFX allocation with an acyclic envy graph without decreasing social welfare. Note that, in this algorithm, RemoveEnvyCycles ensures that when FindUnenviedPlayer is called, an unenvied player does exist.

(a) **[25 points]** Prove that at the beginning of each iteration of the while loop, the partial allocation is 1/2-EFX.

(b) **[15 points]** Prove that the algorithm terminates.
Hint: Consider what happens to the social welfare in each round.

3. In the context of sortition we discussed *allocation rules*, which receive a set of volunteers N, a panel size k, a set of features F, set of values V_f for each $f \in F$, and quotas $u_{f,v}, \ell_{f,v}$ for all $f \in F$ and $v \in F_v$; they output a distribution over panels that satisfy the given quotas if one exists. In particular, we saw that allocation rules like Leximin and Maximum Nash Welfare lead to seemingly fair selection probabilities. But do these rules satisfy appealing axiomatic properties? On a high level the answer is negative, but that is mostly due to strong, general impossibility results that hold in this domain; below you are asked to establish those negative results.

(a) **[25 points]** An allocation rule guarantees *population monotonicity* if, when additional volunteers are added to an instance — that is, there are two sets of volunteers N and N'

such that $N \subseteq N'$, and the panel size, features, values and quotas remain unchanged — the selection probabilities of all previously existing volunteers weakly decrease.

Prove that no allocation rule satisfies population monotonicity.

(b) [**15 points**] An allocation rule guarantees *committee monotonicity* if, when an instance is modified by increasing the size of the panel — that is, there are two panel sizes k and k' such that $k' \geq k$, and the volunteers, features, values and quotas remain unchanged (and the instance remains feasible) — the selection probabilities of all volunteers weakly increase.

Prove that no allocation rule satisfies committee monotonicity.